

Summary of the Film
'Mesh'
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1. Platonic Solids

In an archeological site some archeologists once unearthed stones with a high degree of symmetry. While nobody really knows why somebody created such artefacts, one thing is clear - such symmetry can also be found in nature, e.g. in crystals. These symmetries are related to a certain class of mathematical objects the ancient Pythagoreans school mathematicians were interested in, i.e. the Platonic solids.

As many of us know, there are five Platonic solids. The Pythagorean school mathematicians have also invented and made good use of the notion of 'duality' when studying such solids.

Although we call these solids 'Platonic', Plato did not invent them. Instead, he tried to understand such solids using the Pythagoras Theorem, as coordinates system were probably not known to him at that time.

2. The book 'Timaeus'

The great Greek philosopher Plato was very much interested in these objects, he even tried to establish a kind of 'atomism' based on such solids, claiming that everything in this world is build up of such solids, somewhat similar to the old Chinese Ying-Yang theory. In Plato's theory, the tetrahedron represents fire, the cube earth, the octahedron air, icosahedron water and the dodecahedron the universe.

In a book called Timaeus, Plato also described a theory claiming that everything, including humans could be described by triangles. (A sort of reductionism, or unified theory of nature) This theory also explains how the aging process occurs - a young man is made of strong and fresh triangles, hence is able to digest the triangles in a fruit. On the other hand, the triangles in an old man are weak and fragile and hence cannot digest the triangles in a fruit.

3. Kepler

The great astronomer J. Kepler applied Plato's ideas to astronomy and proposed a very sophisticated theory concerning planetary motions. In his theory, the orbits of Mercury, Venus, earth, Jupiter and Saturn are said to be situated on spheres circumscribing or touching some Platonic solids.

While interesting in itself, Kepler's theory does not conform to the data observed, which were made possible by the invention and improvement of a new device known as the telescope. The interesting part of the story is that the lenses in telescopes are related to conic sections and conic sections ultimately turned out to be the key to understanding planetary orbits.

As an aside, it is worth mentioning that conic sections were known to the Greek mathematicians, but their ways of studying and understanding conic sections did not provide the real tool to explain planetary motions. A completely new kind of mathematics, Calculus, turned out to be essential to the unraveling of the planetary mystery.

4. Discrete Measurements

The question how continuous change can be measured is a theoretical as well as practical question. In the time of Kepler, there was no good algorithm known to measure the volume of something as simple as a wine barrow to arbitrary accuracy. To this end, different approximations were given. And it was not until Newton and Leibniz invented Calculus that this question could be given a satisfactory answer.

The idea of Calculus was 'revolutionary' at the time of its inception, because one has started to get interested now on 'infinitesimal changes' or 'sum of infinitesimal objects', rather than discrete changes (just think about a closed curve without corners versus a polygon!).

In Calculus, one has to deal with arithmetic of 'infinitesimals', which is itself a rather difficult concept to grasp.

Indeed discrete geometry and the geometry of smooth objects seem to be quite different, as can be shown by bizarre examples like Schwarz's lantern. As such, we can ask whether it is possible to bridge these two types of mathematics.

5. Comeback of discrete geometry

With the increasing popularity of computers, discrete mathematics made finally its comeback. Indeed, all images on the computer screens are but discrete approximations of their smooth continuous counterparts, hence discrete thoughts became once again important.

6. Curvature for discrete and smooth objects

One can define curvature on a smooth curve. When one thinks carefully about this definition, one discovers a close relationship between this definition and the angular relationship of external angles of a triangle (or a polygonal curve).

Generalizing this idea, one can define ‘curvature’ on a polygonal surface. This curvature is related to something called the ‘Gaussian curvature’ in geometry, a concept first introduced by Gauss.

For practical purposes, another curvature which related to the notion of surface tension in Physics turns out to be important. This curvature is known as the ‘mean curvature’ and appears in contexts like soap films and soap bubbles.

7. Mesh

As mentioned, we want to a link between discrete geometry and smooth (or continuous) geometry. One simple idea is to start from a discrete approximation of the smooth and continuous and subsequently refine our approximation.

Indeed this can be done in some intelligent ways, so that the computation cost is not too ‘expensive’. One key idea dates back to make this possible comes from a French automobile engineer.

8. Reconstructing surface

To recover a good approximation of the object we aimed at describing, we first obtain sample points from the object and subsequently perform some kind of ‘processing’. Such processing can be done in two ways, in order that edges remain edges and faces remain faces. If we do not separately deal with faces and edges, the recovered surface would not have sharp edges and hence does not resemble the original object.

9. Discrete Soap Bubble?

All these ideas, such as approximating a smooth continuous surface with a discrete polyhedral surfaces, defining discrete curvatures etc. have at least one more application (apart from the aforementioned industrial application of reconstructing a surface).

What we mean is the construction of ‘soap bubbles’ from discrete polyhedral surfaces, just like building complicated objects from Lego pieces. In pretty much the same way, one starts from simple pieces, glue them together in some nice ways and is able to finally arrive at more complicated ‘soap bubbles’.

Indeed one can construct a ‘new soap bubble’ called the Penta-surface on a computer screen using this idea. But what about a proof? A theoretical proof is still not available.

10. Concluding Remarks

History of mathematics is sometimes ironical. With the success and achievements of Calculus, discrete geometric objects have for a while been forgotten by many. It was not until the advent of computer that this once important branch of mathematics could be revitalized. Not only this, modern computer visualization has also helped setting up a

linkage between discrete and continuous (or smooth) geometry as well as an arena on 'experimental mathematics'.

Some hyperlinks to websites related to the film or to Plato's works. You can also find related works from these websites.

- www.mesh-film.de
- <http://page.mi.fu-berlin.de/polthier/>
- [http://en.wikipedia.org/wiki/Timaeus_\(dialogue\)](http://en.wikipedia.org/wiki/Timaeus_(dialogue))
- <http://www.dartmouth.edu/~matc/math5.geometry/unit6/unit6.html>

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