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數聞 IMOment

不可能的藝術
THE ART OF THE IMPOSSIBLE

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概率與 IMO
PROBABILITY AND IMO

2016 年第五十七屆國際數學奧林匹克籌備委員會
Organising Committee of the 57th International Mathematical Olympiad 2016

聯絡
CONTACT
info@imohkc.org.hk
math@edb.gov.hk
facebook.com/imo2016
twitter.com/imo2016

2016 年第五十七屆
國際數學奧林匹克籌備委員會

香港將於 2016 年 7 月舉辦第五十七屆國際數學奧林匹克（IMO），迎接來自超過 100 個國家或地區的中學生數學精英。希望《數聞》可在我們邁向 2016 年 IMO 期間帶動同學和公眾對數學的興趣，更希望這種氣氛歷久不衰。

Hong Kong is proud to be organizing the brightest secondary school mathematics talents from over 100 countries or regions at the 57th International Mathematical Olympiad (IMO) in July 2016. We hope that IMOment will promote interest in mathematics among students and the public in this period leading up to IMO 2016, and beyond.

Organising Committee of the 57th International Mathematical Olympiad 2016

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李祐榮：衣帶漸寬終不悔
Albert Li: Living For My Love

/盧安迪 ANDY LOO

李祐榮畢業於英華書院，曾代表香港參加 2011年 和 2012年 國際數學奧林匹克，共奪一金一銀。他剛從香港中文大學數學系畢業，並將成為美國耶魯大學數學系研究生。

我自小學便認識李祐榮，也多次跟他一起代表香港出外參賽。記得有一次跟李祐榮同房，早上醒來，只見李祐榮坐在床上望著我，他跟我說的第一句話不是「早安」，而是「設 n 是一個正整數！」李祐榮對數學的敏銳觸覺和全情投入，永遠是我學習的榜樣。

盧：盧安迪　李：李祐榮

Albert Li Yau-wing, a graduate of Ying Wa College, represented Hong Kong at the International Mathematical Olympiad in 2011 and 2012, winning one gold and one silver medal. He just graduated from the Chinese University of Hong Kong with a bachelor degree in mathematics, and will be going to graduate school in mathematics at Yale University in the United States.

I have known Albert since we were in primary school, and we have represented Hong Kong together at many competitions. One time he was my roommate. When I opened my eyes in the morning, Albert was sitting on his bed looking at me. The first sentence he uttered was not "Good morning," but "Let n be a positive integer!" Albert's mathematical prowess and his wholehearted dedication to the subject have earned my ever-increasing respect over the years.

盧：你最初是怎樣發現你對數學的興趣的？

李：我於幼稚園時發現數學十分有趣，因為當時我經常取姐姐的數學課本來看。其實姐姐較我年長四歲。當年姐姐背乘數表時，我也懂得背誦。因此，我在幼稚園時已掌握加減乘除。

Andy: When and how did you first discover your interest in math?

Albert: I found mathematics to be very interesting in kindergarten, when I often borrowed my sister's math textbooks to read. Actually my sister is four years older than me. When my sister was reciting the multiplication table, I learned it too. So I knew addition, subtraction, multiplication and division back when I was in kindergarten.

盧：你什麼時候開始參加數學比賽？

李：小學三年級時，香港發生沙士疫症，學校停課，我便在這段期間，完成整本十屆小學數學奧林匹克比賽試題。及後在小學五年級參加第一次數學奧林匹克比賽。

盧：你為何喜歡數學呢？

李：小時候的我喜愛數學是因為享受解決數學問題所帶來的成功感；而現在我喜愛數學是因為它的抽象，這特性往往能夠將不同範疇如幾何、數論等連繫起來。

盧：你對哪個數學領域最感興趣？

李：由中學到現在，數論和代數一直都是我最感興趣的數學範疇。因為數論背後的理論往往比其他範疇更抽象和優雅，而代數則把拓撲學、幾何、數論等範疇聯繫起來。

盧：可否分享一下你參加奧數的難忘經歷？

李：我最難忘的是二零一一年的中國數學奧林匹克比賽，那是於吉林省長春市舉行，還記得出外參觀當地景點時，氣溫低至攝氏零下三十多度，沒有到過這麼冷地方的我步行數分鐘已感到不妙，結果要入醫療室保暖。

盧：你認為你在數學上的成功建基於哪些因素呢？

李：我認為自己對數學的興趣是成功最重要的因素。濃厚的興趣驅使我花在數學的時間較別人多，也較別人勤力。同時，這份興趣亦令我在面對難題時不會輕言放棄，從而達致成功。

Andy: When did you start to take part in math competitions?

Albert: When I was in Primary 3, the SARS epidemic broke out in Hong Kong, and school was suspended. It was during that time that I completed the entire volume of 10 years of primary school math Olympiad past papers. Then in Primary 5, I took part in my first math Olympiad competition.

Andy: Why do you like mathematics?

Albert: When I was smaller, I liked mathematics for the satisfaction of successfully solving a problem. Now, I like mathematics because of its abstractness, which is what often links up different fields such as geometry and number theory.

Andy: Which areas of math are you most interested in?

Albert: Number theory and algebra have been my favorite mathematical fields since secondary school. The underlying ideas of number theory are often more abstractness and elegant than those in other areas. On the other hand, algebra reveals connections between topology, geometry, number theory and other fields.

Andy: Are there any particularly unforgettable stories from your math Olympiad experience?

Albert: My most unforgettable experience is the 2011 Chinese Mathematical Olympiad, which was held in Jilin City of the Changchun Province. During our excursion, the temperature was as low as −30 ºC, and I had never been to such a cold place before. Just after walking outdoors for a few minutes, I couldn't stand it any longer, and was sent to the medical room.

Andy: What do you think are the most important factors that contributed to your mathematical success?

Albert: I think the most crucial factor for my success is my deep interest in mathematics, which motivated me to spend more time and work harder on math than others do. At the same time, this passion also keeps me going in the face of hard problems, eventually leading to achievements.

盧：你在求學過程中有沒有遇上特別難忘的挫折？你又怎樣克服呢？

李：在剛升上中學的時候，我發覺到小學奧數和中學奧數有著本質上的不同，小學奧數較著重速算技巧；而中學奧數則強調邏輯思考，這不同之處令我有點習慣不來。透過不停地做練習，我終於適應下來。

盧：你有沒有一道最喜歡的奧數題目呢？

李：在眾多奧數題目中，我特別喜歡以下這條融合了組合和幾何的題目，它是荷蘭於1988年提交給 IMO 委員會的試題，可惜最終未能被選中。

給定平面上的 1988 個點，沒有三點共線。現把其中 1788 個點塗成藍色，其餘 200 個點塗成紅色。求證存在符合以下條件的直線：這條直線把平面分成的兩部分，各有 894 個藍點和 100 個紅點。

盧：你參與奧數活動的最大得著是什麼呢？

李：我認為從參與數學活動中學會的邏輯推論技巧最為重要，不論待人處世，還是研究數學，邏輯推論是不可缺少的，有理才能說服他人。

盧：你的未來計畫是什麼？

李：在大學畢業後，我打算繼續深造，而研究的領域將會是代數數論。

盧：對一個對數學有興趣，並打算進一步學習數學的同學，你有什麼忠告？

李：在追求數學的道路上往往遇到很多挫折，但千萬不要因這些挫折而變得灰心，「鍥而捨之，朽木不折；鍥而不捨，金石可鏤」。

Andy: Has there been any particularly memorable obstacle or setback in your academic pursuit? How did you overcome it?

Albert: When I first entered secondary school, I found that the math Olympiad in the secondary level was rather different from that in the primary level. The latter focuses more on speed calculation whereas the former stresses logical thinking. It took me some time to adapt to this transition, mainly by doing exercises continually.

Andy: Do you have a favorite math Olympiad problem?

Albert: Among the numerous math Olympiad problems, my favorite is the following problem, which puts combinatorics with geometry. It was proposed by the Netherlands for IMO 1988, but unfortunately wasn't selected.

Given a set of 1988 points in the plane, no three of which are collinear. 1788 of the points are colored blue and the remaining 200 are colored red. Prove that there exists a line in the plane such that each of the two parts into which the line divides the plane contains 894 blue points and 100 red points.

Andy: What are the most important things you have learned from your math Olympiad activities?

Albert: I think the most important thing I learned is the ability of logical deduction. It is indispensable not only for mathematical research but also for everyday interaction in life. Only with sound reason can one convince others.

Andy: What are your future plans?

Albert: After graduating from university, I plan to further my studies and do research in algebraic number theory.

Andy: Do you have any advice for high school students who are interested in math, and are possibly aspiring to pursue further studies in math?

Albert: No matter how many setbacks you encounter in the pursuit of mathematics, do not become defeated. As the Chinese saying goes, "Without persistence, one cannot even break rotten wood. With persistence, one can carve diamond."

概率與 IMO

PROBABILITY AND THE IMO

羅家豪/ LAW KA-HO

雖然概率題目不時在初中的數學比賽出現，但卻較少見於國際數學奧林匹克（IMO）和其他中學程度的大型數學比賽。然而，儘管我們很少遇到要求直接計算概率的題目，但概率方法的應用卻屢見不鮮。例如，1998 年亞太區數學奧林匹克（APMO）就有以下這道題目：

1998 年 APMO 第一題：

設 F 為所有如下所述的 n 元組($A_1, A_2, \ldots, A_n$) 的集合：每個 A_i（$i = 1, 2, \ldots, n$）都是 $\{1, 2, \ldots, 1998\}$ 的子集。以 $|A|$ 表示集合 A 的元素數量。求以下算式的值：

$$\sum_{(A_1, A_2, \ldots, A_n)} |A_1 \cup A_2 \cup \cdots \cup A_n|$$

解答：

先注意到 $\{1, 2, \ldots, 1998\}$ 這個集合有 2^{1998} 個子集，這是因為對於 1998 個元素的每一個，我們都可選擇納入子集與否。因此，題中的算式共有 $(2^{1998})^n = 2^{1998n}$ 項相加。

現在，我們計算各項的平均值。對於 $i = 1, 2, \ldots, 1998$，易見 i 屬於 $A_1 \cup A_2 \cup \cdots \cup A_n$ 當且僅當 i 屬於 A_1、A_2、…、A_n 其中至少一個。此事發生的概率是 $1-2^{-n}$。由於表示 $A_1 \cup A_2 \cup \cdots \cup A_n$ 項數分佈的概率分佈為二項分佈，因此算式中各項的平均值為 $1998(1-2^{-n})$，故答案為 $2^{1998n} \cdot 1998(1-2^{-n})$。

另一個常用技巧是通過證明某些結構存在的概率大於零，以解決一些存在性問題。

問題：

在一個 100×100 棋盤中，每格均寫著 1、2、…、5000 其中一個整數，而且每個整數在棋盤上剛好出現兩次。證明我們能在棋盤上選取 100 個格子，滿足以下三項條件：

(1) 每行均有剛好一格被選。
(2) 每列均有剛好一格被選。
(3) 被選的格子中的整數互不相同。

While probability problems occur from time to time in junior mathematical competitions, their occurrence in the IMO or other major mathematical competitions at secondary school level is relatively rare. However, while we seldom encounter problems directly asking for the computation of probabilities, the use of probabilistic methods finds much more place. For example, there was such a problem on the Asian Pacific Mathematics Olympiad (APMO) 1998:

APMO 1998 Problem 1.

Let F be the set of all n-tuples ($A_1, A_2, \ldots, A_n$) where each A_i, $i = 1, 2, \ldots, n$, is a subset of $\{1, 2, \ldots, 1998\}$. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{(A_1, A_2, \ldots, A_n)} |A_1 \cup A_2 \cup \cdots \cup A_n|$$

Solution.

Note that the set $\{1, 2, \ldots, 1998\}$ has 2^{1998} subsets because we may choose to include or not to include each of the 1998 elements in a subset. Hence there are altogether $(2^{1998})^n = 2^{1998n}$ terms in the summation.

Now we compute the average value of each term. For $i = 1, 2, \ldots, 1998$, i is an element of $A_1 \cup A_2 \cup \cdots \cup A_n$ if and only if i is an element of at least one of $A_1, A_2, \ldots, A_n$. The probability for this to happen is $1-2^{-n}$. The probability distribution on the number of terms of $A_1 \cup A_2 \cup \cdots \cup A_n$ follows binomial distribution, hence the average value of each term in the summation is $1998(1-2^{-n})$, and so the answer is $2^{1998n} \cdot 1998(1-2^{-n})$.

Another common technique is to solve some existence problems by showing that certain structures exist with positive probability.

Problem.

In each cell of a 100×100 table, one of the integers 1, 2, …, 5000 is written. Moreover, each integer appears in the table exactly twice. Prove that one can choose 100 cells in the table satisfying the three conditions below:

(1) Exactly on cell is chosen in each row.
(2) Exactly on cell is chosen in each column.
(3) The numbers in the cells chosen are pairwise distinct.

解答：

選取 $\{1, \ldots, 100\}$ 的一個隨機排列 a_1、…、a_{100}，並選取第 i 行的第 a_i 個格子。顯然，這個選擇滿足條件 (1) 和 (2)。對於 $j = 1$、…、5000，兩個寫著 j 的格子同時被選的概率要麼是 0（如果那兩個格子位於同一行或同一列），要麼就是 $\frac{1}{100}\times\frac{1}{99}$（如果它們不同行也不同列）。故此，這個選擇滿足條件 (3) 的概率不小於 $1-5000\times\frac{1}{100}\times\frac{1}{99}>0$，證畢。

有興趣的讀者可參閱 *Mathematical Excalibur* 第 13 期第 2 號中題為 *Probabilistic Method* 的文章（可在搜索引擎以“math excalibur”查找），文中有更多入門例子。

雖然「某些結構存在的概率大於零」這個命題聽似簡單，實際上卻可能頗難驗證。儘管我們只需確立該概率的一個下限，但難處在於我們所需的結構需要滿足幾個條件，而其中有些條件或會同時成立。

為克服這個困難，以下是一個有用的結果：

Lovász Local Lemma (LLL)：

假設有數個事件，每個事件發生的概率均不大於 p，而每個事件最多跟 d 個其他事件相關，其中 $epd \le 1$（這裡 e 是自然對數的基數）。那麼，這些事件皆不發生的概率是正數。

以上結果是由匈牙利數學家 László Lovász 證明的。Lovász 在 1963 年至 1966 年間四度參加 IMO，共獲三金一銀。更甚者，他在 1965 年和 1966 年兩屆 IMO 都勇奪滿分和特別獎！

LLL 對於解答 IMO 的不少組合問題都大有幫助。例如，考慮以下問題：

2014 年 IMO 第六題：

平面上的一組直線，若其中任兩條不平行、任三條不共點，則稱這組直線位於一般位置。位於一般位置的直線組，將平面分割成若干區域，其中有些區域的面積是有限的；這些區域稱為此直線組的有限區域。

Solution.

Take a random permutation $a_1, \ldots, a_{100}$ of $\{1, \ldots, 100\}$ and choose the a_i-th cell in the i-th row. Such choice satisfies (1) and (2). For $j = 1, \ldots, 5000$, the probability of choosing both cells written j is either 0 (if they are in the same row or column) or $\frac{1}{100}\times\frac{1}{99}$ (otherwise). Hence the probability that such choice satisfies (3) is at least $1-5000\times\frac{1}{100}\times\frac{1}{99}>0$, and so the desired conclusion follows.

Interested readers may refer to the article *Probabilistic Method* in Volume 13, Number 2 of *Mathematical Excalibur* (just google "math excalibur") for more elementary examples.

While the statement 'certain structures exist with positive probability' sounds simple, its actual verification can be difficult, even though all we need is a positive lower bound for the probability. The main difficulty lies with the fact that the desired structure often needs to satisfy multiple properties, some of which may occur simultaneously.

In order to overcome this difficulty, one useful tool is the following.

Lovász Local Lemma (LLL).

Suppose there are several events, each occurring with probability at most p, and that each event is dependent on at most d other events, where $epd \le 1$. (Here e is the base of the natural logarithm.) Then the probability that none of the events happens is positive.

The about result is due to the Hungarian mathematician László Lovász. Lovász joined the IMO four times from 1963 to 1966, obtaining one silver and three gold medals. Furthermore, he obtained perfect scores as well as a special prize at each of IMO 1965 and 1966!

It turns out that the LLL is of great help in a number of combinatorics problems in the IMO. For example, consider the following problem.

IMO 2014 Problem 6.

A set of lines in the plane is in **general position** if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its **finite regions**.

證明：對任意足夠大的 n，皆可以在位於一般位置的 n 條直線組裡，選取至少 $\sqrt{n}$ 條直線塗上藍色，使得此直線組沒有任何有限區域的邊界完全是藍色。

註：證出的結果中，如果 $\sqrt{n}$ 換成了 $c\sqrt{n}$，會依常數 c 之值給予分數。

部分解答：

隨機地把各條直線編號為 1 至 n，並把他們分成 $c\sqrt{n}$ 組，其中每組有 $\frac{\sqrt{n}}{c}$ 條直線，然後在每組中隨機選一條直線塗上藍色。對於每個有限區域，考慮以下事件：「其邊界上的三條編號最小的直線都被塗上藍色」。我們只需證明這些事件皆不發生的概率為正數。

以上每項事件發生的概率皆為 $\left(\frac{c}{\sqrt{n}}\right)^3$。而且每項事件均與少於 $\frac{6n\sqrt{n}}{c}$ 項其他事件相關（因為這三條直線中的每條都是少於 $2n$ 個有限區域的邊界，並屬於某一組共 $\frac{\sqrt{n}}{c}$ 條直線）。由於

$$e\left(\frac{c}{\sqrt{n}}\right)^3\frac{6n\sqrt{n}}{c}=6ec^2,$$

故當 $c=\frac{1}{\sqrt{6e}}$ 時，我們便可由 LLL 得出所需結果。

上述論證只提供了 $c=\frac{1}{\sqrt{6e}}$ 這情況下的解答，但原題需證明 $c=1$ 時的更強命題。然而，這足以展示運用 LLL 作簡單運算已可帶來殊不簡單的結果。

事實上，有一個較複雜的概率方法，可證明時的結果，有興趣的讀者不妨試試。

除了上述各例子外，還有其他 IMO 題目和候選題目也跟 LLL 有關，或可用概率方法解答，包括 2012 年 IMO 第三題。讀者可參閱 *Mathematical Excalibur* 第 17 期第 1 號的 IMO 2012 (Leader Perspective) 一文，2012 年香港 IMO 代表隊領隊梁達榮博士在文中討論了那道題目，並介紹了關於概率技巧的其他參考資料。

Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

Partial Solution.

Randomly number the lines 1 to n, split them into $c\sqrt{n}$ groups with $\frac{\sqrt{n}}{c}$ lines each and randomly colour one line in each group blue. For each finite region, consider the event 'the three lowest-numbered lines on its boundary are coloured blue'. It suffices to show that the probability that none of these events happens is positive.

Each event occurs with probability $\left(\frac{c}{\sqrt{n}}\right)^3$. Also, each event depends on less than $\frac{6n\sqrt{n}}{c}$ other events (each of the three lines belongs to less than $2n$ finite regions and to a group with $\frac{\sqrt{n}}{c}$ lines). Now

$$e\left(\frac{c}{\sqrt{n}}\right)^3\frac{6n\sqrt{n}}{c}=6ec^2.$$

So the result follows from the LLL if $c=\frac{1}{\sqrt{6e}}$.

The above gives only a partial solution with $c=\frac{1}{\sqrt{6e}}$ instead of a complete solution (which requires $c=1$); nevertheless it illustrates the power of the LLL as a straightforward computation already leads us to something non-trivial.

It is known that a more sophisticated probabilistic treatment could prove the case. Interested readers may try it out.

Apart from the above example, there are other IMO problems and shortlisted problems that are related to the LLL or can be solved using probabilistic techniques. One example is IMO 2012 Problem 3. Interested readers may read the article IMO 2012 (Leader Perspective) in Volume 17 Number 1 of *Mathematical Excalibur*, in which Dr Leung Tat-Wing, leader of the Hong Kong team in IMO 2012, would discuss the problem and point readers to further references about probabilistic techniques.

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THE ART OF THE IMPOSSIBLE

/盧安迪 ANDY LOO

我們常認為，我們無法使所有人滿意，是因為世上不夠資源。例如，或許所有 IMO 參賽者都喜歡住單人房，但由於房間數目有限，所以只能給他們住雙人房。然而，你有沒有想過，即使有無限資源，我們仍然不能使所有人滿意？

We are often under the impression that we cannot satisfy everybody just because there are not enough resources in the world. For example, maybe all IMO participants prefer to live in single rooms, but we can only put them in double rooms because we don't have enough rooms. However, have you ever thought that we may still not be able to satisfy everyone even when we have infinite resources?

假設我們要為三位住客分配酒店房間。我們有無限房間，每間可容納任意多人。住客們的偏好如下：

Suppose we are to assign hotel rooms to three guests, A, B and C. We have infinitely many rooms and each room can accommodate any number of people. The guests' preferences are as follows:

	A	B	C
最喜歡 Most preferred	跟 B 住雙人房 Double with B	跟 C 住雙人房 Double with C	跟 A 住雙人房 Double with A
	跟 C 住雙人房 Double with C	跟 A 住雙人房 Double with A	跟 B 住雙人房 Double with B
	住單人房 Single	住單人房 Single	住單人房 Single
最不喜歡 Least preferred	跟 B、C 住三人房 Triple with B, C	跟 C、A 住三人房 Triple with C, A	跟 A、B 住三人房 Triple with A, B

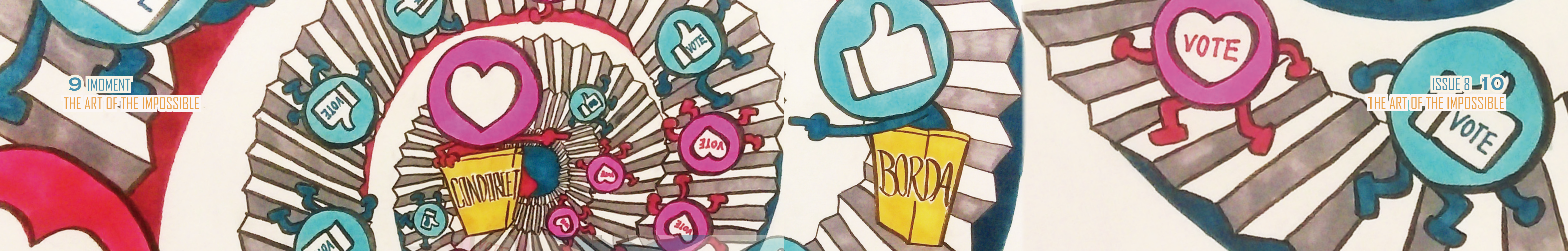
表一 Table 1

我們可否找到一種安排，使所有住客都滿意，即令他們沒有換房的動機？顯然，我們不應把三人放在同一房間，因為任何其他安排對三人來說都比這更好。如果讓三人都住單人房又如何呢？這也不能讓他們安頓下來，因為（舉例說）A 會想搬去 B 的房間，而 B 也會樂意接受 A。但當 A、B 住雙人房，C 住單人房時，B 會想搬去 C 的房間，而 C 也會樂意接受 B。但當 B、C 住雙人房，A 住單人房時，C 會想搬去 A 的房間，而 A 也會樂意接受 C。但當 C、A 住雙人房，B 住單人房時，A 會想搬去 B 的房間，而 B 也會樂意接受 A！

Can we find an arrangement such that all three guests are satisfied, in the sense that they will have no incentive to move? Clearly we shouldn't put all of them in the same room, since anything else would be better for everyone. What if we put them all in single rooms? This will not settle them, because, for example, A will want to move into B's room and B will be glad to accept A. But if A and B are in a double room while C is in a single room, then B will want to move into C's room and C will be glad to accept B. But if B and C are in a double room while A is in a single room, then C will want to move into A's room and A will be glad to accept C. But if C and A are in a double room while B is in a single room, then A will have an incentive to move into B's room and B will be glad to accept A!

至此，我們已考慮了所有可能的安排，發現沒有一個能使所有住客滿意。注意住客的偏好的「循環」結構：撇除「住三人房」這個糟糕的選擇，A的偏好可以寫成 (B, C, A)，其中「A」代表自己一個人住。同樣地，B 和 C 的偏好可分別寫成 (C, A, B) 和 (A, B, C)。

Thus, we have visited all possible arrangements and found that none of them can make all guests satisfied. Observe the "cyclic" structure contained in the guests' preferences: excluding the terrible option of a triple room, A's roommate preference can be written as (B, C, A), where "A" means staying alone. Likewise, B's and C's roommate preferences are (C, A, B) and (A, B, C) respectively.



其實，這個「循環」的特性也是另一個「不可能」的例子的關鍵。我們很容易以為只要民主地作出一個選擇，便會令多數人滿意，即不會有過半數選民想以另一選項代替勝出的選項。但事實又是否如此呢？

假設 IMO 評審團要在三道題目 A、B、C 中選出一題放進 IMO 試卷。評審團的成員可分為三組 X、Y、Z，其大小和偏好如下：

In fact, such a cyclic feature is the key to another impossibility example. It is tempting to think that as long as a choice is made democratically, most people are satisfied in the sense that no more than half of the voters would want to replace the winner with another choice. But is this really the case?

Suppose the IMO Jury has to select one of three problems, A, B and C, to put in the contest paper. Members of the jury can be classified three groups, X, Y and Z, with sizes and preferences as follows:

組別 Group	X	Y	Z
大小 Size	25%	35%	40%
最喜歡 Most preferred	A	B	C
	B	C	A
最不喜歡 Least preferred	C	A	B

表二 Table 2

哪道題目應該當選呢？如果 A 當選，則 Y 組和 Z 組（75%）會想以 C 取代它。如果 B 當選，則 Z 組和 X 組（65%）會想以 A 取代它。如果 C當選，則 X 組和 Y 組（60%）會想以 B 取代它。所以，無論哪道題目當選，總會有過半數成員不滿意，想以另一道題目取代它！

但設想經過一番唇槍舌劍後，Y 組被說服了，決定把自己偏好中的 B、C 對換。現在，對於 C 以外的任何選項，總有至少半數成員喜歡 C 多於該選項。在這情況下，我們可宣布 C 當選──這種選取贏家的方法稱為孔多塞法。我們剛才在表二的原本情況中已看到，在這制度下不一定存在贏家，這是孔多塞法的一個弱點。

另一方面，有多種投票制度都保證能選出贏家。波達法就是常見的例子之一：如果選舉中有 n 個選項，便請每名選民列出他的偏好次序，並給第一名 n 分、第二名 $n-1$ 分、......、最後一名 1 分。把所有選民所給的分數加起來，總分最高的選項就是贏家。

Which problem should be chosen? If A is chosen, groups Y and Z (75%) will prefer to replace it with C. If B is chosen, groups Z and X (65%) will prefer to replace it with A. If C is chosen, groups X and Y (60%) will prefer to replace it with B. So, no matter which problem is chosen, there will always be more than half of the jury being unhappy and preferring another problem!

But suppose after some vigorous debate, group Y has been convinced to swap the order of B and C in their preference. Now, for every single alternative other than C, there is always at least half of the jury who prefers C to that alternative. In this case, we can declare C as the winner of the election – this criterion for determining the winner is called the Condorcet method. As we have seen with the original scenario in Table 2, one drawback of the Condorcet method is that a winner does not necessarily exist.

On the other hand, there are plenty of systems that guarantee we can always find a winner. One common example is the Borda count: If there are n choices in an election, let each voter list his preference, and give n points to his first choice, $n-1$ points to his second choice, ..., 1 point to his last choice. Add up the points that each choice receives from all the voters. The choice that receives the most points is the winner.

毫無疑問，波達法總能產生一個贏家（表二中的贏家是 C），但也有一個弱點：受制於「策略性投票」，也就是說，選民有不誠實投票的動機，以致結果未能反應選民的真正偏好。這怎麼可能發生呢？例如在表二中，X 組可假扮自己的偏好是 (B, A, C) 而非 (A, B, C)，因為（正如讀者可以驗證）在這情況下 B 會當選，而 X 組會認為這個結果比 C 當選更佳！

至於孔多塞法，則不存在策略性投票的問題（見「挑戰園地」）。於是，孔多塞法和波達法互有長短。那麼是否存在一個「完美」的投票制度，既是總能產生贏家，又不受制於策略性投票，並符合另外一些基本條件呢？答案是否定的，這就是著名的 Gibbard-Satterthwaite 定理。常言道政治是可能的藝術，或許設計投票規則可稱為「不可能的藝術」。

可是，我們無需絕望。雖然沒有在一般情況下「完美」（即符合上述條件）的投票制度，但如果我們把選民的偏好限制在某些類型，一個投票制度可能在這些情形下是「完美」的。近年，Maskin 和 Dasgupta 證明了，孔多塞法和波達法正是最常「完美」的制度：如果任何其他制度在選民偏好集合 S 裡是「完美」的，則孔多塞法和波達法其中一個在一個更大的集合 T 裡是「完美」的。

更震撼的是，在孔多塞法能產生贏家的情況裡，孔多塞法是「完美」的，而恰恰當孔多塞法不能產生贏家時，波達法是「完美」的！其實，孔多塞和波達這兩位 18 世紀法國學者，不但在投票制度上意見分馳，在許多學術和現實問題上都往往針鋒相對。這麼多年後，終於發現兩人提倡的投票制度竟乃相輔相成，他們在九泉之下，或可相逢一笑泯恩仇吧！

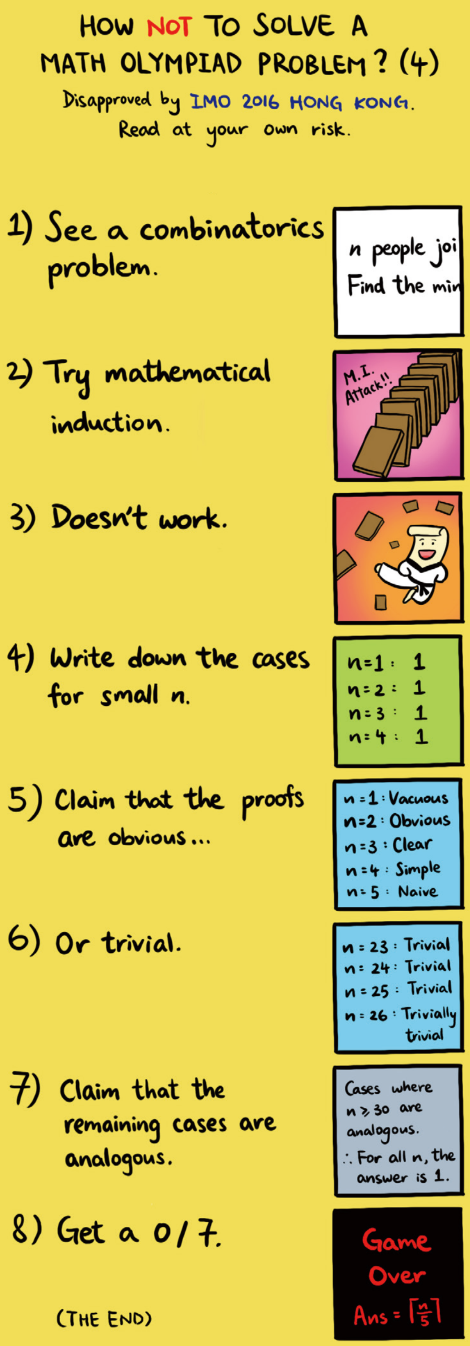
There is no doubt that the Borda count always generates a winner (in Table 2 the winner is C), but it also has a drawback: it is subject to strategic voting, which means voters may have an incentive to vote dishonestly, so the results may not reflect voters' true preferences. How could this happen? For example, in the original Table 2, group X may want to pretend that their preference is (B, A, C) instead of (A, B, C), because, as the reader can check, B would then become the winner, and group X would find this outcome more desirable than C winning!

Meanwhile, the Condorcet method is not subject to strategic voting (see Challenge Corner). So, the Condorcet method and the Borda count each have their advantages and disadvantages. Is there a "perfect" voting system for single-winner elections that always produces a winner, is not subject to strategic voting, and satisfies some other basic requirements? The answer is no, and this is the famous Gibbard-Satterthwaite theorem. It is often said that politics is the art of the possible, but it can perhaps be said that designing voting rules belongs to the art of the impossible.

Yet, all is not lost. While no voting system can be generally "perfect" (i.e. satisfy the above-mentioned conditions), a system may be "perfect" if we restrict voters' preferences to certain types. In recent years, Maskin and Dasgupta proved that the Condorcet method and the Borda count are in fact the systems that are "perfect" the most often: if any other voting system is "perfect" for a certain set of preferences, then either the Condorcet method or the Borda count is "perfect" for a strictly larger set of preferences.

Even more astonishingly, the Condorcet method is "perfect" when it can generate a winner, and the Borda count is "perfect" precisely when the Condorcet method fails to generate a winner! Actually, Condorcet and Borda were French academics in the 18th century who were rivals not only in voting rule proposals but also on a full range of intellectual and practical subjects. If they know that their voting methods turn out to be complementary after all these years, perhaps they can finally smile at each other and shake hands in heaven.

笑一笑 Laugh Out Loud



挑戰園地 Challenge Corner

第七期挑戰園地的解答及得獎名單，可見:
For the solutions and list of awardees of the Challenge Corner of the 7th issue, please see:
http://www.edb.gov.hk/tc/curriculum-development/kla/ma/IMO/IMOment.html

1. 設 $ABCD$ 為凸四邊形，且不是平行四邊形。設 E 為 BD 的中點、F 為 AC 的中點。求證：若 L 為線段 EF 上的任何一點，則三角形 ALB 和 CLD 的面積之和，等於四邊形 $ABCD$ 的面積的一半。

Let $ABCD$ be a convex quadrilateral that is not a parallelogram. Let E be the midpoint of BD and F be the midpoint of AC. Prove that for any point L on the line segment EF, the sum of the areas of triangles ALB and CLD is half the area of the quadrilateral $ABCD$.

2. 求證：對於任意整數 a、b、c、d，以下算式可被 12 整除：

Prove that for any integers a, b, c and d, the following expression is divisible by 12:

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

3. 某校有 1000 名學生，他們的編號分別是 1 至 1000。每名學生皆須在法語和德語兩者之間選修一科，並規定如果任何一組 20 名學生的編號組成等差數列，則該組學生不能全部選修同一語文。是否有可能滿足以上要求？（可參閱本期《概率與 IMO》一文。）

There are 1000 students in a school and they are numbered 1 to 1000. Each student must choose to study one of French or German, with the additional restriction that any group of 20 students whose numbers form an arithmetic sequence cannot all choose the same language. Is it possible to satisfy the requirement? (You may refer to the article Probability and the IMO in this issue.)

4. 參見本期《不可能的藝術》一文。設有奇數名選民。對於單贏家的選舉，孔多塞法的定義是：每名選民在選票上為所有選項排序。如果存在一個選項 P，使得對於任何其他選項 Q，都有過半數選民把 P 排在 Q 之上，則 P 為贏家。求證：如果所有選民誠實投票的結果是有贏家，則沒有一名選民有單方面地不誠實投票的動機。

Refer to the article The Art of the Impossible in this issue. Assume there is an odd number of voters. The Condorcet method for single-winner elections is defined as follows: Each voter votes by reporting a ranking of all the choices. If there exists a choice P such that for any other choice Q, more than half of the voters rank P higher than Q, then P is the winner. Prove that if a winner exists in the case of all voters voting honestly, then no single voter will have a unilateral incentive to vote dishonestly.

歡迎香港中、小學生讀者電郵至 info@imohkc.org.hk 提交解答（包括證明），並於電郵中列明學生中英文姓名、學校中英文名稱及學生班級。每一名學生只可發送一份電郵。首 20 名答對最多題目的同學將獲贈紀念品，但每間學校最多有 3 名同學得獎。解答可以中文或英文提交。打字及掃描文件皆可接受。截止日期為 2016年7月16日。得獎者將於 2016 年 8 月份公布。2016 年第五十七屆國際數學奧林匹克籌備委員會對本活動安排有最終決定權。如有疑問，可電郵至 info@imohkc.org.hk 查詢。

Hong Kong secondary and primary school student readers are welcome to submit solutions (with proofs) via email to info@imohkc.org.hk, specifying the student's name in Chinese and in English, the school's name in Chinese and in English, and the student's class in the email. Each student may send at most one email. Souvenirs will be awarded to the first 20 students solving the most questions on the condition that each school can have at most 3 awardees. Solutions can be submitted in Chinese or English. Both typed and scanned files are acceptable. Submission deadline is 16 July 2016, and the awardees will be announced in August. The decision of the Organising Committee of the 57th International Mathematical Olympiad on any matter of this activity is final. Enquiries may be emailed to info@imohkc.org.hk.