數學概念的學與教

吳銳堅

2019.07.04 & 2019.07.09

證明

解難

解題

理論

運算

技巧

概念

表達

概念是什麼？

What Is a Concept?

Dean R. Spitzer

What precisely is a *concept*? There seems to be general agreement among psychologists and educators that concept formation is the basis by which man orders his experience; that it represents some sort of cognitive grouping; and that it is an extremely important element of human learning. After these initial agreements, however, there is little continued support in the literature for a common definition of "concept."

One fundamental problem in formulating a suitable psychological definition of "concept" is that the term has entered the realm of general language and is commonly used with a very indefinite and flexible meaning. "Concept" has come to refer, in common usage, to any idea, process or thing which cannot be defined readily in another way. An instructor says: "Today we will talk about the concept of aerodynamics." A mother says: "You don't have a concept of neatness." Examples of different common usages of the term "concept" abound. Common use is often tantamount to abuse. A similar state of affairs exists among members of the scientific community and has caused considerable ambiguity and other problems of communication.

This article explores some of the key notions of the construct "concept" from the psychological and educational literature in order to demonstrate the need for standardization of definition and a more unified front in future investigations involving this important element in the study of cognition. This is not a mere problem of semantics or terminology but it represents a basic disagreement on a fundamental problem in the study of human intellectual development. The necessity of at least considering divergent views of the term "concept" is particularly pressing today, due to current interest in the product and process of concept formation by all those concerned with learning. Articles in this magazine in recent years have demonstrated this point.

Webster's New Collegiate Dictionary does little to help us toward clarification of the problem. The first definition given is: "Something conceived in the mind, a thought or notion"; the second definition is: "An abstract idea generalized from particular instances." Neither of these definitions does much to present a clear idea of what a concept is. This dictionary selection does indicate a starting point for the investigation, however, and this point is the notion of "generalization." Certainly there have been many psychological definitions of "concept" which have centered on "generalization," and there is some consensus that some degree of generalization is involved in concept formation. One of the most frequently expressed values of concepts is that they allow the individual to form general notions of his world, on the basis of specific experiences with it. This notion of "abstraction of common properties" from experience is a commonly held product of concept formation but it is of little assistance in formulating a definition. It oftens appears that the nature of the construct is being confused with one of its important benefits. Few would contradict the fact that concepts allow us to generalize, but generalization from particulars is hardly the whole story.

The most fundamental disagreement among theorists concerns the complexity of the construct. On one extreme, there are those who believe that concepts are spontaneous capabilities which the infant can almost immediately master, on the basis of his earliest perceptions (Lewis, 1963; Montessori, 1972; Ricciuti, 1965). On the other extreme, there are others who feel strongly that concepts represent a high level mental process, and results from considerable prerequisite experience and maturity of thought (Gagne, 1965; Tennyson and Merrill, 1971; Vygotsky, 1962). Within the bounds of this fundamental disagreement, a vehement conflict exists between the "pre-language" and the "post-language" positions, concerning whether or not language facility is a prerequisite for concept formation. This controversy appears to be the most clearly defined area of disagreement.

The pre-language position is supported primarily by experts in early childhood development and child psychologists, who view concepts as being the result of simple learning. Advocates of this position see concept formation as fundamentally a "process of recognition," based on responses to certain perceptual characteristics. It is the perceptual basis of this view of the construct that allows the very young child to deal effectively with it. Many advocates of this position see perceptual recognition as being a spontaneous developmental ability, which appears naturally with appropriate experience. Researchers who have defined the construct in this way include Ricciuti (1965), who studied children from 12 to 24

Dean R. Spitzer is Assistant Professor, Department of Educational Communications, State University of New York at Albany.

Spitzer (1975)

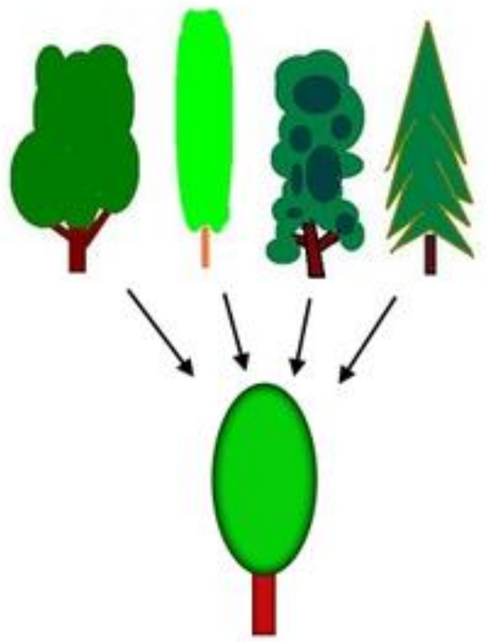
概念的概念

概念的概念

- 人類組織經驗的基礎

概念的概念

- 分類／概括
（學習中極重要的元素）

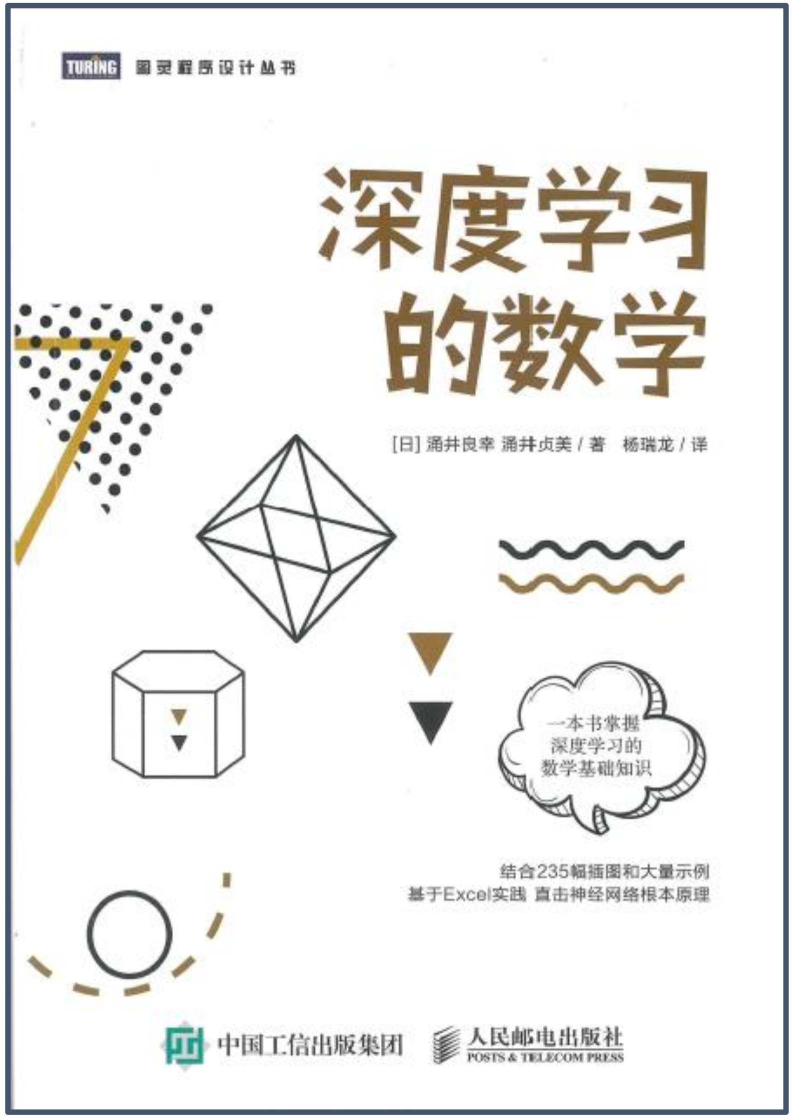


前 言

近年来，我们在媒体上到处可见人工智能（AI）这个词，而深度学习是人工智能的一种实现方法。下面我们就来简单地看一下深度学习具有怎样划时代的意义。

下面是三张花的图片，它们都具有同一个名字，那究竟是什么呢？





概念的建構

- 名稱

如何學會「紅」這個概念？

如何學會「 紅 」這個概念？

定義法

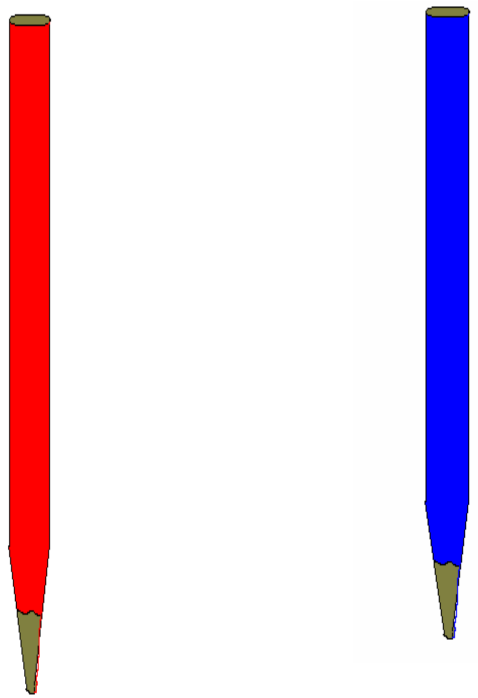
波長 647 – 760 nm

如何學會「紅」這個概念？

指稱法

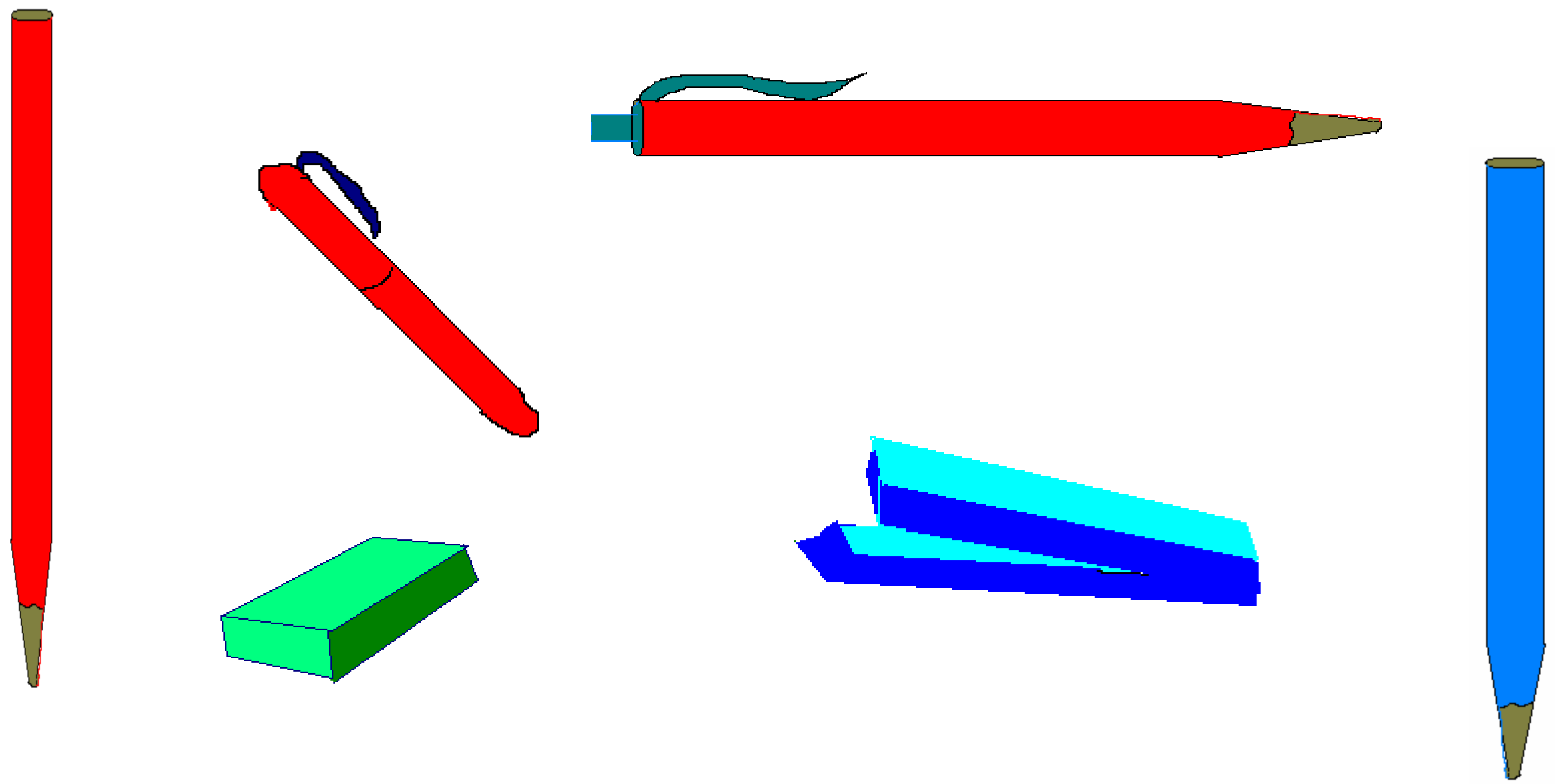
爸爸

如何學會「紅」這個概念？



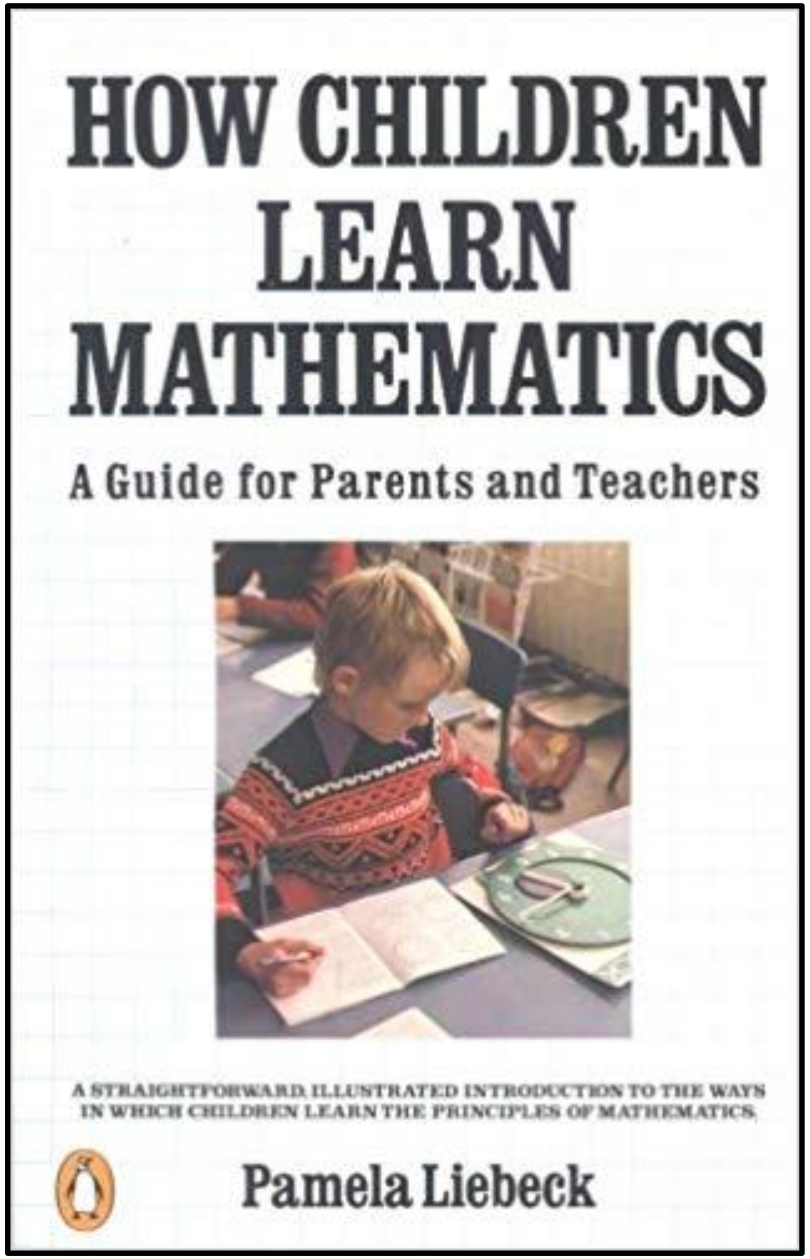
指稱法

如何學會「紅」這個概念？

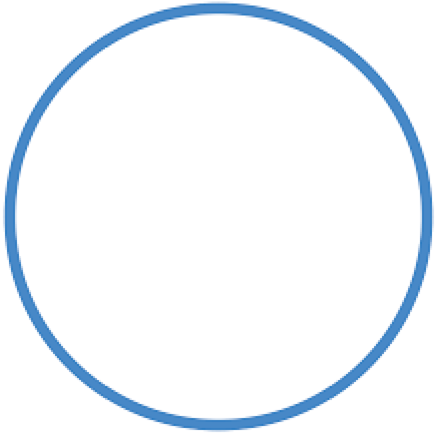


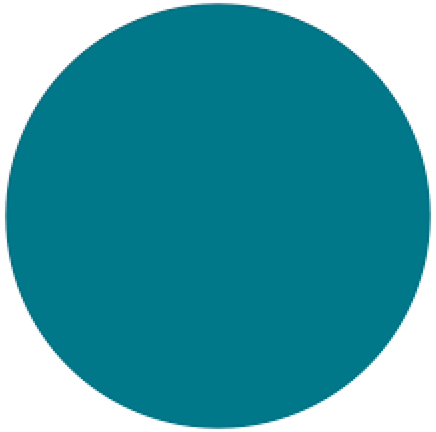
概念是會發展的

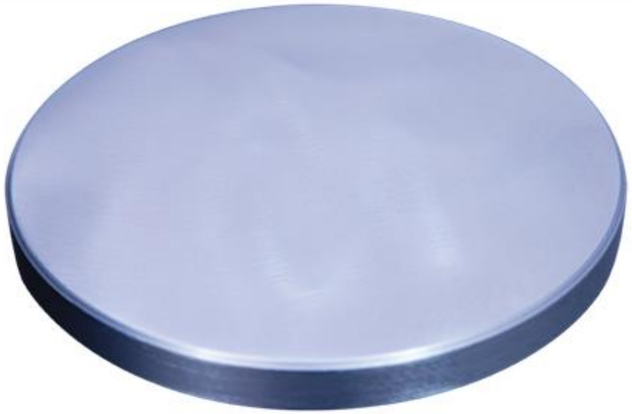
- E (Experience) 體驗
- L (Language) 語言
- P (Picture) 圖畫
- S (Symbol) 符號



圓的概念







掌握概念

是有不同程度的

KS3

1.2 理解乘方的概念

2.1 理解有向數的概念

3.1 認識近似值的概念

4.1 認識 n 次方根的概念

4.2 認識有理數和無理數的概念

5.1 理解百分變化的概念

6.1 理解率、比及比例的概念

KS3

7.3 認識數列的概念

7.4 認識函數的初步概念

11.1 理解多項式的概念

12.1 理解恆等式的概念

14.1 理解不等式的概念

15.1 認識量度中誤差的概念

15.2 認識最大絕對誤差、相對誤差和百分誤差的概念

17.1 認識直立角柱、直立圓柱、直立角錐、直立圓錐、正角柱、正角錐、多面體和球形的概念

KS3

19.1 理解直線上的鄰角、對頂角和同頂角的概念

19.2 理解同位角、內錯角和同旁內角的概念

20.1 理解正多邊形的概念

21.1 理解全等三角形的概念

22.1 理解相似三角形的概念

22.3 認識相似平面圖形的概念

28.1 認識離散數據和連續數據的概念

30.1 理解平均數、中位數和眾數/眾數組的概念

31.1 認識必然事件、不可能事件和隨機事件的概念

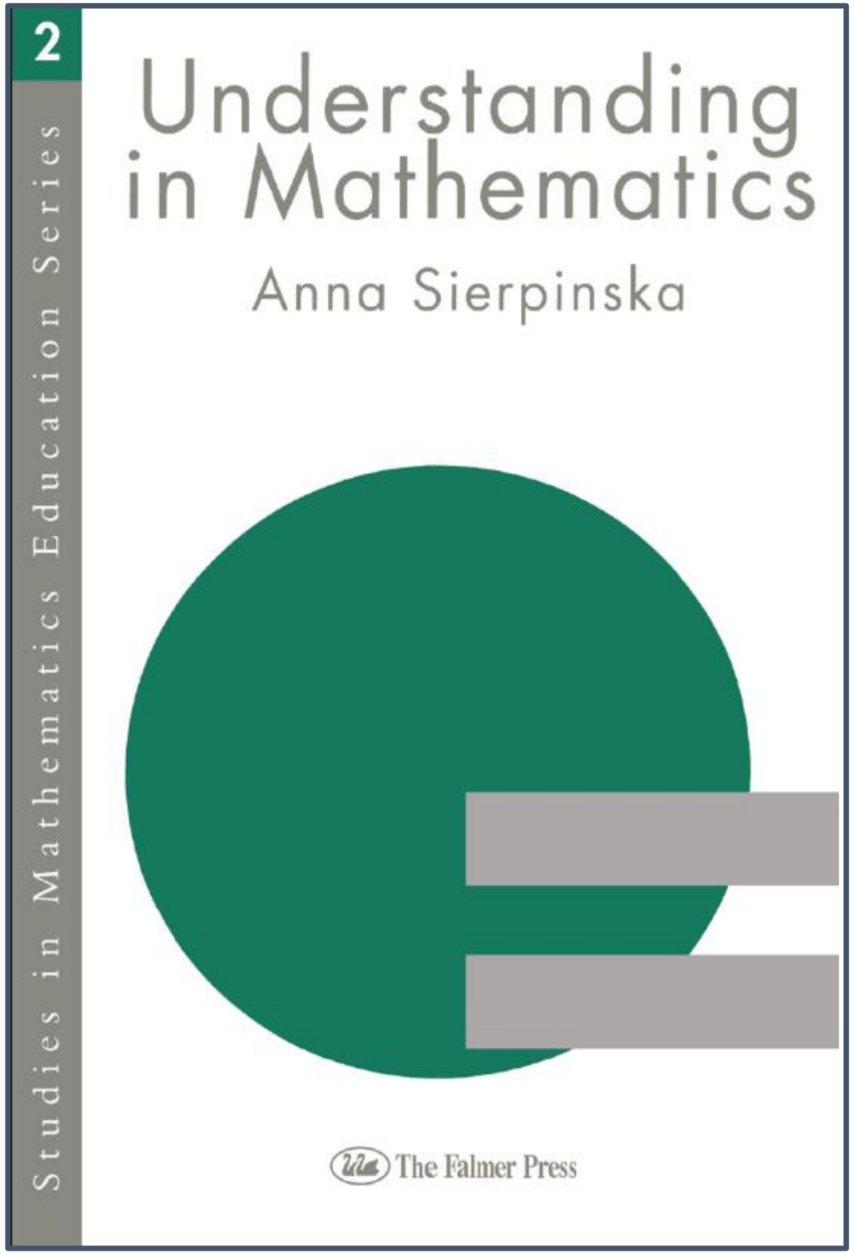
31.2 認識概率的概念

22.1 **理解**相似三角形的概念

22.3 **認識**相似平面圖形的概念

30.1 **理解**平均數、中位數和衆數/衆數組的概念

31.2 **認識**概率的概念



Sierpinska, A. (1994). *Understanding in mathematics*. London: The Falmer Press.

The most widely accepted ideas in mathematics education is that students should *understand mathematics*

Hiebert and Carpenter, 1992

3.2 理解估算策略

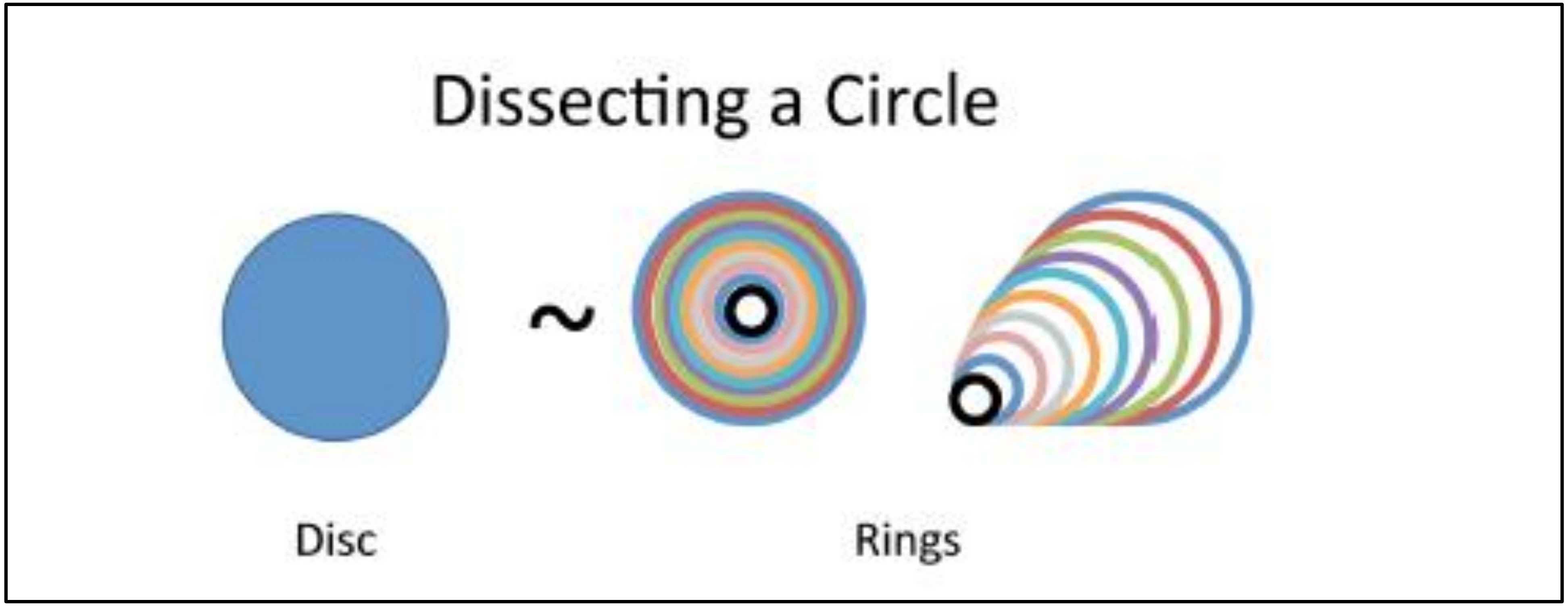
10.1 理解正整數指數定律

22.2 認識相似三角形的判別條件

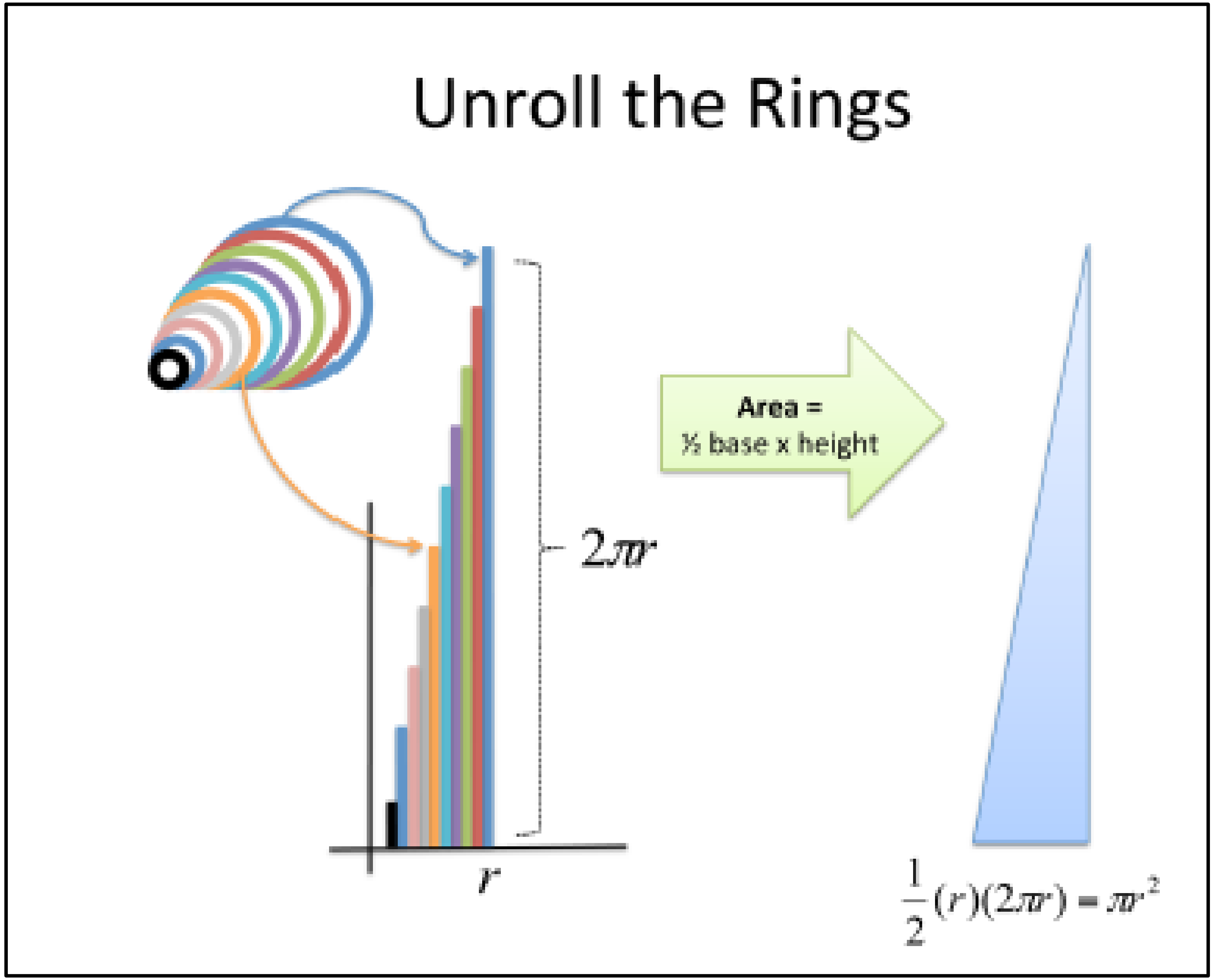
25.1 理解畢氏定理

25.2 認識畢氏定理的逆定理

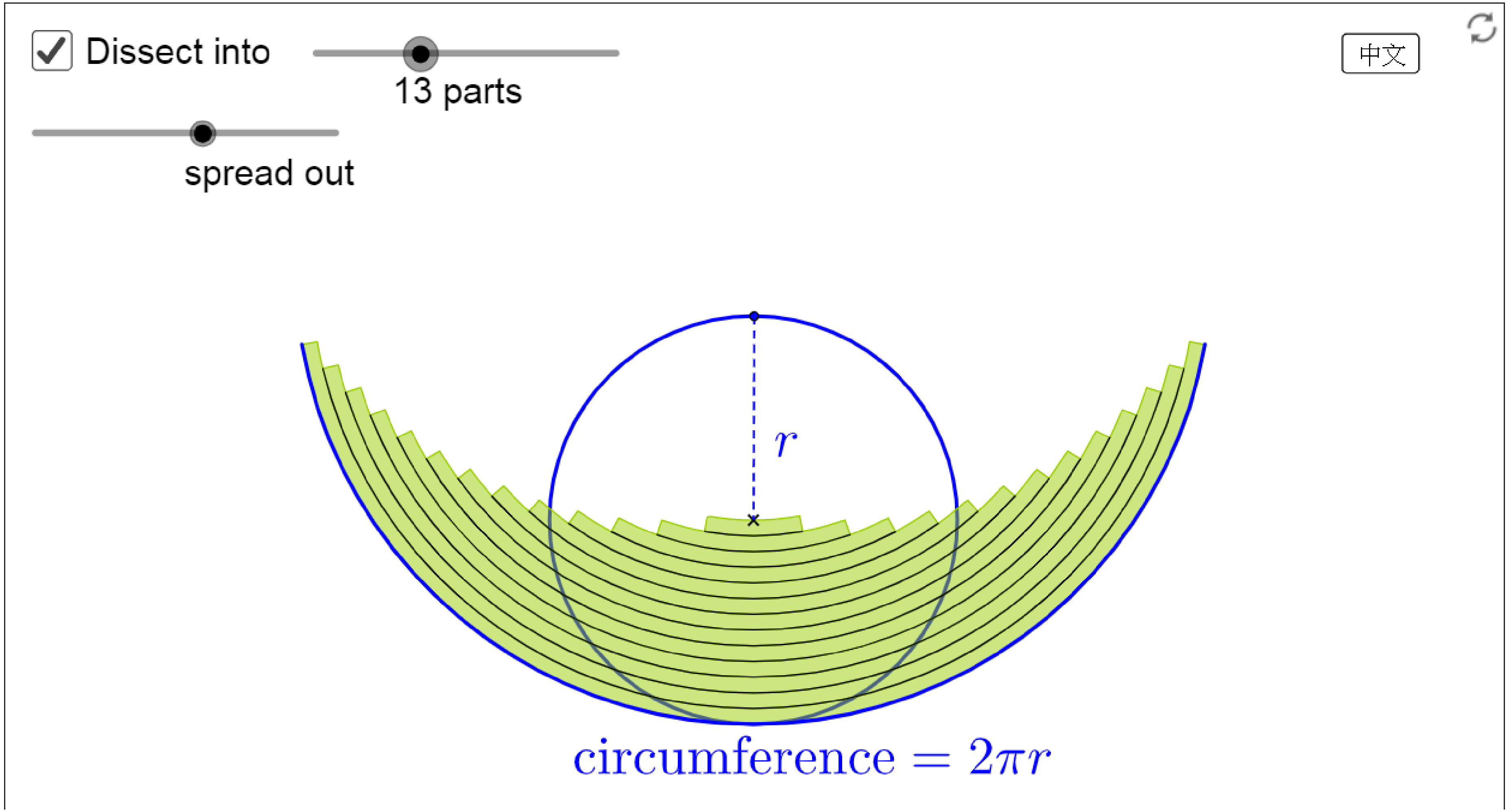
理解的意義



(Azad, 2013; 2015)

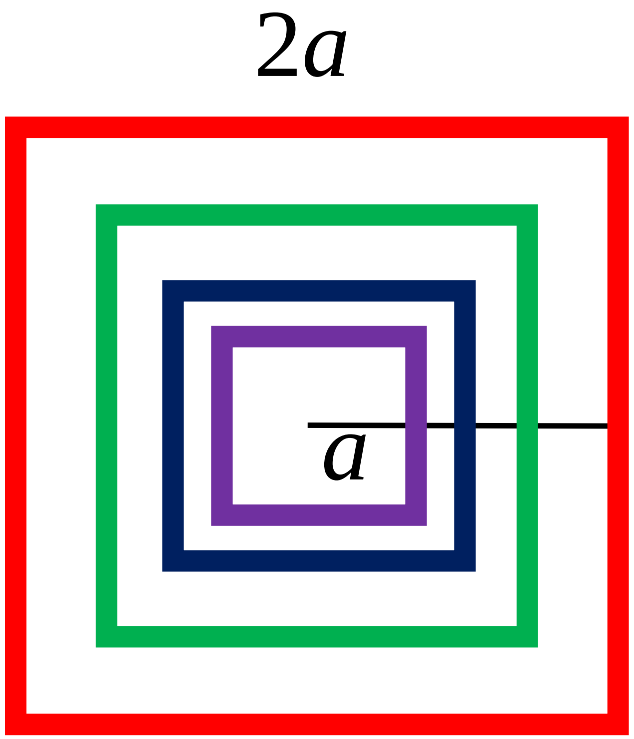


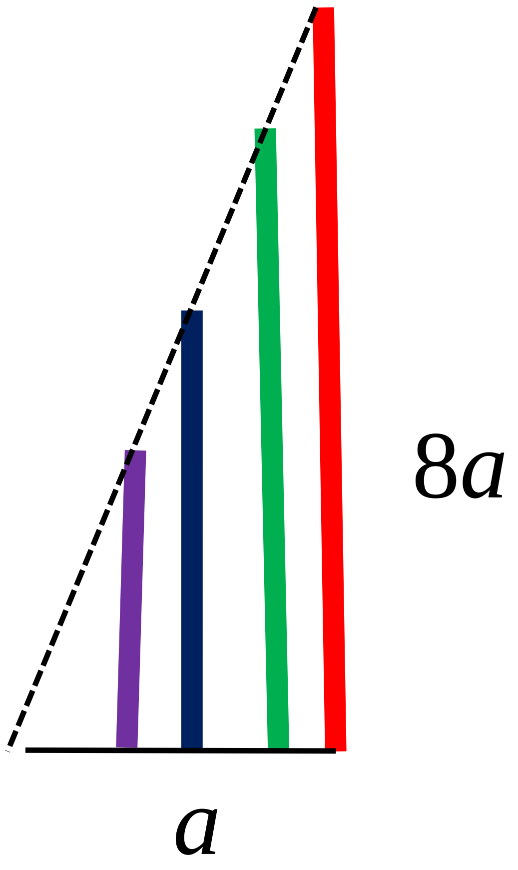
(Azad, 2013; 2015)

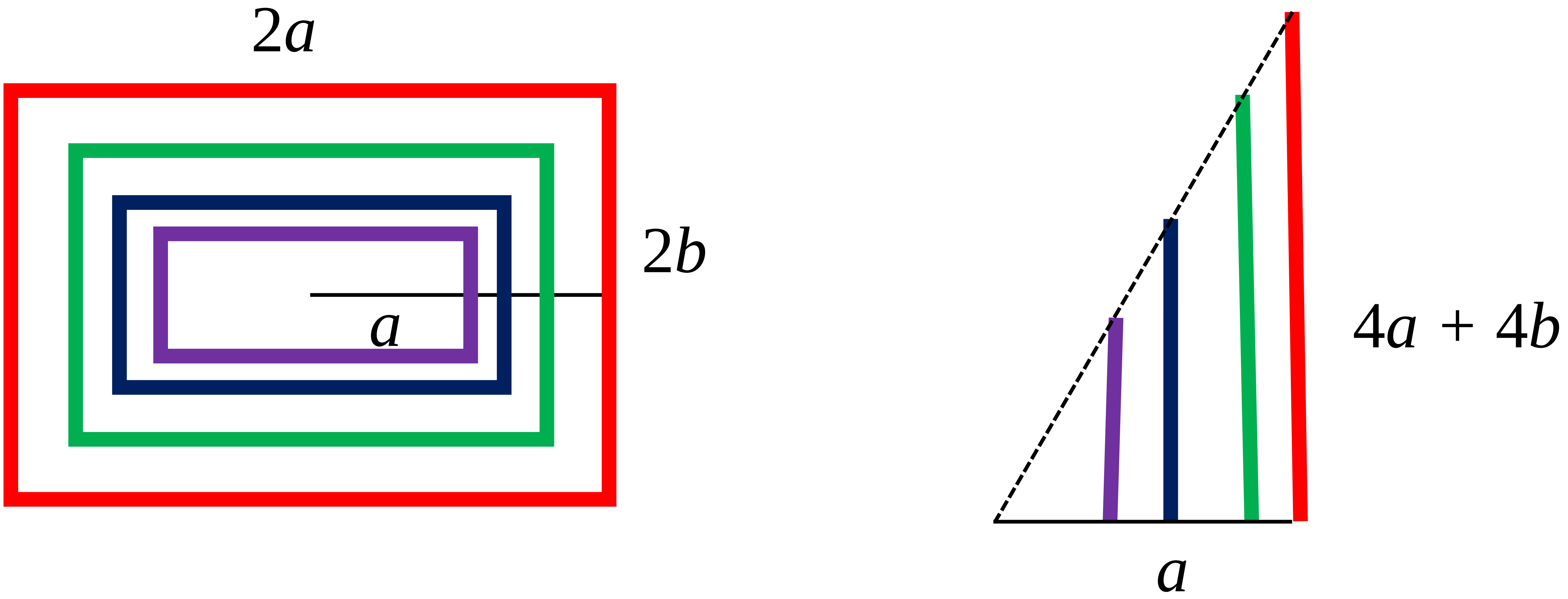


https://www.geogebra.org/m/r5VBs842

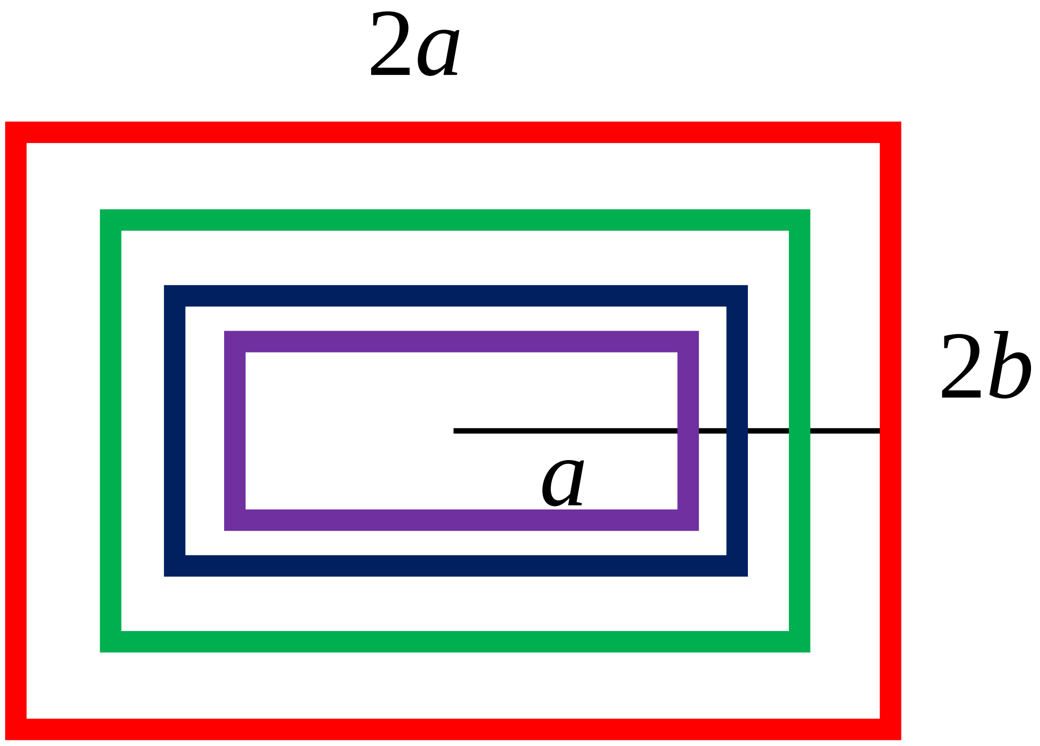
面積＝$(a)(8a)/2 = 4a^2$

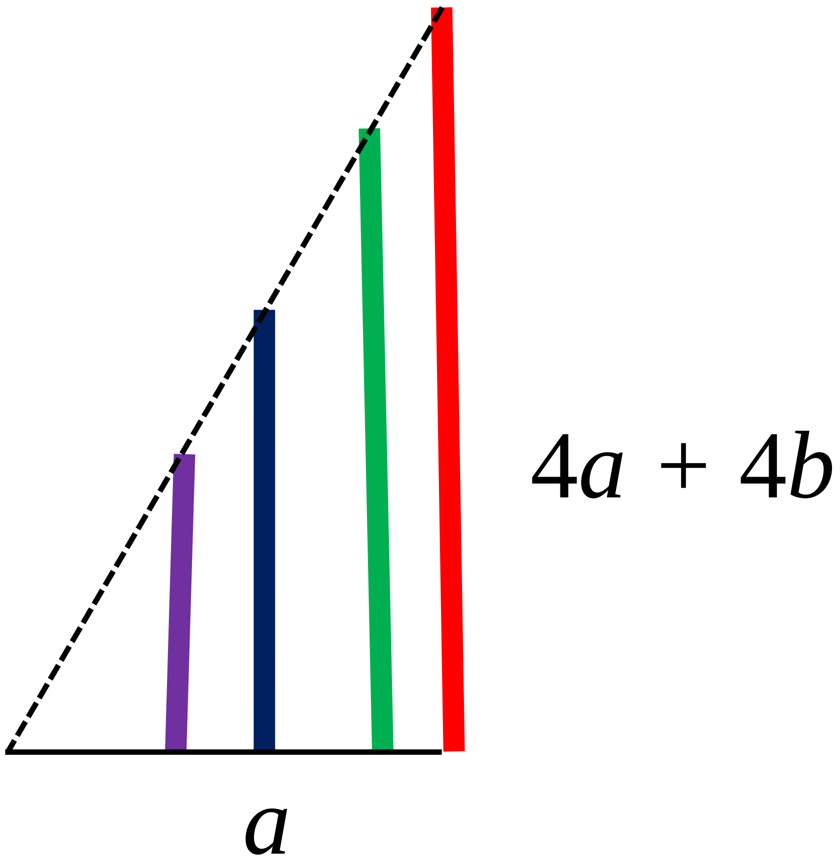





2a
2b
a
4a + 4b
a

面積$=(a)(4a+4b)/2=2a(a+b)$





意義建構(meaning-making)
vs
程序知識(procedual knowledge)

捨難取易

學、教、評估

美國某大型學校區(30 000名學生）：

Find the square root of 25.

$\sqrt{25}$ ＝

A. 4
B. 5
C. 6
D. 7

計數機 ✓

8年級學生　連續5年答對率為100%

What number has a square root of approximately 13.42 ?
Explain how you know.

之後連續兩年答對率僅為 17%

Brahier, D.J. (2001). *Assessment in middle and high school mathematics: A teacher's guide*. New York: Eye on Education.

信心建立

求平方根的方法

- 機械化程度
- 應用範圍
- 速度
- 原理

METHOD 1

計算機

METHOD 1

- 機械化程度
- 應用範圍
- 速度
- 原理

METHOD 2

質因數分解法

$$\begin{array}{r|r} 5 & 225 \\ \hline 5 & 45 \\ \hline 3 & 9 \\ \hline & 3 \end{array}$$

$$225 = 5 \times 5 \times 3 \times 3$$

$$= 15^2$$

METHOD 2

質因數分解法

5	225
5	45
3	9
	3

- 機械化程度
- 應用範圍
- 速度
- 原理

METHOD 3

連減法

36

36 – 1 = 35

35 – 3 = 32

32 – 5 = 27

27 – 7 = 20

20 – 9 = 11

11 – 11 = 0

METHOD 3

連減法

- 機械化程度
- 應用範圍
- 速度
- 原理

36

36 – 1 = 35

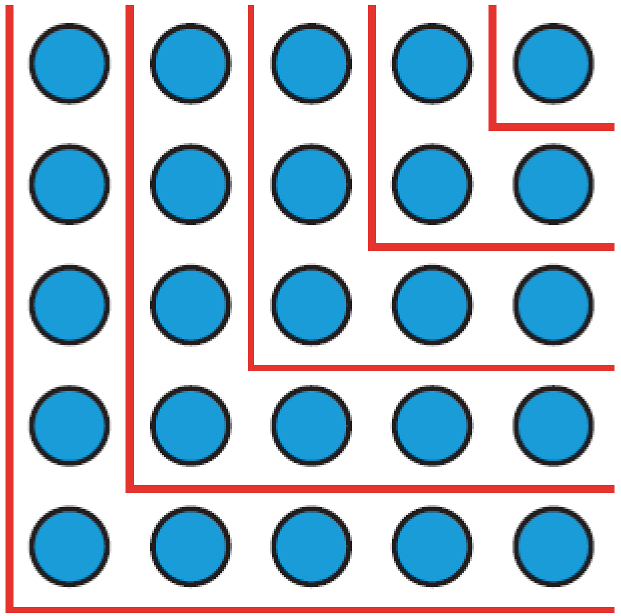
35 – 3 = 32

32 – 5 = 27

27 – 7 = 20

20 – 9 = 11

11 – 11 = 0



$$1+3+5+\ldots+(2n-1)=n^2$$

METHOD 4

連除法

求 283024 的平方根

求 283024 的平方根

STEP 1

28 30 24

求 83024 的平方根

求　83024 的平方根

STEP 1’

8 30 24

STEP 2

28	30	24

STEP 2

a	28	30	24
×			
a	a^2		

STEP 2

3	28	30	24
×			
3	9		

STEP 2

5	28	30	24
×			
5	25		

STEP 2

5	28	30	24
5	25		
	3	30	24

STEP 2

	5		
5	28	30	24
5	25		
	3	30	24

STEP 2

		5		
5		28	30	24
+				
5		25		
1	0	3	30	24

STEP 3

			5		
		5	28	30	24
		5	25		
1	0	$a \times a$	3	30	24

STEP 3

				5		
		5	28		30	24
		5	25			
1	0	3		3	30	24
		×				
		3		3	09	

				5	3	
			5	28	30	24
			5	25		
	1	0	3	3	30	24
			3	3	09	
1	0	6			21	24

				5	3	
			5	28	30	24
			5	25		
	1	0	3	3	30	24
			3	3	09	
1	0	6	a		21	24
			a			

				5	3	2
			5	28	30	24
			5	25		
	1	0	3	3	30	24
			3	3	09	
1	0	6	2		21	24
			2		21	24

				5	3	2
			5	28	30	24
			5	25		
	1	0	3	3	30	24
			3	3	09	
1	0	6	2		21	24
			2		21	24

求 97344 的平方根

			3	1	2
		3	9	73	44
		3	9		
	6	1	0	73	44
		1		61	
6	2	2		12	44
		2		12	44

求 3 的平方根

			1.	7	3	
		1	3.	00	00	00
		1	1			
	2	7	2	00	00	00
		7	1	89		
3	4	3		11	00	
		3		10	29	

METHOD 4

連除法

- 機械化程度
- 應用範圍
- 速度
- 原理

METHOD 5

連減法（升級版）

Square roots by subtraction

Frazer Jarvis

When I was at school, my mathematics teacher showed me the following very strange method to work out square roots, using only subtraction, which is apparently an old Japanese method. I'll start by writing down the algorithm in a fairly formal way, which may, for example, make it easier to implement on a computer.

Although this method converges much more slowly than Newton's method for finding square roots, for example, this method also has its advantages. Perhaps the main advantage from the computational point of view is that, when finding square roots of integers, no infinite decimals are involved at any step, which can cause loss of precision due to rounding errors.

Algorithm to compute the square root of an integer n

Initial step

Let $a = 5n$ (this multiplication by 5 is the only time when an operation other than addition and subtraction is involved!), and put $b = 5$.

Repeated steps

(R1) If $a \geq b$, replace a with $a - b$, and add 10 to b.

(R2) If $a < b$, add two zeroes to the end of a, and add a zero to b just before the final digit (which will always be '5').

Conclusion

Then the digits of b approach more and more closely the digits of the square root of n.

Jarvis, P. (1994). Square roots by subtraction.

求 n 的平方根

STEP 1 $(5n, 5)$

求 n 的平方根

STEP 2

RULE A $\quad a \geq b$

$$(a, b)$$

$$\downarrow$$

$$(a-b, b+10)$$

RULE A

(20 , 5)

RULE A

$$(20, 5)$$

$$\downarrow$$

$$(20 - 5, 5 + 10)$$

RULE A

(20 , 5)

↓

(15 , 15)

RULE A

(20 , 5)

↓

(15 , 15)

↓

(0 , 25)

RULE A

$$(45, 5)$$

↓

?

RULE A

(45 , 5)

↓

(0 , 35)

求 n 的平方根

STEP 2

RULE B $\quad a < b$

$$(a, b)$$

$$\downarrow$$

$$(\overline{a\,00}, \overline{b_1\,05})$$

RULE B

(5 , 15)

↓

(500 , 105)

RULE B

(20 , 145)

↓

(2000 , 1405)

RULE A+B

$$(5n, 5)$$

$$\downarrow$$

$$\downarrow$$

$$(a, b)$$

b 的數位 $\rightarrow \sqrt{n}$ 的數位

求 36 的平方根

(36×5 , 5)

求 36 的平方根

(180 , 5)

求 36 的平方根

(180 , 5)

(175, 15)

求 36 的平方根

(180 , 5)

(175, 15)

(160, 25)

求 36 的平方根

(180 , 5)

(175, 15)

(160, 25)

(135, 35)

求 36 的平方根

(100 , 45)

(180 , 5)

(175, 15)

(160, 25)

(135, 35)

求 36 的平方根

(180 , 5)

(175, 15)

(160, 25)

(135, 35)

(100 , 45)

(55, 55)

(0, 65)

求 36 的平方根

(180 , 5)

(175, 15)

(160, 25)

(135, 35)

(100 , 45)

(55, 55)

(0, 65)

(0, 605)

(0, 6005)

求 36 的平方根

(180 , 5)

(175, 15)

(160, 25)

(135, 35)

(100 , 45)

(55, 55)

(0, 65)

(0, 605)

(0, 6005)

求 1 的平方根

(5, 5)

求 1 的平方根

(5 , 5)

(0, 15)

(0, 105)

(0, 1005)

求 25 的平方根

(125 , 5)

(120, 15)

(105, 25)

(80, 35)

(45 , 45)

(0, 55)

(0, 505)

(0, 5005)

(0, 50005)

求 3 的平方根

(15 , 5)

求 3 的平方根

(15 , 5)

(10, 15)

(1000, 105)

(895, 115)

(780 , 125)

求 3 的平方根

(15 , 5)

(10, 15)

(1000, 105)

(895, 115)

(780 , 125)

(655, 135)

(520, 145)

(375, 155)

(220, 165)

(55 , 175)

求 3 的平方根

(15 , 5)	(655, 135)	(5500, 1705)
(10, 15)	(520, 145)	(3795, 1715)
(1000, 105)	(375, 155)	(2080, 1725)
(895, 115)	(220, 165)	(355, 1735)
(780 , 125)	(55 , 175)	(35500, 17305)

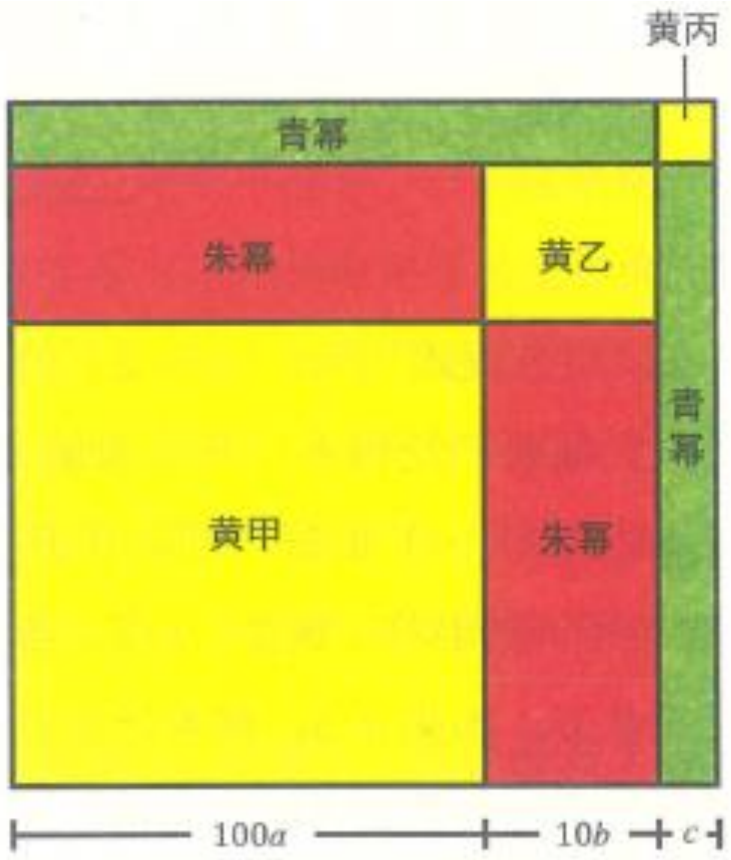
求 3 的平方根

(15 , 5)	(655, 135)	(5500, 1705)	(18195, 17315)
(10, 15)	(520, 145)	(3795, 1715)	(880, 17325)
(1000, 105)	(375, 155)	(2080, 1725)	(88000, 173205)
(895, 115)	(220, 165)	(355, 1735)	……
(780 , 125)	(55 , 175)	(35500, 17305)	

METHOD 5　連減法（升級版）

- 機械化程度
- 應用範圍
- 速度
- 原理

Other
s
METHODS



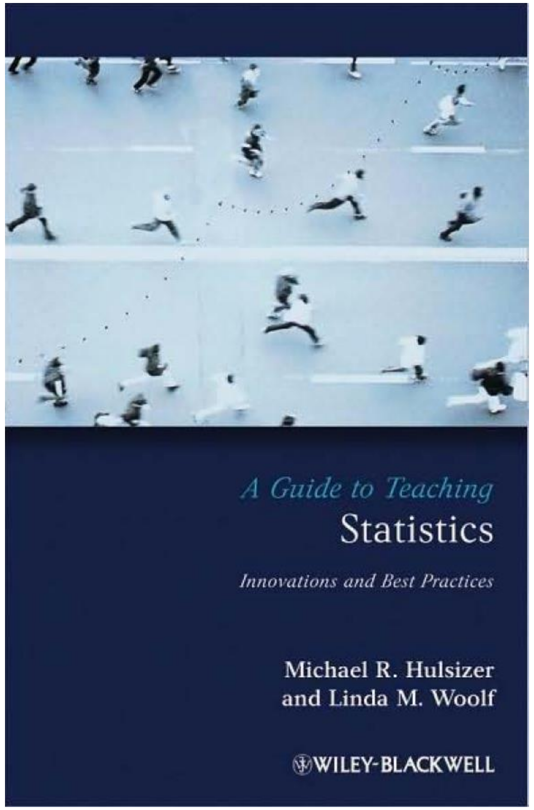
運算及機械式學習會妨礙學習數學時的意義建構

Goos, Tillman, & Vale, 2007, p.259.

懂得計算算術平均數的學生中
只有少數懂得選擇運用平均數比較兩組數據

懂得計算算術平均數的學生中
只有少數懂得選擇運用平均數比較兩組數據

EXAMPLE



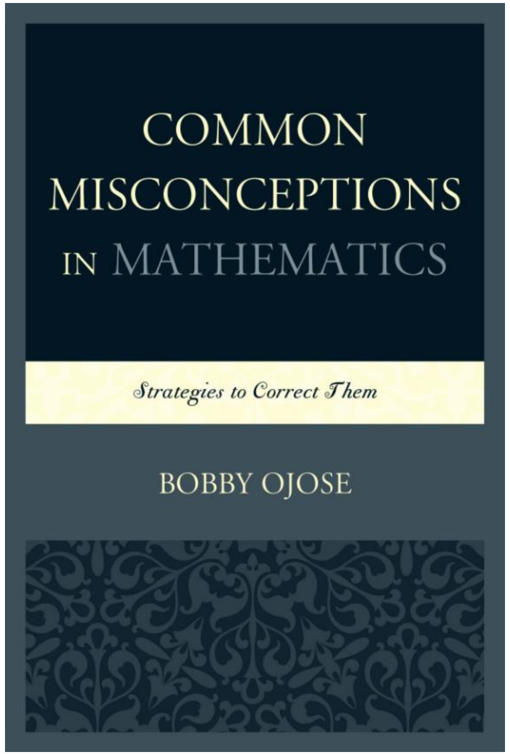
p.80

意義建構

意義建構

概念的多重表達

Ojose, 2015, p.50



四則運算的概念

10 – 4

1. 哥哥有10 個蘋果，吃了4個，剩下幾個？

10 – 4

1. 哥哥有10 個蘋果，吃了4個，剩下幾個？

2. 妹妹有10 個蘋果，吃了4個，剩下幾個？

3. 妹妹有10 個橙，吃了4個，剩下幾個？

4. 小明有10 元，用了4 元，剩下幾元？

10 – 4

1. 哥哥有10 個蘋果，吃了4個，剩下幾個？

5. 哥哥有10 個蘋果，妹妹有4個蘋果，哥哥比妹妹多幾多個蘋果？

10 – 4

1. 哥哥有10 個蘋果，吃了4個，剩下幾個？

5. 哥哥有10 個蘋果，妹妹有4個蘋果，哥哥比妹妹多幾多個蘋果？

6. ……

$20 \div 4$

$20 \div 4$

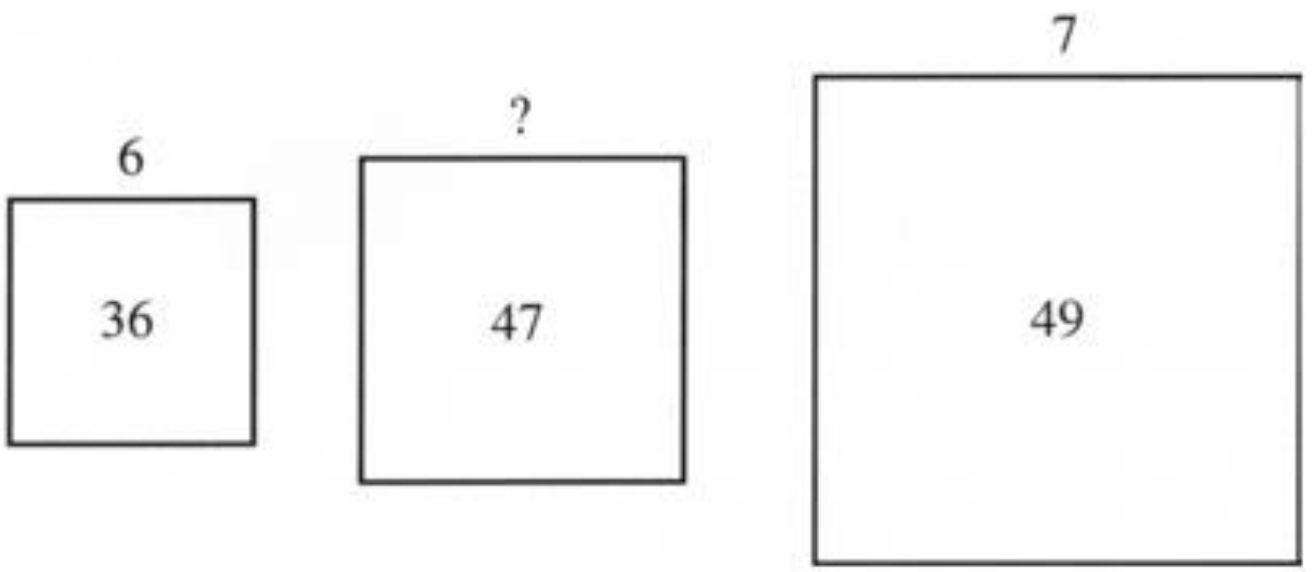
$$1 \div \frac{1}{6}$$

平方根的概念

平方 vs 平方根

平方根的概念

平方 vs 平方根



平方根的概念

平方 vs 平方根

Square roots

$\sqrt{4}=2$

4的平方根是 2、-2

4的平方根是 $\pm\sqrt{4}$

4的平方根是 2 和/或 -2

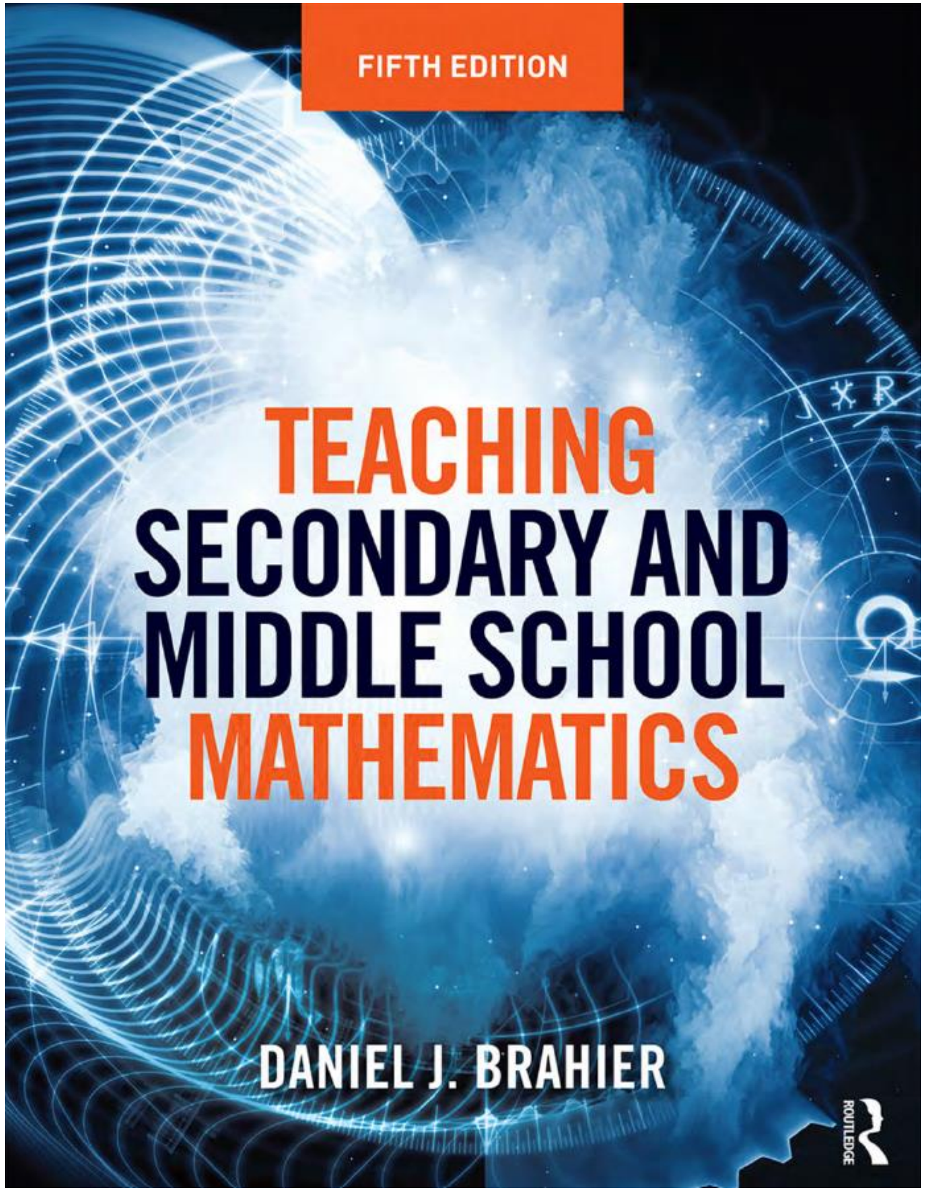
平方根的概念

平方 vs 平方根

Square roots

非平方數的平方根

負數的平方根



Brahier, D.J. (2016). *Teaching secondary and middle school mathematics* (5th ed.). New York & London: Routledge.

In a mathematics class, Anne Kelley is helping her students estimate the value of the square root of a number and compare the estimate with an answer on the calculator.

Ms. Kelley: Can anyone give me an estimate of $\sqrt{52}$?

Luke: It's about 7.5.

Ms. Kelley: Explain your thinking on that, Luke.

Luke: Well, I know that $\sqrt{49}$ is 7 and that $\sqrt{64}$ is 8. So, I knew it had to be between those, since 52 is between 49 and 64.

Else: That's true, but 52 is a lot closer to 49 than it is to 64, so I would guess it would be more like 7.2 or 7.3.

Ms. Kelley: Does anyone else have any thoughts on this one?

Terry: Well, look at it this way . . . the average of 49 and 64 is 56.5. So, we can figure that $\sqrt{56.5}$ is halfway between 7 and 8 . . . um . . . 7.5, right?

Brahier, 2016, p.169.

Ms. Kelley: How can we check to see how close 7.5 is to the answer?

Luke: Find 7.5^2 and see if it's 56.5.

Ms. Kelley: Good. Why don't you go ahead and find 7.5^2 on your calculators.

(The class keys 7.5 into their calculators and squares the number.)

Joshua: No, 7.5^2 is equal to 56.25, not 56.5.

Terry: Close enough, wouldn't you say, Ms. Kelley?

Ms. Kelley: I don't know. Why don't you go ahead and take $\sqrt{52}$.

(The class finds the square root with their calculators.)

Elise: I knew it! It's only 7.21-something. It's like I said; it's closer to 7 than to 8.

Ms. Kelley: Do we know "exactly" what it's equal to?

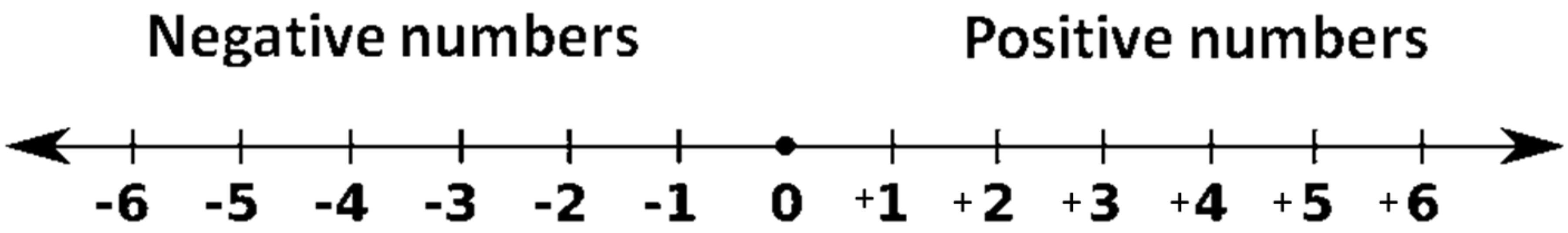
Luke: Yea, it's exactly 7.211102551. That's what my calculator shows anyway.

Terry: So does mine. But I still think it's close enough if we call it 7.21.

Lakisha: Does it go any further than 7.211102551? I'm just wondering because we've seen other times where the calculator cuts a number off because it doesn't have enough space . . . like that one time we tried to change a fraction to a repeating decimal that had, like, 12 decimal places before it started to repeat.

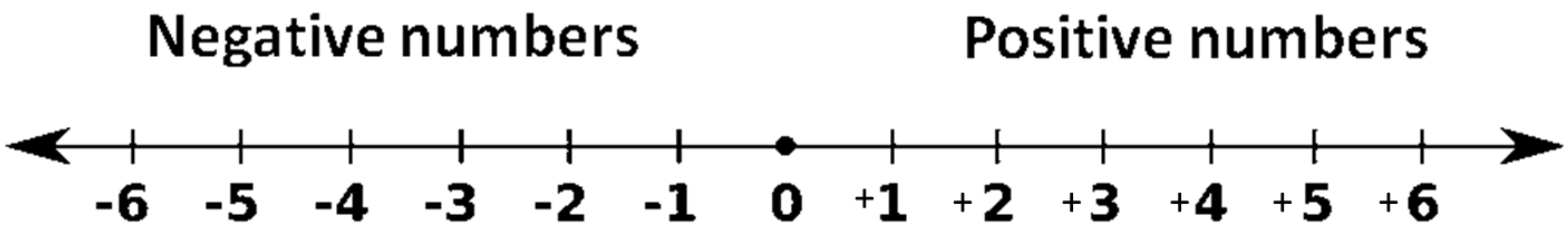
(The students in the class start to look confused and wonder if Lakisha is on to something. They look up at Ms. Kelley and wait for her response.)

有向數的概念



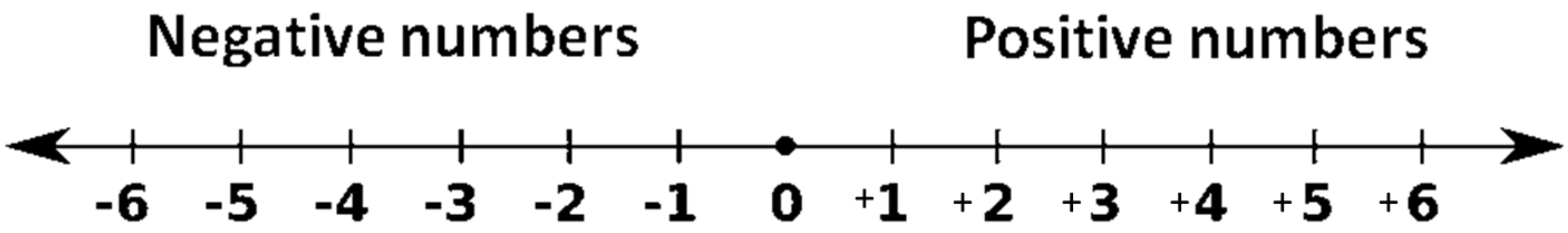
有向數的概念

$2 \quad +2 \quad -2$



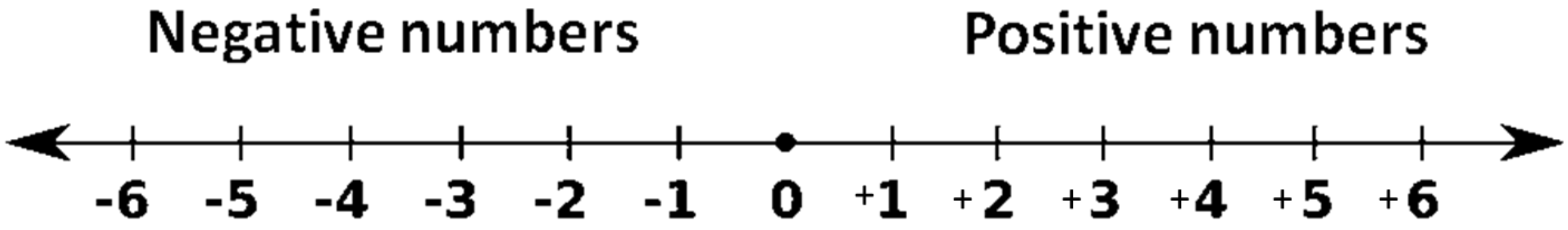
有向數的概念

$$(+2) + (-3) - (-4)$$



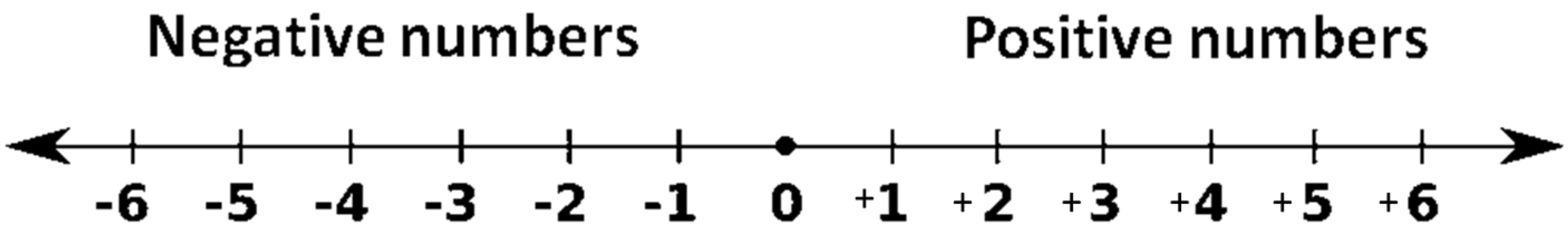
有向數的概念

$$(+2) + (-3) - (-4)$$



有向數的概念

$$({}^{+}2) + ({}^{-}3) - ({}^{-}4)$$



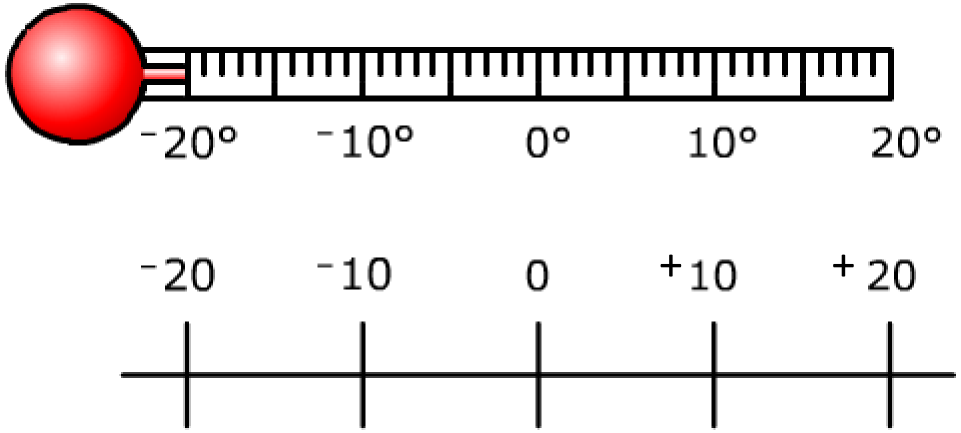
有向數的概念

First floor

Ground floor

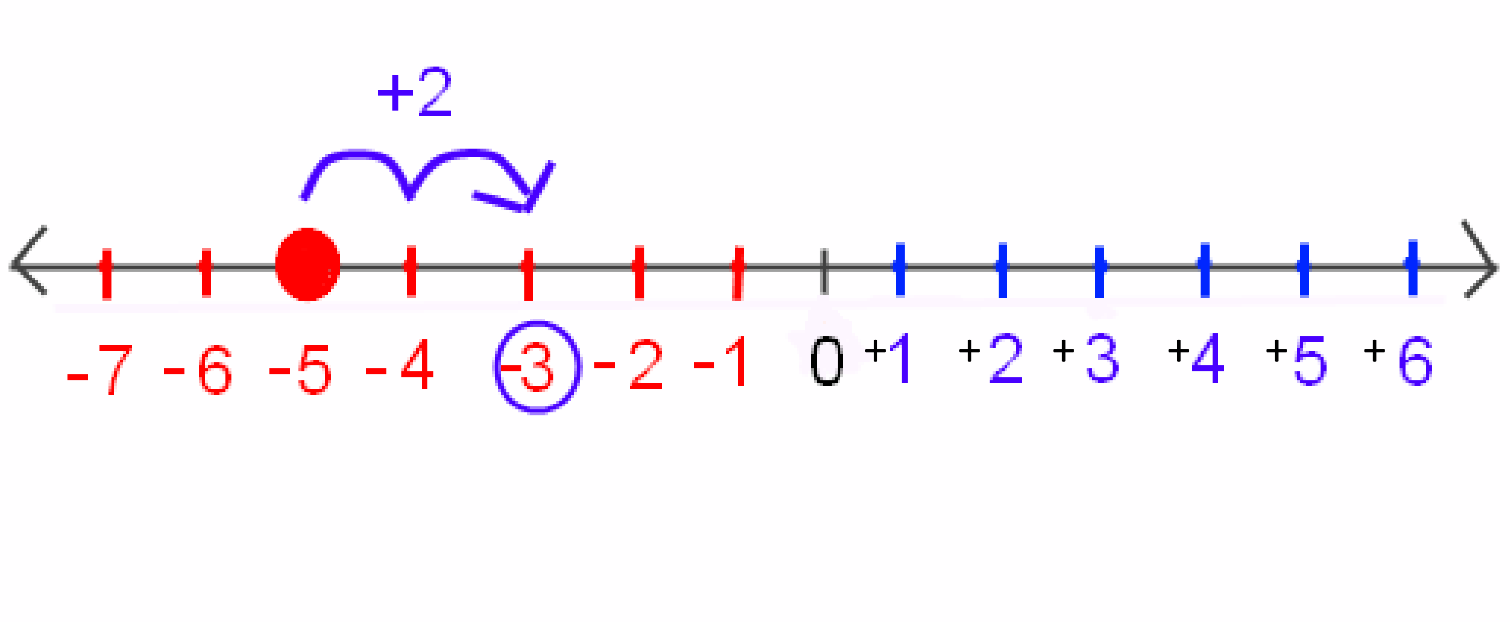
First basement (underground)

有向數的概念



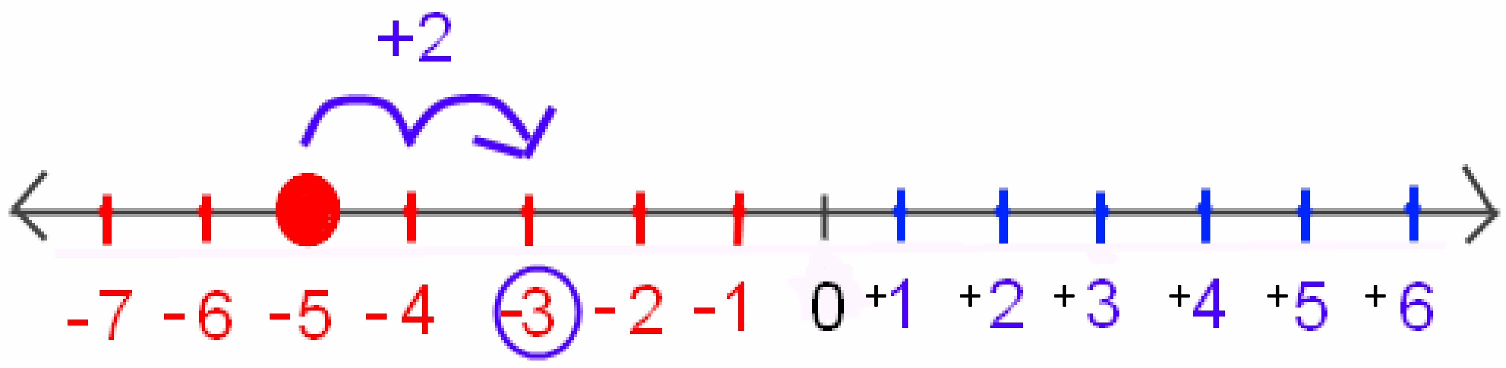
有向數運算的概念

$$(-5) + 2 = -3$$



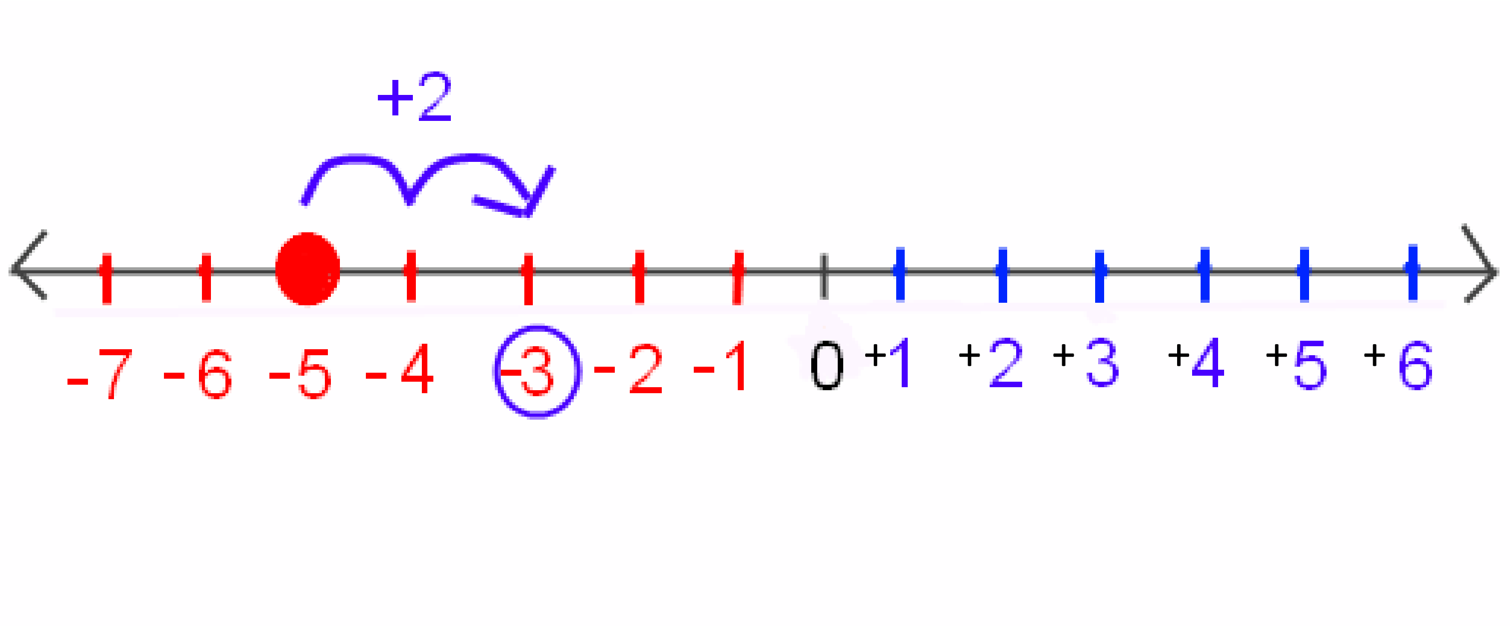
有向數運算的概念

$$(-5) + 2 = -3$$



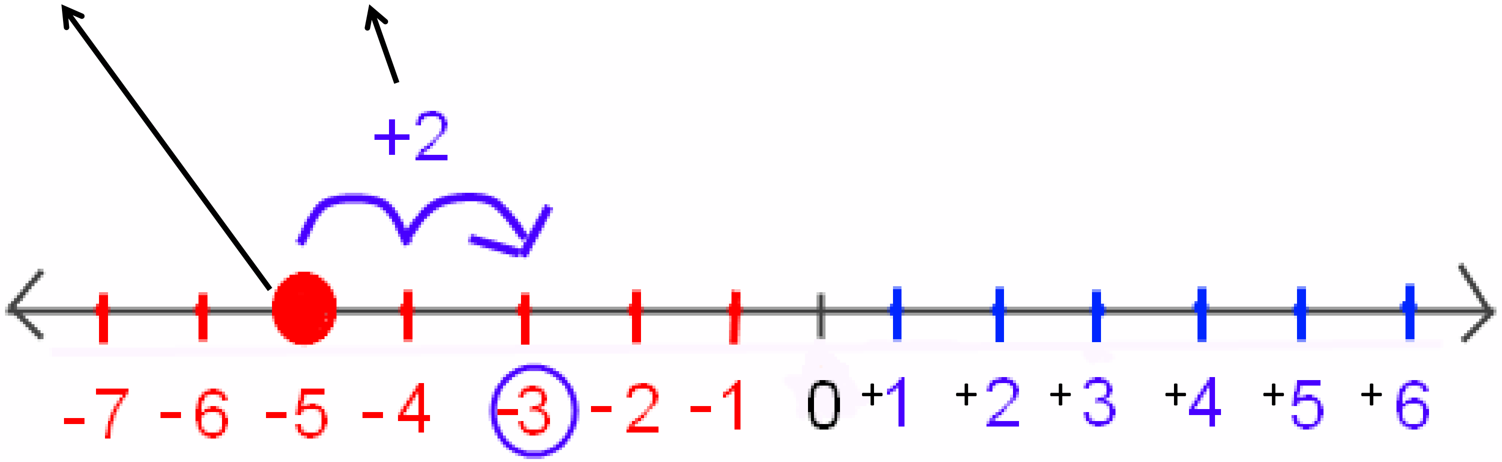
有向數運算的概念

$$(-5) + (+2) = -3$$



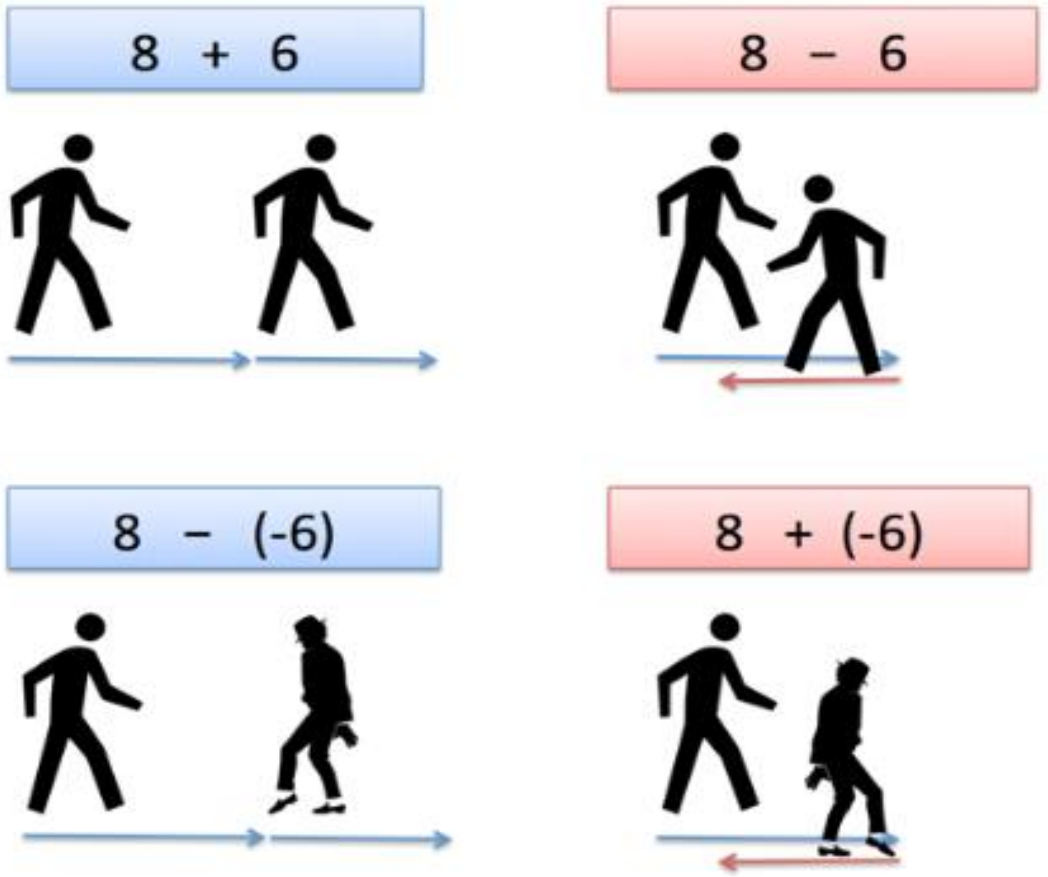
有向數運算的概念

$$(-5) + (+2) = -3$$



有向數運算的概念

The addition/subtraction tells us which way to face, and the positive/negative tells us if our steps will be forward or backward (regardless of the way we're facing).

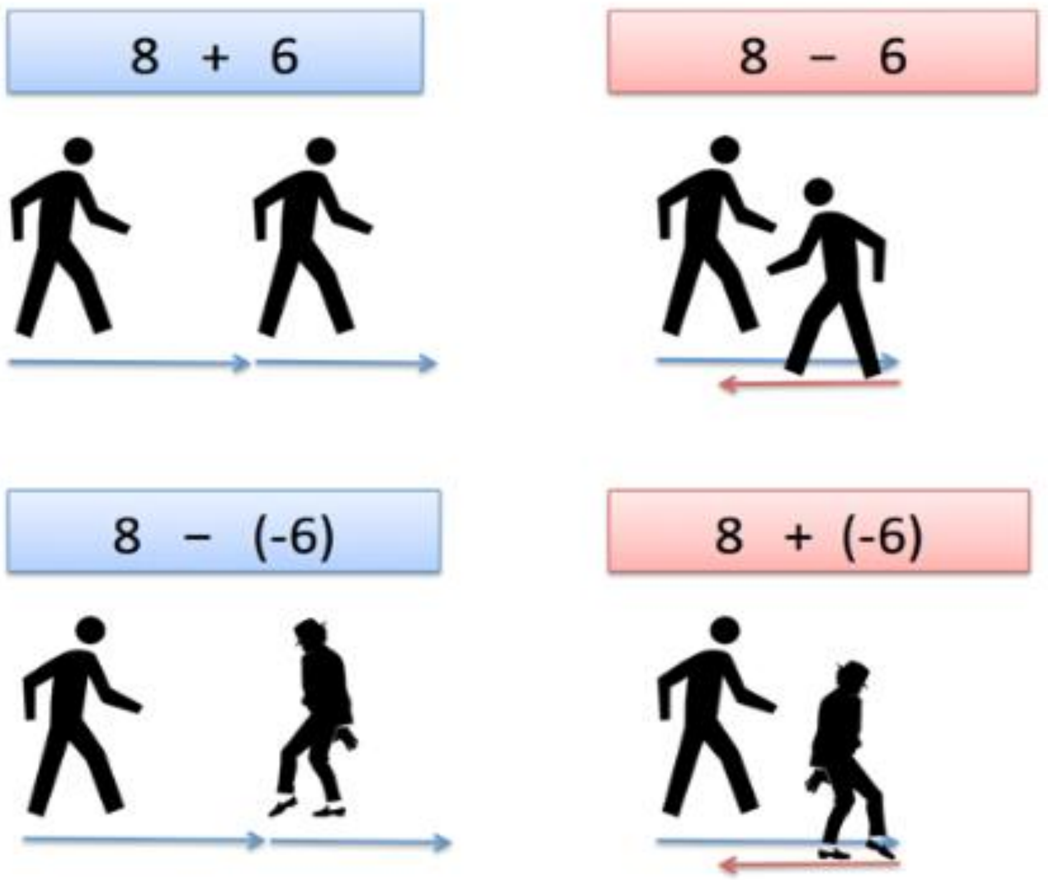


有向數運算的概念

The addition/subtraction tells us which way to face, and the positive/negative tells us if our steps will be forward or backward (regardless of the way we're facing).

$(^{+}8) + (^{+}6)$

$(^{+}8) - (^{+}6)$



$(^{+}8) - (^{-}6)$

$(^{+}8) + (^{-}6)$

有向數運算的概念

模型	計算
加熱空氣	加一個正數
加沙包	加一個負數
減熱空氣	減一個正數
減沙包	減一個負數

$$(+5)-(-2)$$

有向數運算的概念

$(+5)-(-2)$

模型	計算
買優秀球員	加一個正數
買差劣球員	加一個負數
賣優秀球員	減一個正數
賣差劣球員	減一個負數

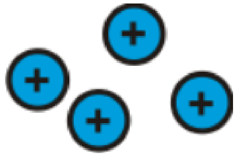
有向數運算的概念

$^{+}4$ 的不同表達方式

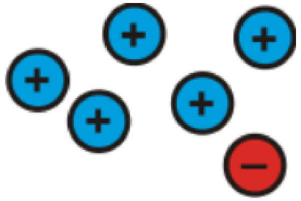
https://nrich.maths.org/5947

This model was introduced to us by Don Steward.

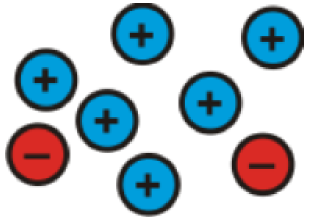
Let this be 4



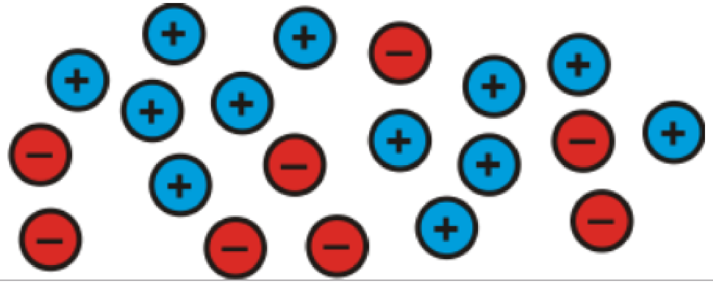
and this



and this



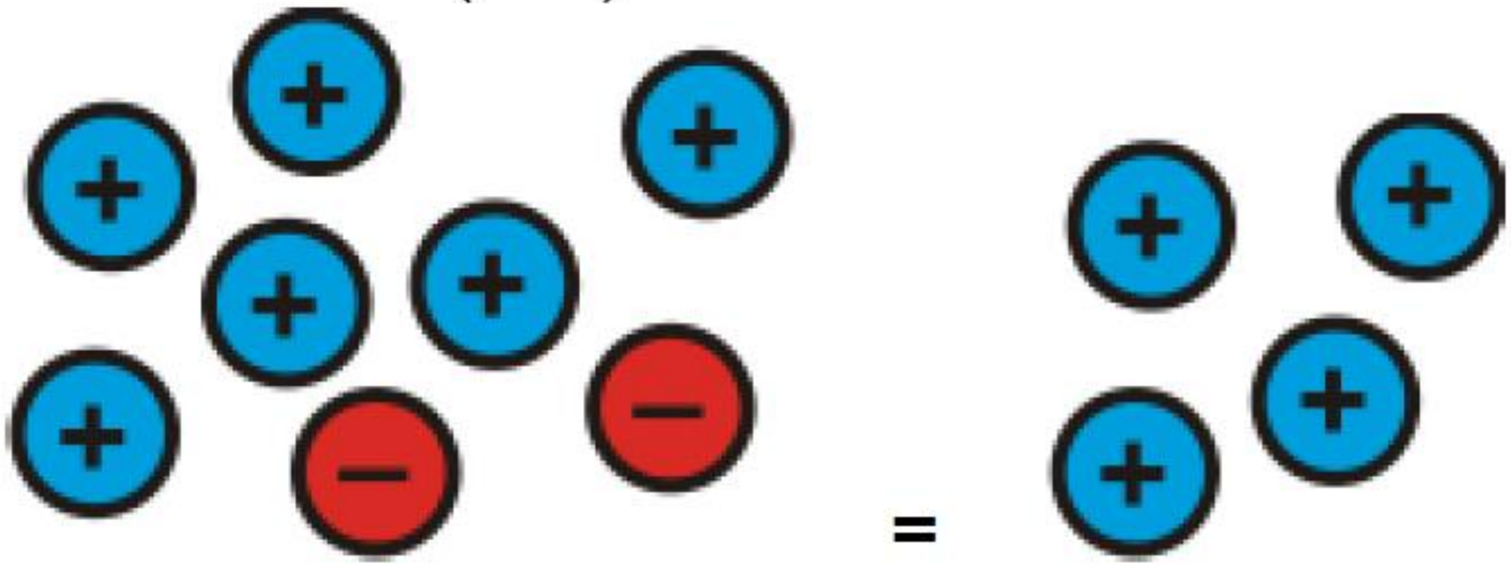
and this



有向數運算的概念

$(^{+}6) + (^{-}2)$

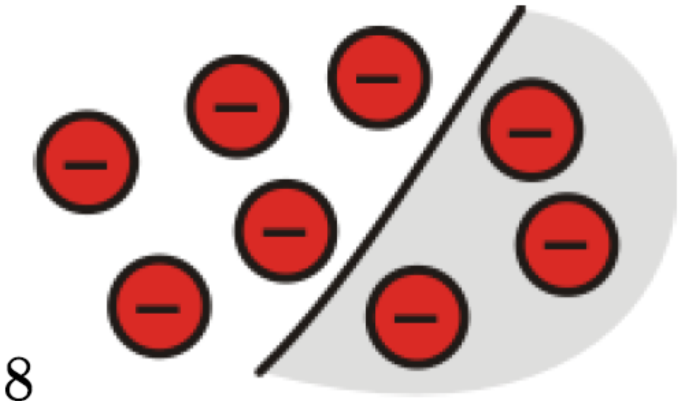
$6 + (^{-}2) = ?$



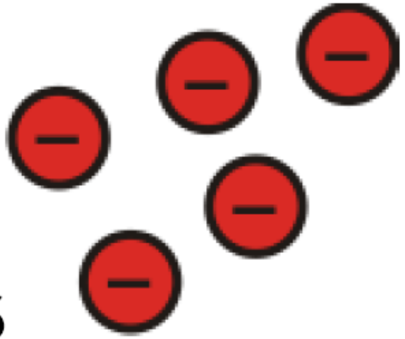
$6 + (^{-}2) = 4$

有向數運算的概念

$(^{-}8) - (^{-}3)$



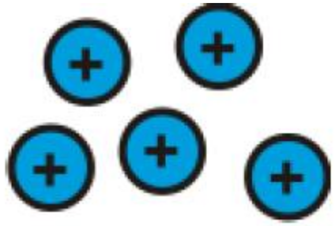
And this represents $^{-}8$
subtract $^{-}3$



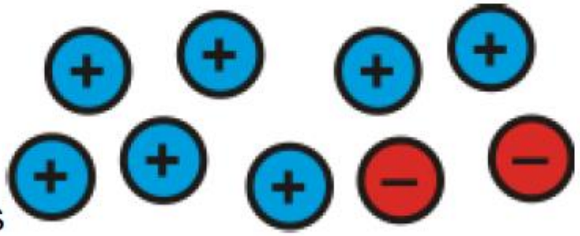
Leaving us with $^{-}5$
So $^{-}8 - (^{-}3) = {}^{-}5$

有向數運算的概念

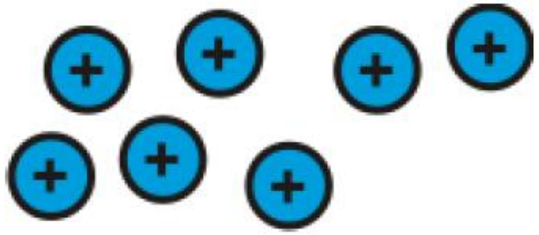
$(^{+}5) - (^{-}2)$



but so does this



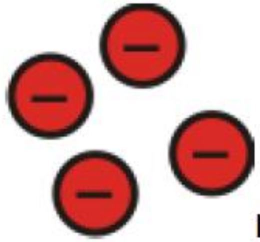
Now we can take $^{-}2$ from 5 and we will be left with:



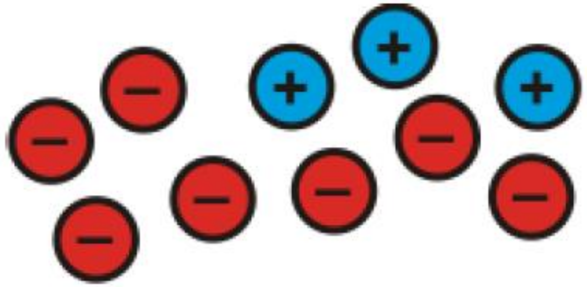
So $5 - (^{-}2) = 7$

有向數運算的概念

$(^-4) - (^-7)$



represents $^-4$



but so does this

Now we can take $^-7$ from $^-4$ and we will be left with:

So $^-4 - (^-7) = 3$

有向數運算的概念

$$(^{+}5) \times (^{+}4)$$

$$(^{+}5) \times (^{-}4)$$

$$(^{-}5) \times (^{-}4)$$

坐標系統的概念

文 1－9

坐标：源于定位，高于定位

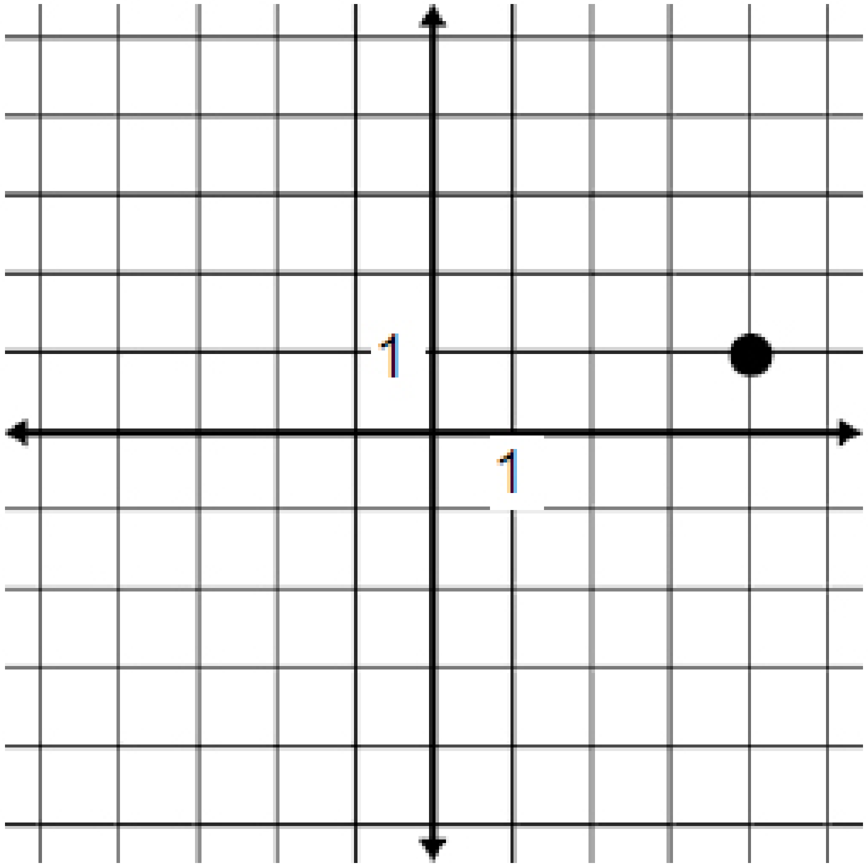
大量的“坐标”教学设计，都把“用一对数确定平面一点的位置”作为教学重点。还常常让学生站成几排，用第几排第几个的一对有序的自然数来表示位置，一时上上下下好不热闹。其实，这些都是生活常识，不教也会。就像打电话，用不着一本正经地在课堂上教，自己看看就会了。

小学数学中引入坐标系，学习的重点和难点是坐标系的建立，尤其是坐标原点的设置。许多教案从电影院找座位引入，当然可以，问题在于这时的电影院排座位的坐标原点在哪里？第一排第一座是原点吗？可是我们还有 0 排 0 座怎么办？电影院用单双号方法排座位，就无法设置原点，也构不成数学意义上的坐标轴。其实，还是把教室中的座位排紧，可以构成符合坐标系要求的座位图。我们不妨设左上角为原点：0 排 0 座。其他座位就都有（自然数）坐标了。如果将它定为第一排第一座，那就需要假想虚拟的原点和坐标轴。这些内容是我们学习的核心。

令人不解的是，许多初中数学的坐标系教学设计，仍然谈“东大街、南马路”马路交界处之类的问题来引入，不厌其烦地举例，停留在小学阶段“确定位置”的水平上。

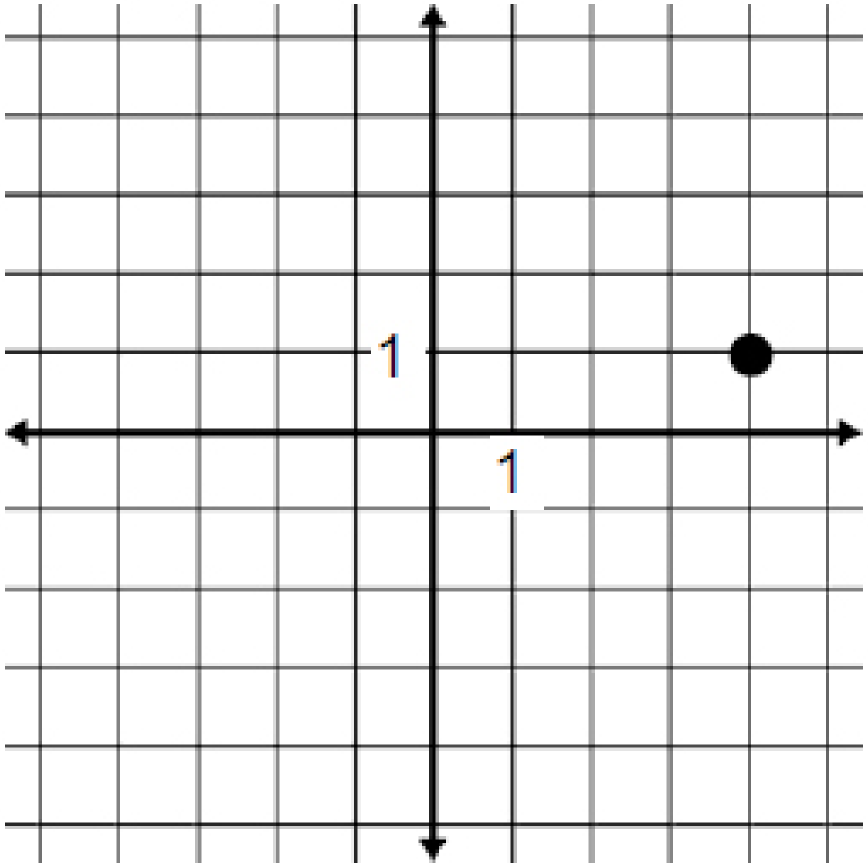
其实，同样用教室中排竖的座位构成的座位图，只要把原点定在任何一个中间同学的座位，以横向的一排作为 x 轴，纵向的一列为 y 轴，就构成一个具有四个象限的直角坐标系了。选择不同的同学座位作为原点，就产生了坐标变换。正负数在这里的作用就充分显示出来了。这是初中坐标系教学的一个关节点。至于将整数坐标扩充到实数坐标点情形，是容易想象的事情[①]。

2D 的概念



(0.2345 , 0.5672)

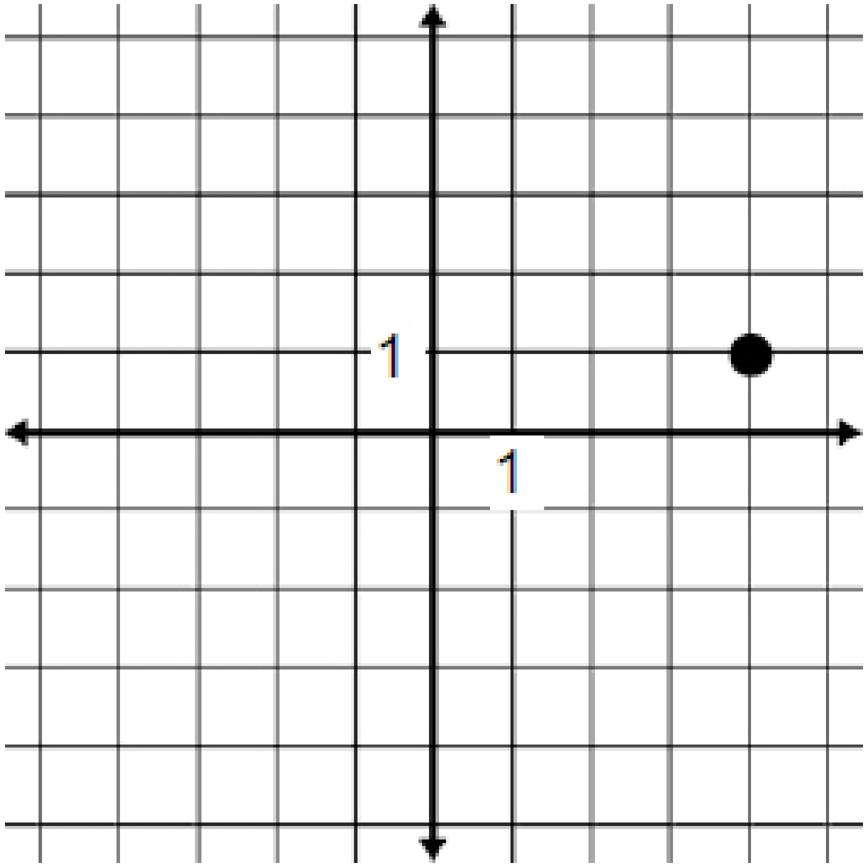
2D 的概念



(0.2345 , 0.5672)

[11.25 36 47 52]

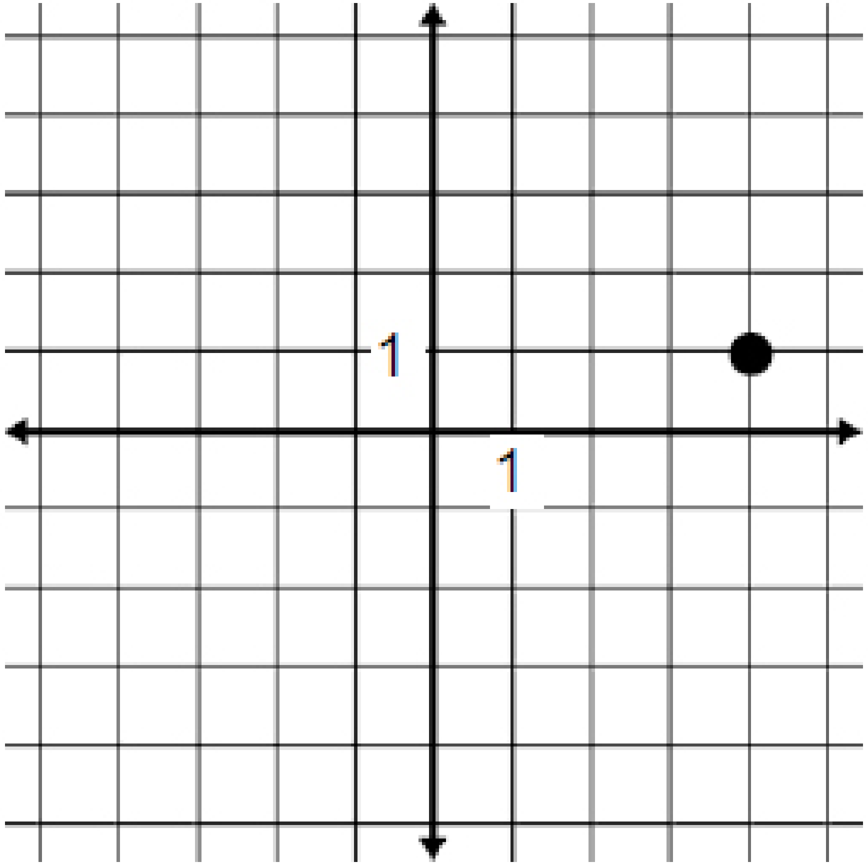
2D 的概念



(⁻0.2345 , 0.5672)

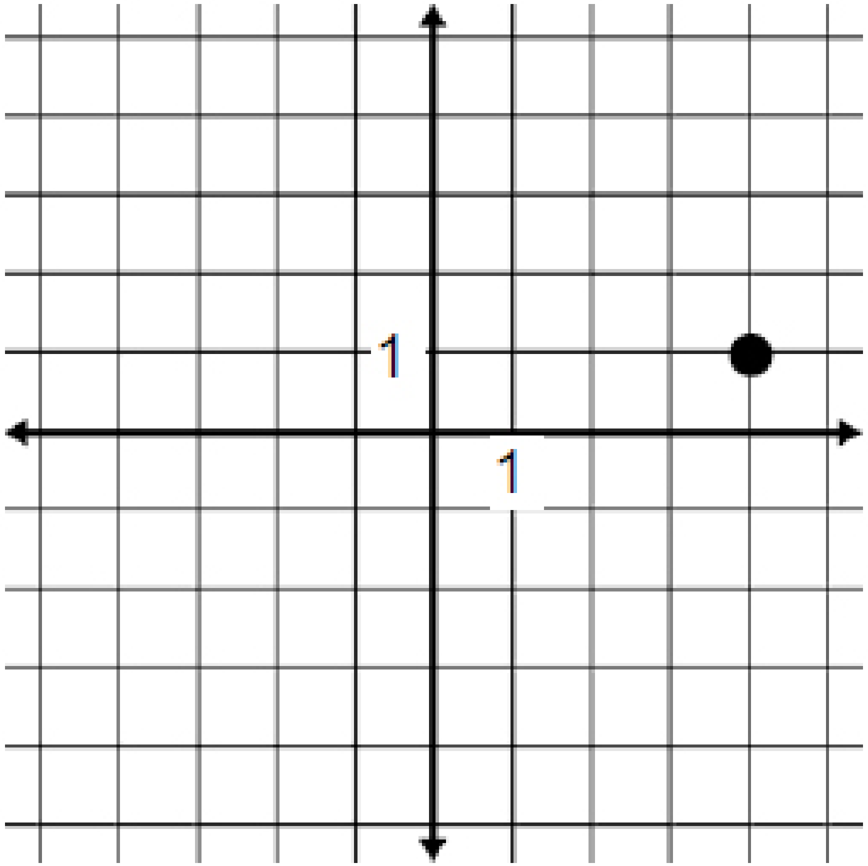
[12.25 36 47 52]

2D 的概念



[500011.341899]

2D 的概念



(0.319 , 50.489)

[500011.341899]

方程圖像的**概念**

二元一次方程的圖像必定是一條直線，且直線上任意一點的坐標必定滿足該方程

方程圖像的概念

二元一次方程的圖像必定是一條直線，且直線上任意一點的坐標必定滿足該方程

方程圖像的概念

二元一次方程的圖像必定是一條直線，且直線上任意一點的坐標必定滿足該方程

$$Ax + By + C = 0$$

方程圖像的**概念**

二元一次方程的圖像**必定是一條直線**，且直線上任意一點的坐標必定滿足該方程

$$0x + 0y + 0 = 0$$

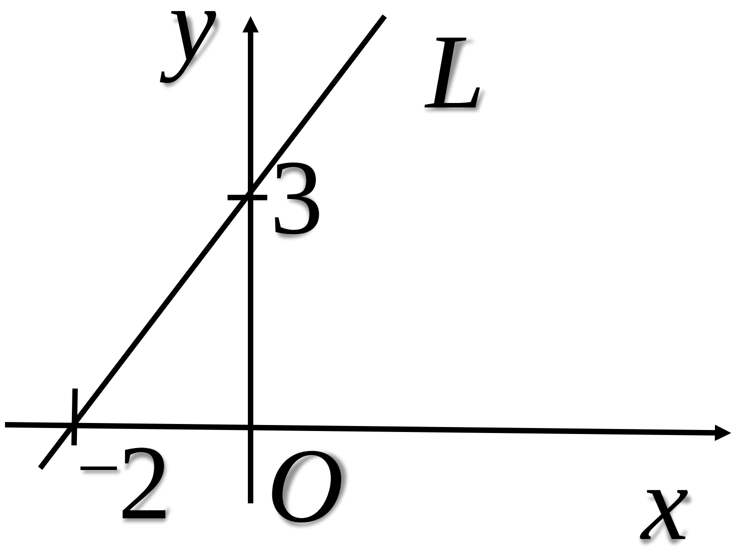
方程圖像的**概念**

二元一次方程的圖像**必定是一條直線**，且直線上任意一點的坐標必定滿足該方程

$$0x + 0y + 0 = 0$$

$$0x + 0y + 3 = 0$$

方程圖像的概念

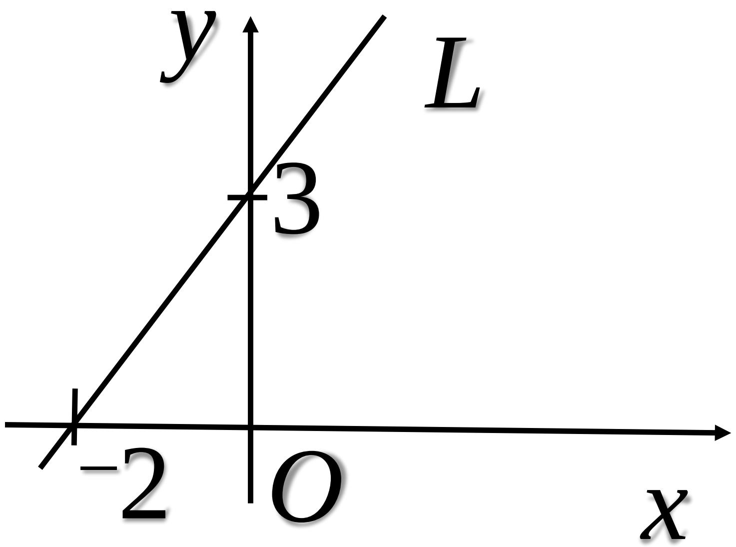


任何滿足 $3x - 2y + 6 = 0$ 的坐標 (x, y) 代表 L 上的點。

反之，任何 L 上的點(x, y)均滿足 $3x - 2y + 6 = 0$。

方程圖像的概念

$$y\,(3x - 2y + 6) = 0$$



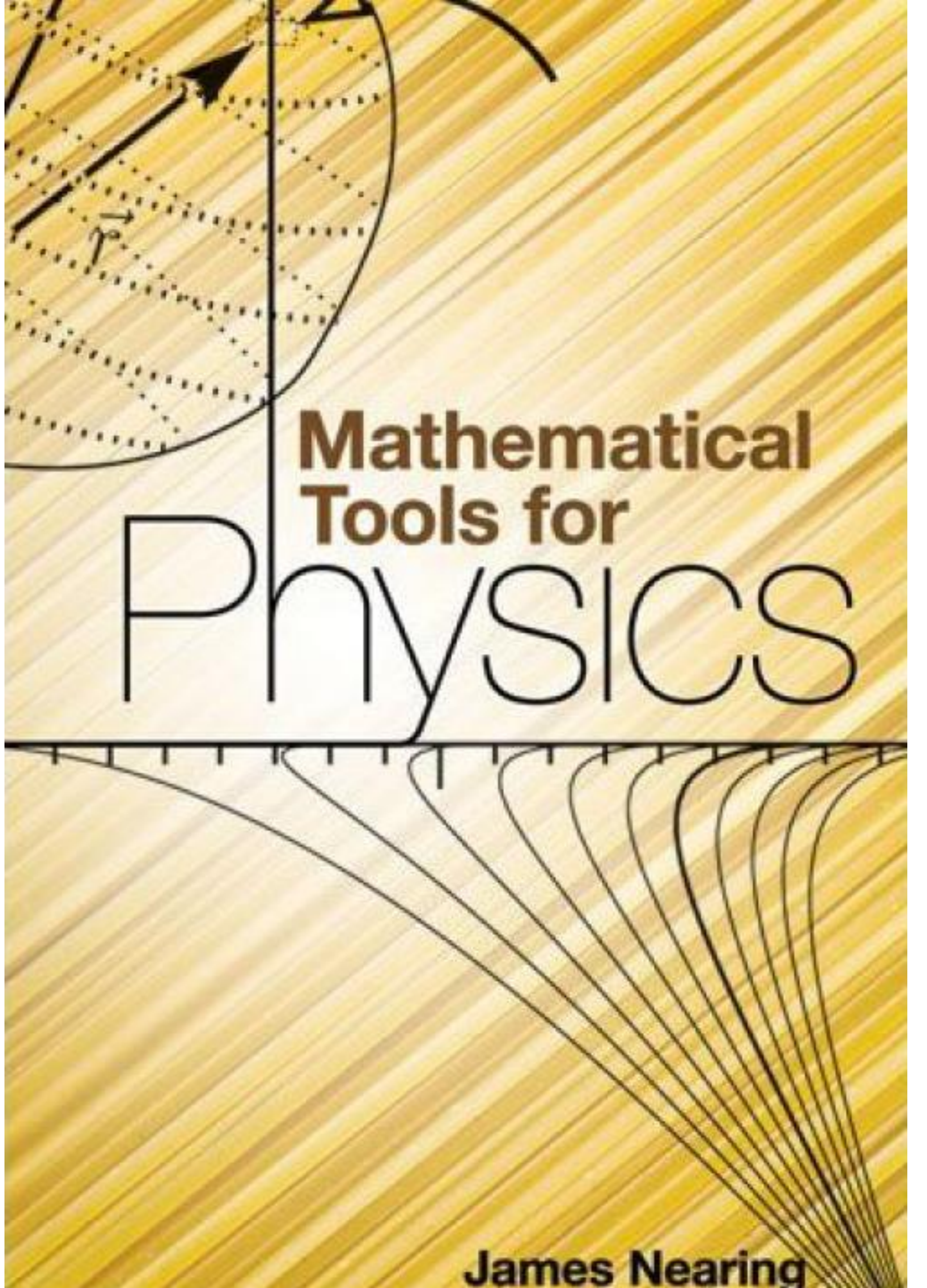
任何滿足 $3x - 2y + 6 = 0$ 的坐標 (x, y) 代表 L 上的點。

反之，任何 L 上的點(x, y)均滿足 $3x - 2y + 6 = 0$。

向量的概念

Nearing(2003):

可追溯至1800's
Peano 1888 流行



向量的概念

Nearing(2003):

可追溯至1800’s
Peano 1888 流行

向量的概念

有大小和方向的就是
向量?

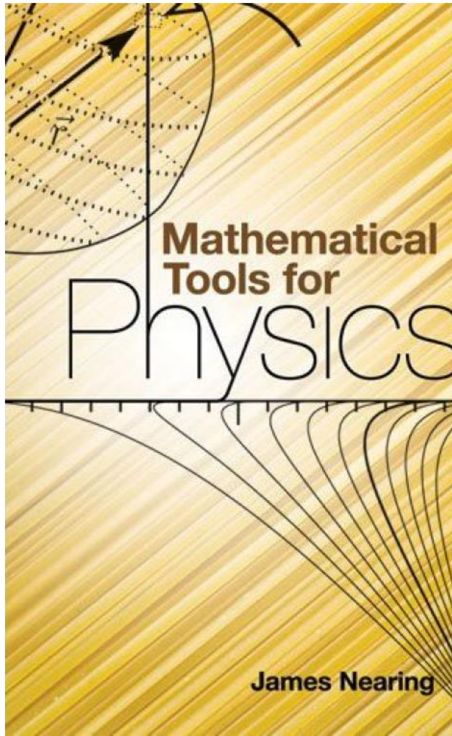
向量的概念

6.1 The Underlying Idea

What *is* a vector?

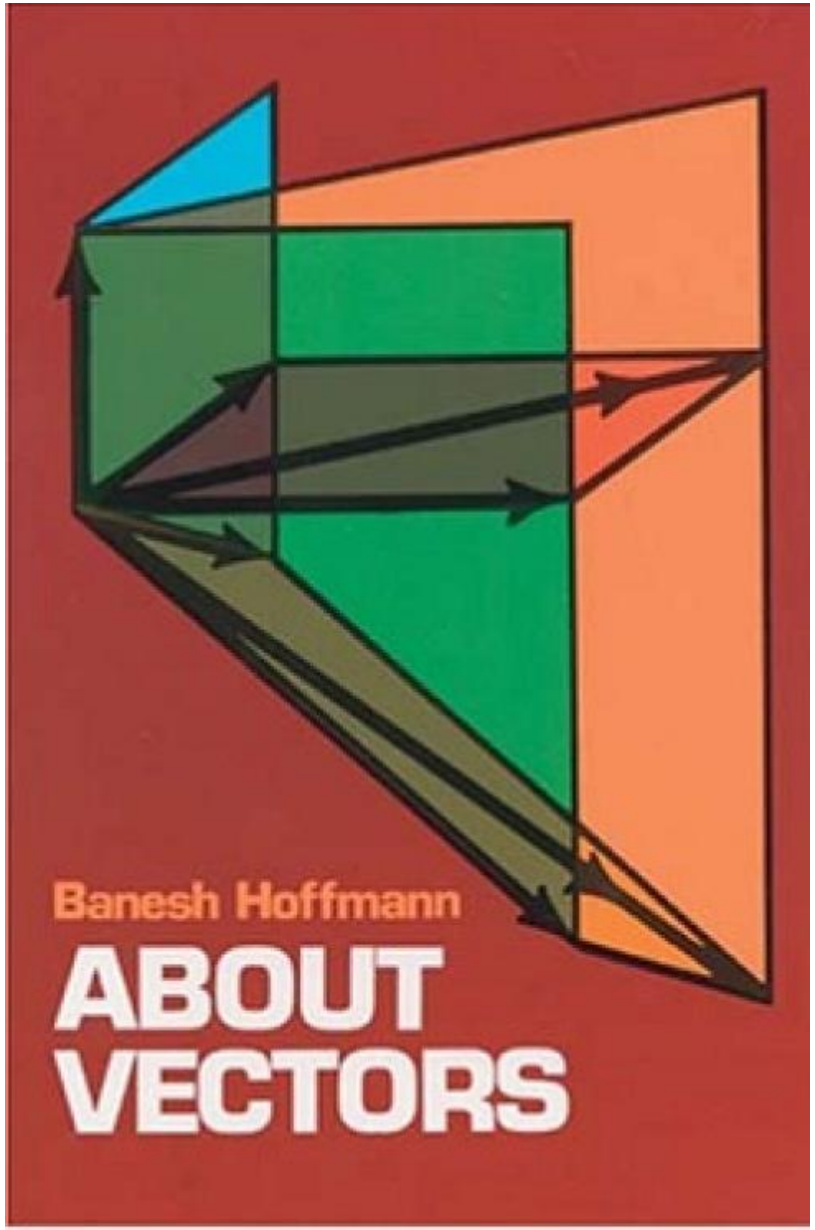
If your answer is along the lines "something with magnitude and direction" then you have something to unlearn. Maybe you heard this definition in a class that I taught. If so, I lied; sorry about that. At the very least I didn't tell the whole truth. Does an automobile have magnitude and direction? Does that make it a vector?

Nearing, 2003, p.142



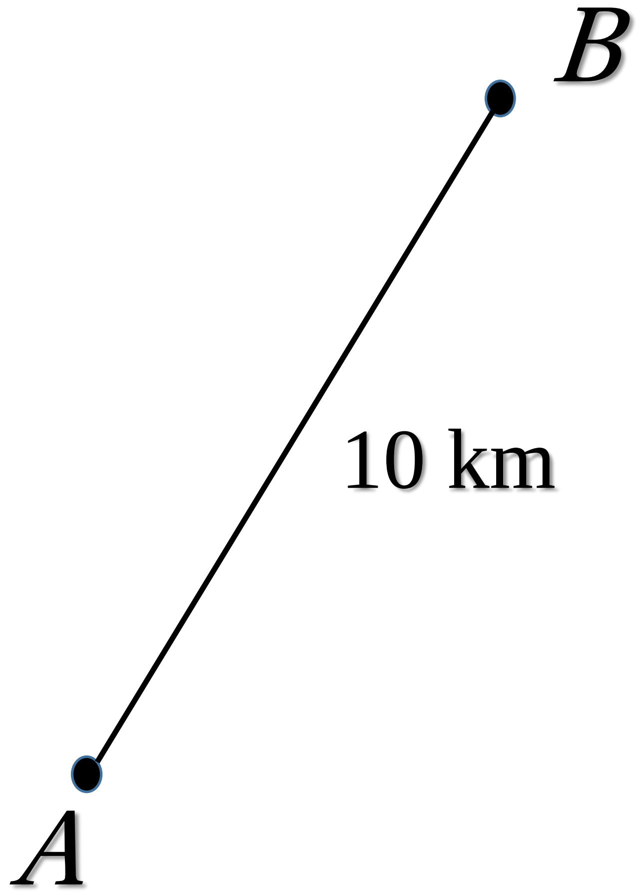
向量的概念

我家在A，我的朋友住在距離我家10km遠的B。我從家出發，步行了2.5h，步行的速率是4km/h，........



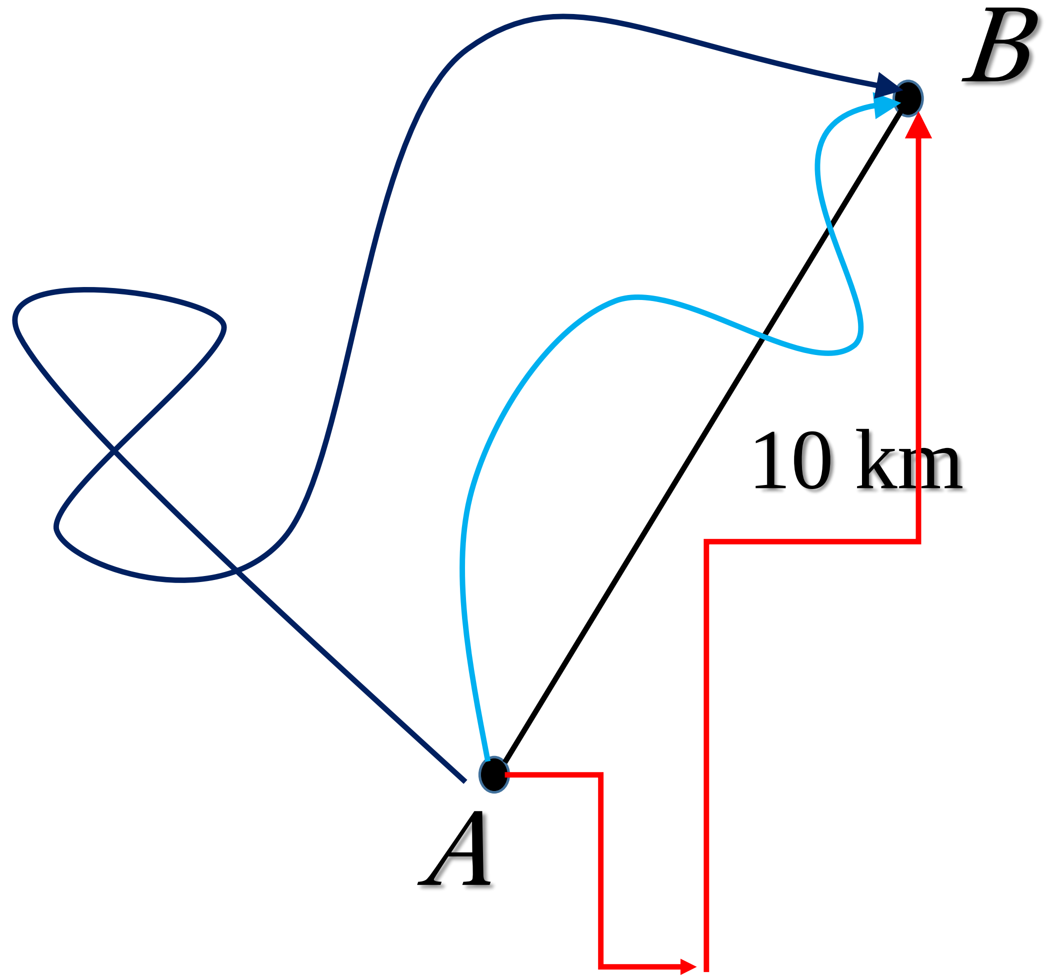
向量的概念

我家在A，我的朋友住在距離我家10 km 遠的B。我從家出發，步行了2.5h，步行的速率是4km/h，……..

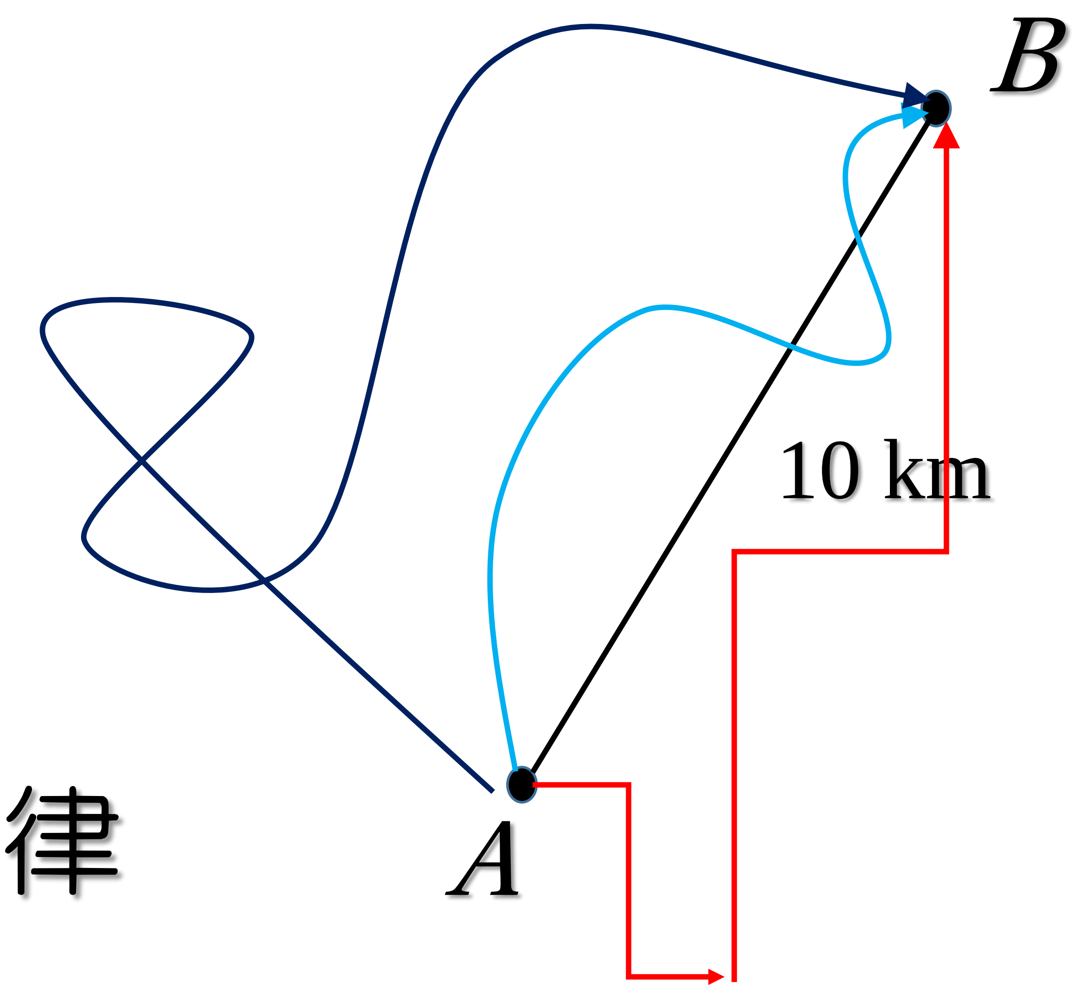


向量的概念

從A到B的不同路程，但都是由A移動到B



向量的概念

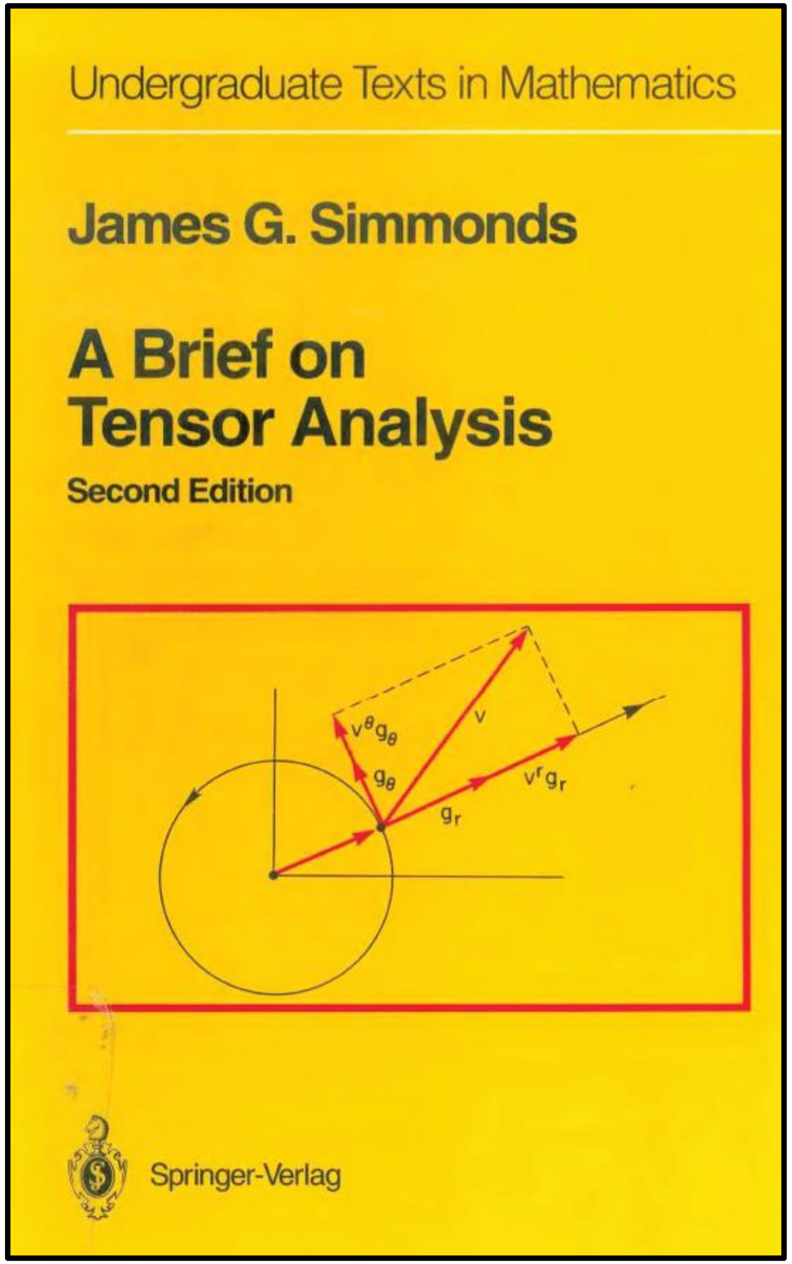


滿足

平行四邊形加法定律

向量的概念

向量是
有向線段的等價類



向量的概念

向量是
有向線段的等價類

Directed Line Segments

Directed line segments, or *arrows*, are of fundamental importance in Euclidean geometry. Logically, an arrow is an ordered pair of points, (A, B). A is called the *tail* of the arrow and B the *head*. It is customary to represent such an arrow typographically as $\overrightarrow{AB}$, and pictorially as a line segment from A to B with an arrow head at B. (To avoid crowding, the arrow head may be moved towards the center of the segment). Assigning a length to an arrow or multiplying it by a real number (holding the tail fixed) are precisely defined operations in E_3.

Two arrows are said to be *equivalent* if one can be brought into coincidence with the other by a parallel translation.[7] In Fig. 1.1, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent, but neither $\overrightarrow{AB}$ and $\overrightarrow{EF}$ nor $\overrightarrow{AB}$ and $\overrightarrow{GH}$ are.

The set of *all* arrows equivalent to a given arrow is called the (geometric) *vector* of that arrow and is usually denoted by a symbol such as **v**. A vector is an example of an *equivalence class* and, by convention, a vector is represented by any one of its arrows.

Equivalence classes are more familiar (and more useful) than you may realize. Suppose that we wish to carry out, on a computer, exact arithmetic

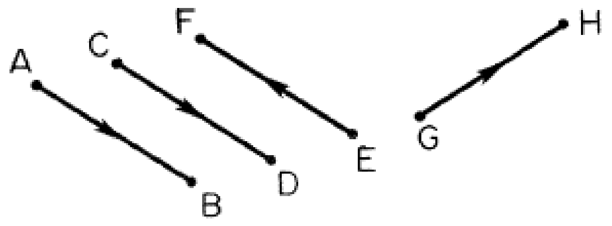


Figure 1.1.

向量的概念

向量空間的公設

6.2 Axioms

The precise definition of a vector space is given by listing a set of axioms. For this purpose, I'll denote vectors by arrows over a letter, and I'll denote scalars by Greek letters. These scalars will, for our purpose, be either real or complex numbers — it makes no difference which for now.*

1 There is a function, addition of vectors, denoted $+$, so that $\vec{v}_1 + \vec{v}_2$ is another vector.

2 There is a function, multiplication by scalars, denoted by juxtaposition, so that $\alpha\vec{v}$ is a vector.

3 $(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$ (the associative law).

4 There is a zero vector, so that for each $\vec{v}$, $\vec{v} + \vec{O} = \vec{v}$.

5 There is an additive inverse for each vector, so that for each $\vec{v}$, there is another vector $\vec{v}\,'$ so that $\vec{v} + \vec{v}\,' = \vec{O}$.

6 The commutative law of addition holds: $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$.

7 $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$.

8 $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$.

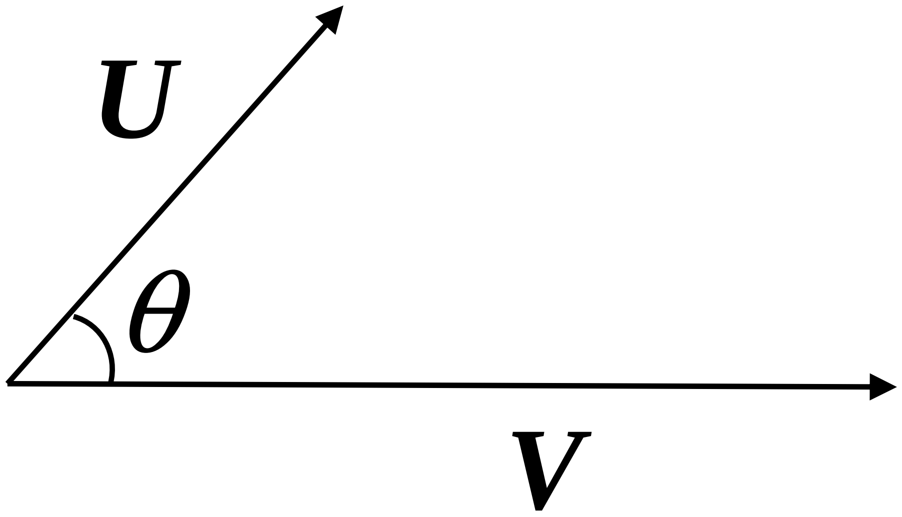
9 $\alpha(\vec{v}_1 + \vec{v}_2) = \alpha\vec{v}_1 + \alpha\vec{v}_2$.

10 $1\vec{v} = \vec{v}$.

向量的純量積的概念

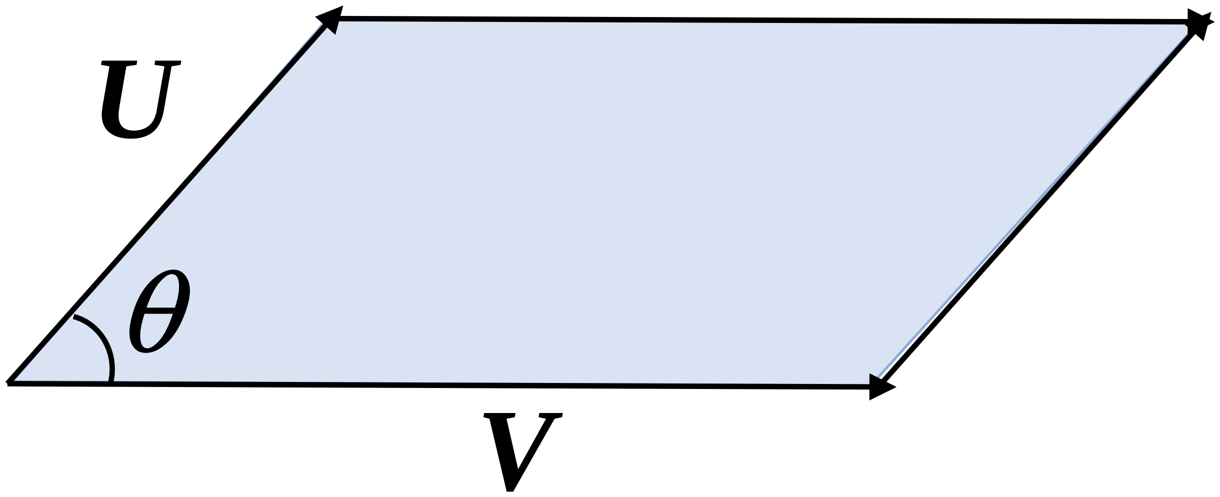
向量的純量積的概念

$$\boldsymbol{U} \odot \mathbf{V} = |\boldsymbol{U}||\mathbf{V}| \sin\theta$$



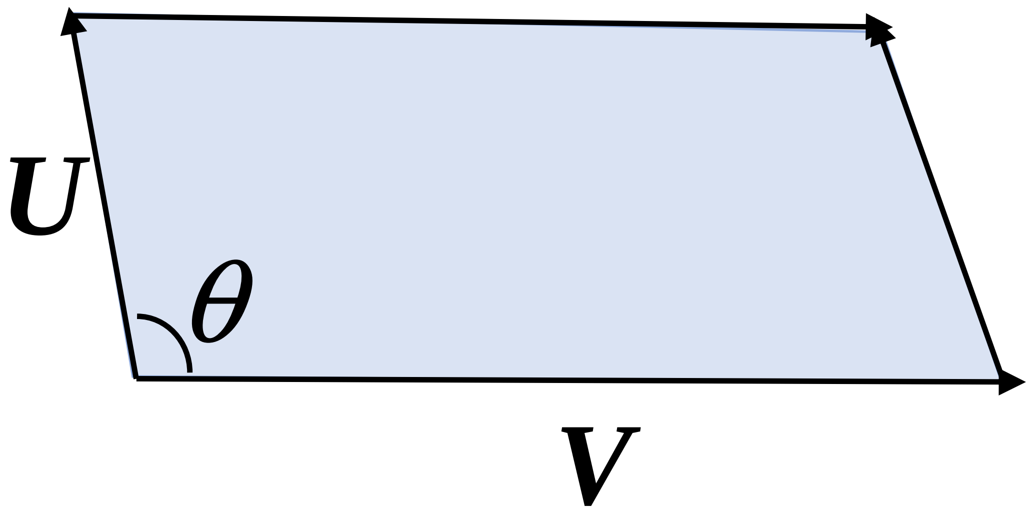
向量的純量積的概念

$$\boldsymbol{U} \odot \mathbf{V} = |\boldsymbol{U}||\mathbf{V}| \sin\theta$$



向量的純量積的概念

$$\boldsymbol{U} \odot \mathbf{V} = |\boldsymbol{U}||\mathbf{V}| \sin\theta$$

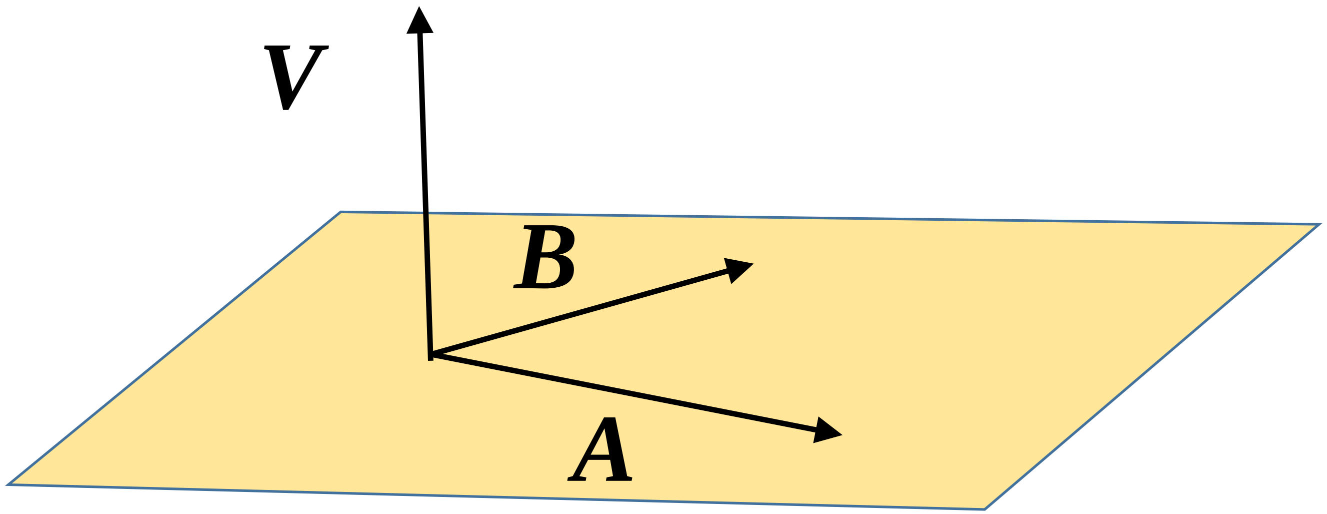


向量的純量積的概念

$$\boldsymbol{V} \odot (\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{V} \odot \boldsymbol{A} + \boldsymbol{V} \odot \boldsymbol{B} \text{ ?}$$

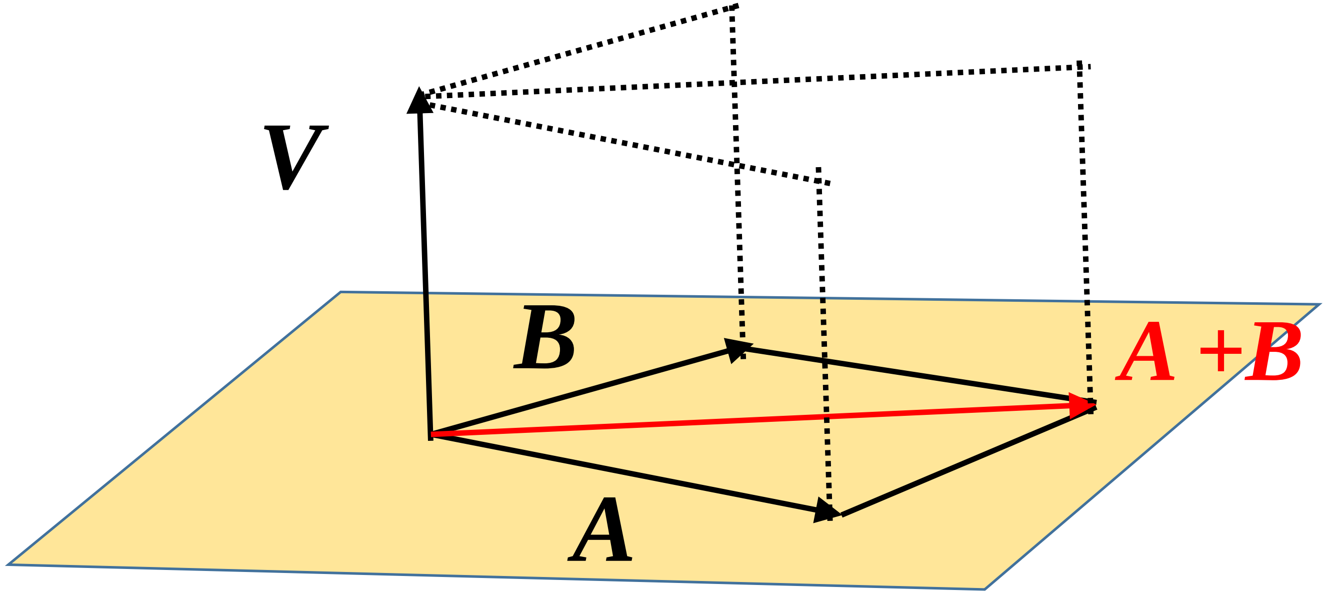
向量的純量積的概念

$$V \bullet (A + B) = V \bullet A + V \bullet B\,?$$



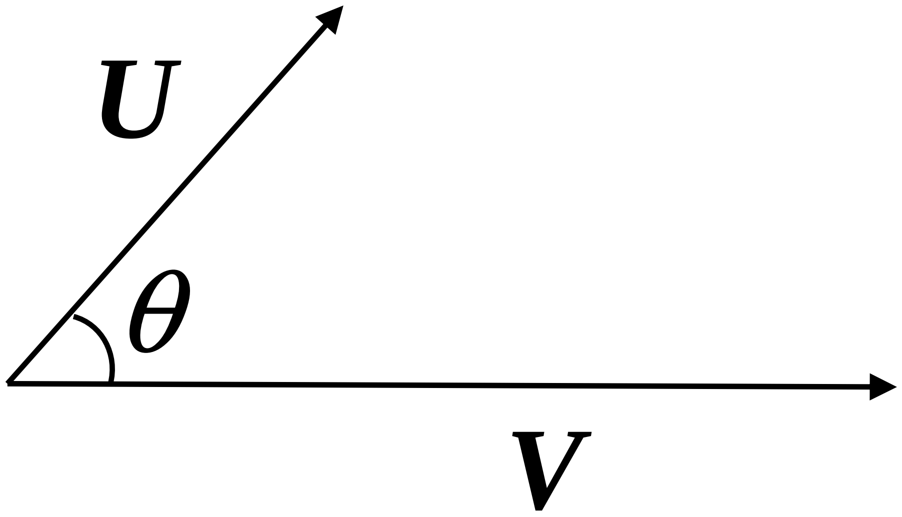
向量的純量積的概念

$$V \odot (A + B) = V \odot A + V \odot B\ ?$$



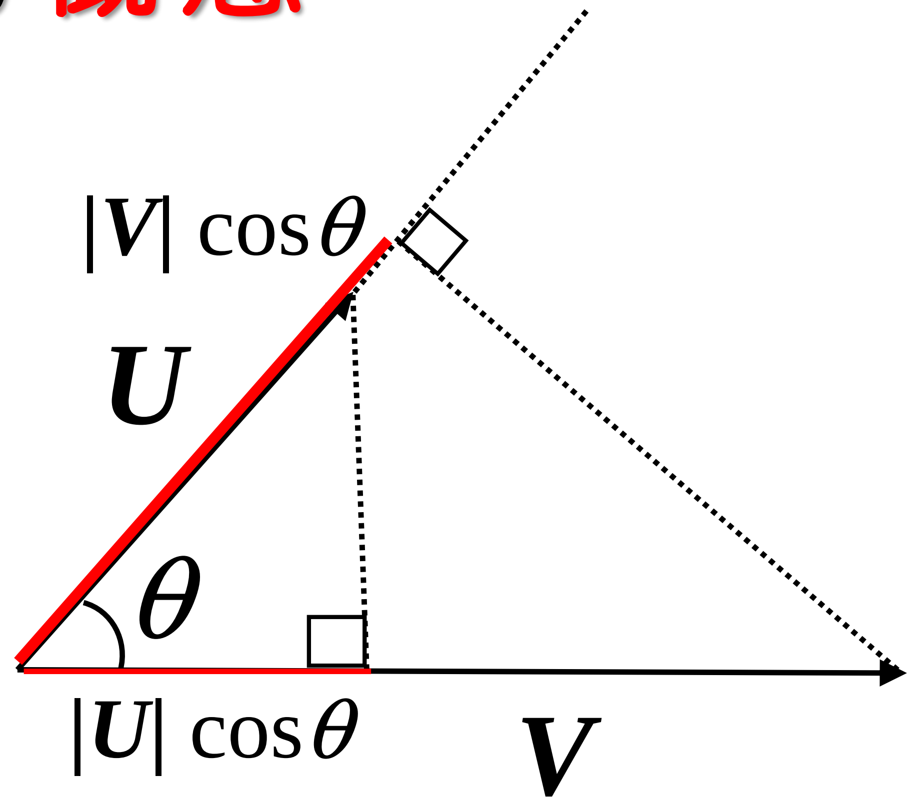
向量的純量積的概念

$$\boldsymbol{U} \bullet \boldsymbol{V} = |\boldsymbol{U}||\boldsymbol{V}| \cos\theta$$



向量的純量積的概念

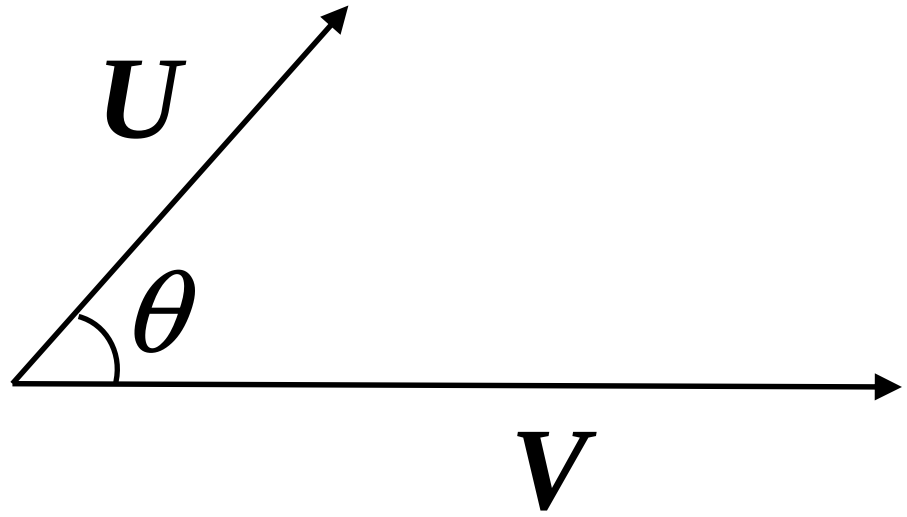
$$\boldsymbol{U} \bullet \boldsymbol{V} = |\boldsymbol{U}||\boldsymbol{V}| \cos\theta$$



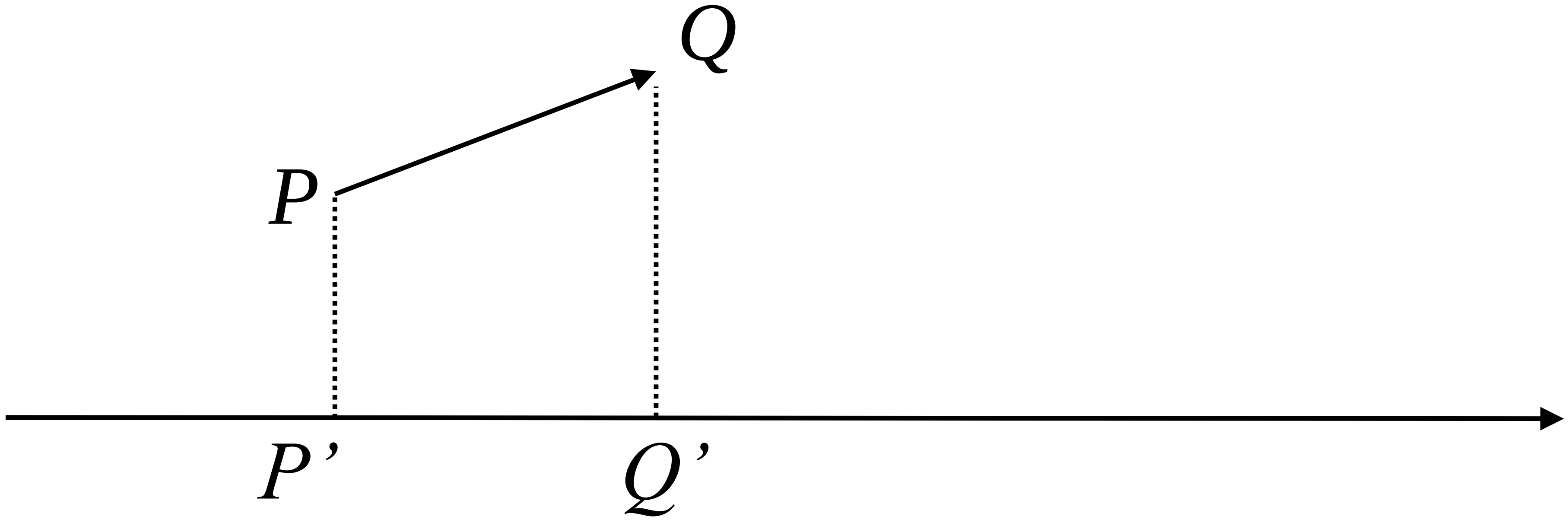
向量的純量積的概念

純量積是純量？

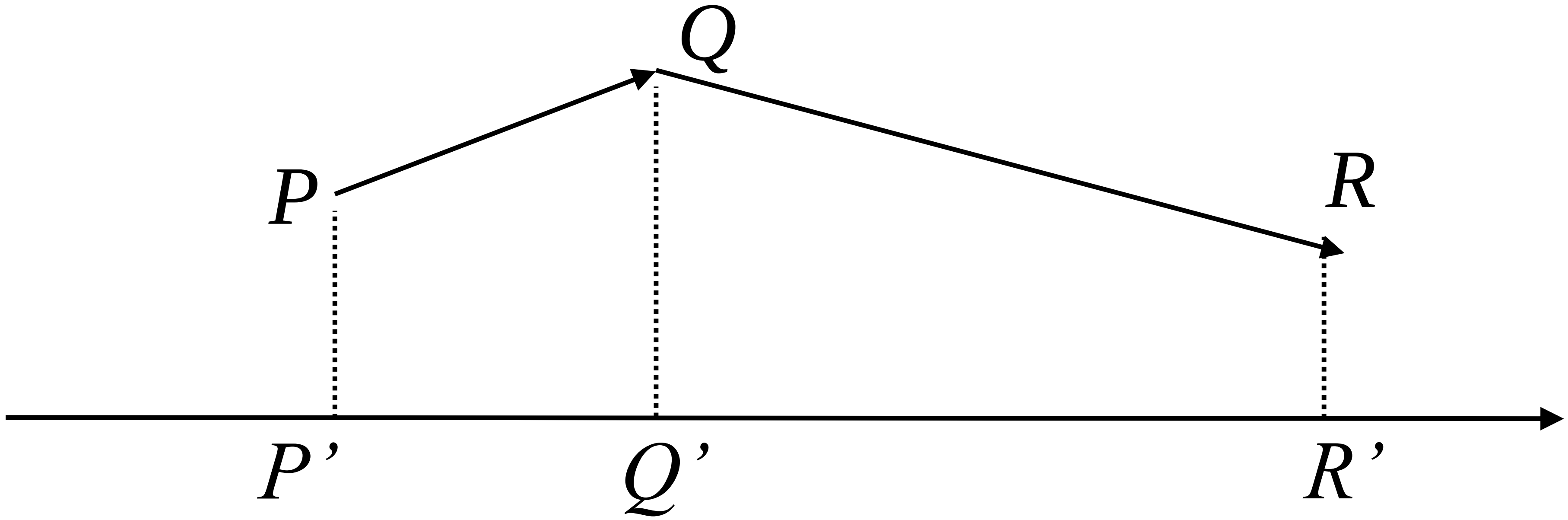
$$\boldsymbol{U} \bullet \boldsymbol{V} = |\boldsymbol{U}||\boldsymbol{V}| \cos\theta$$



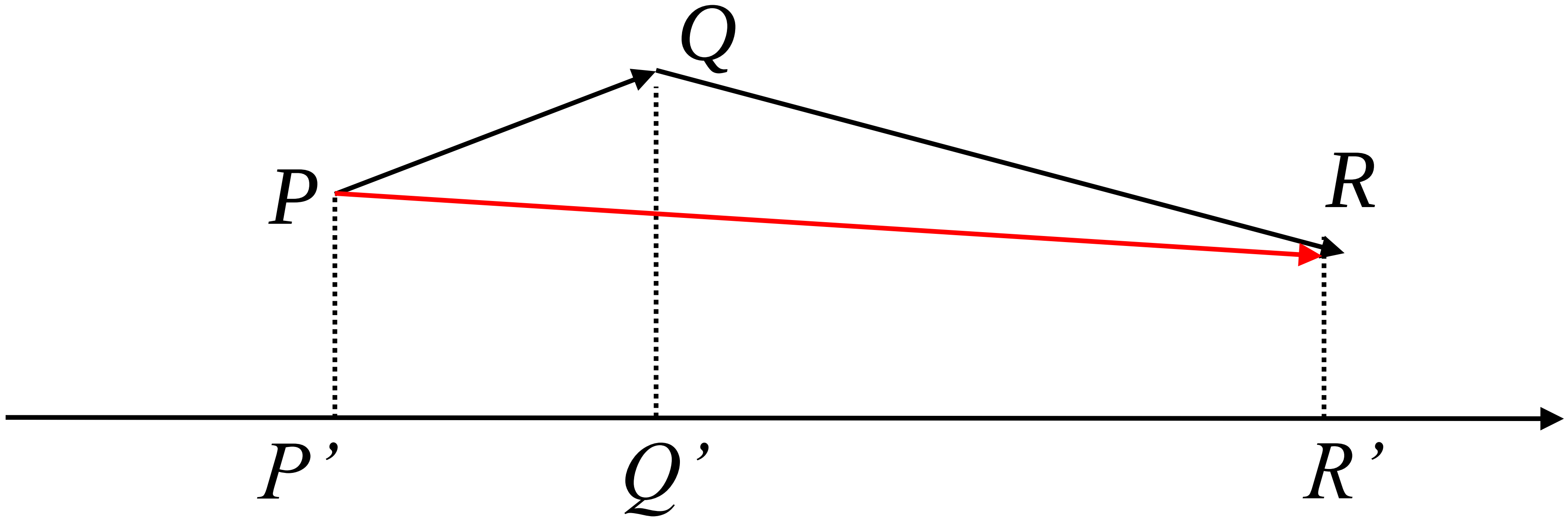
向量的純量積的概念



向量的純量積的概念



向量的純量積的概念

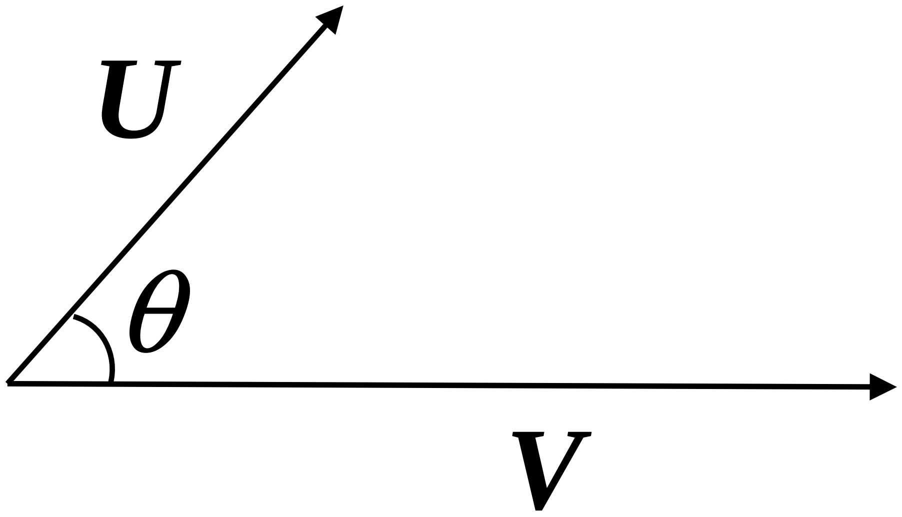


向量的純量積的概念

$$\boldsymbol{V} \bullet (\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{V} \bullet \boldsymbol{A} + \boldsymbol{V} \bullet \boldsymbol{B}\ ?$$

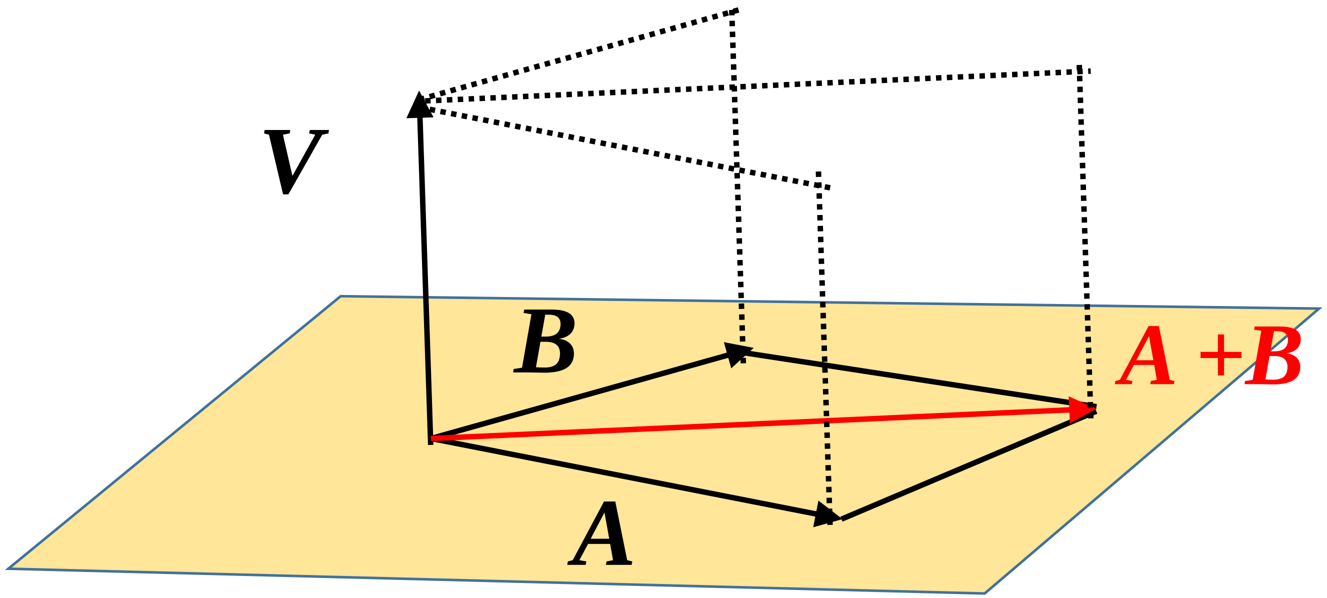
向量的向量積的概念

$$\boldsymbol{U} \times \mathbf{V} = |\boldsymbol{U}||\mathbf{V}| \sin\theta$$



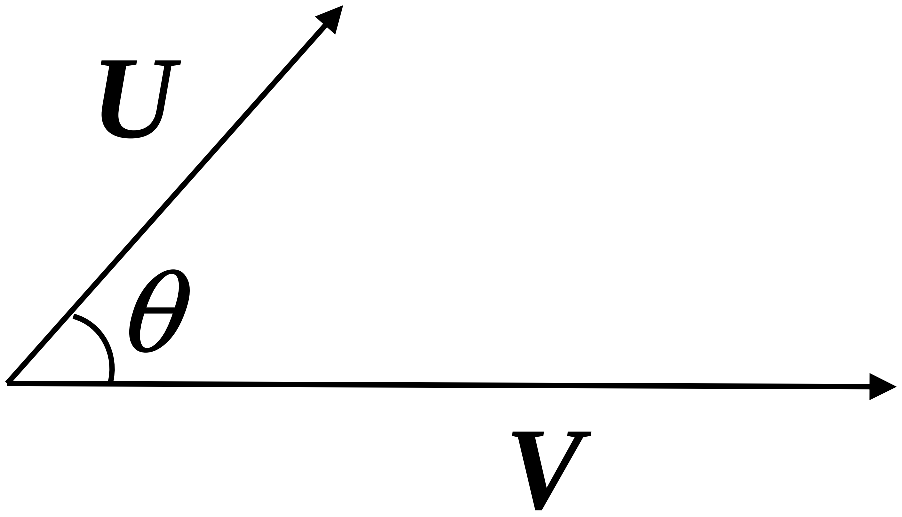
向量的純量積的概念

$$\boldsymbol{V} \times (\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{V} \times \boldsymbol{A} + \boldsymbol{V} \times \boldsymbol{B}$$ ✘



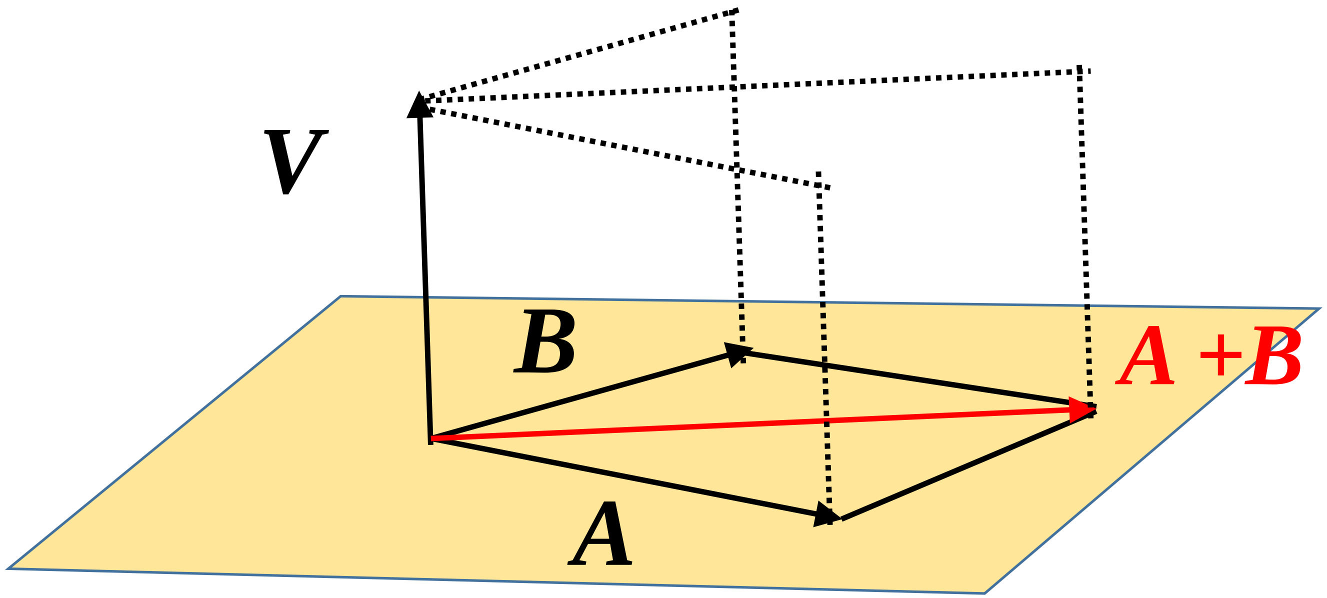
向量的向量積的概念

$$\boldsymbol{U} \otimes \mathbf{V} = |\boldsymbol{U}||\mathbf{V}| \sin\theta \ \boldsymbol{v}$$



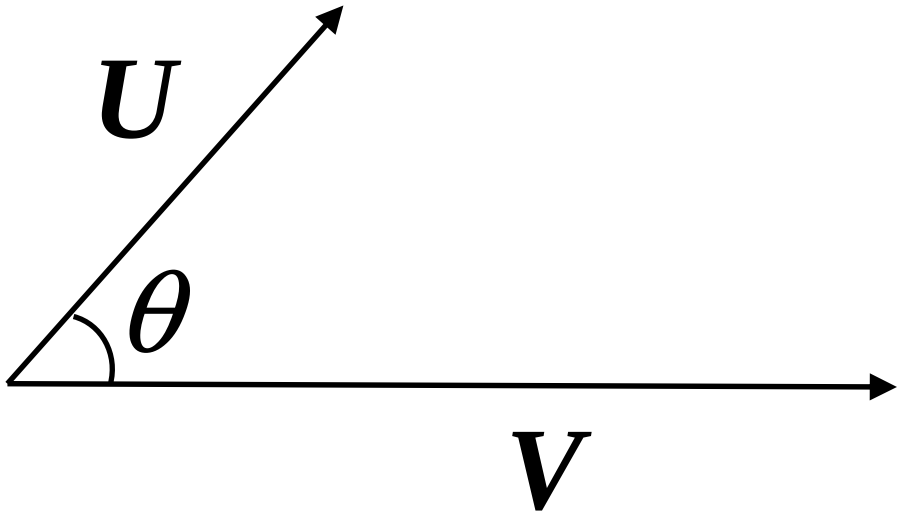
向量的純量積的概念

$$\boldsymbol{V} \otimes (\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{V} \otimes \boldsymbol{A} + \boldsymbol{V} \otimes \boldsymbol{B}$$



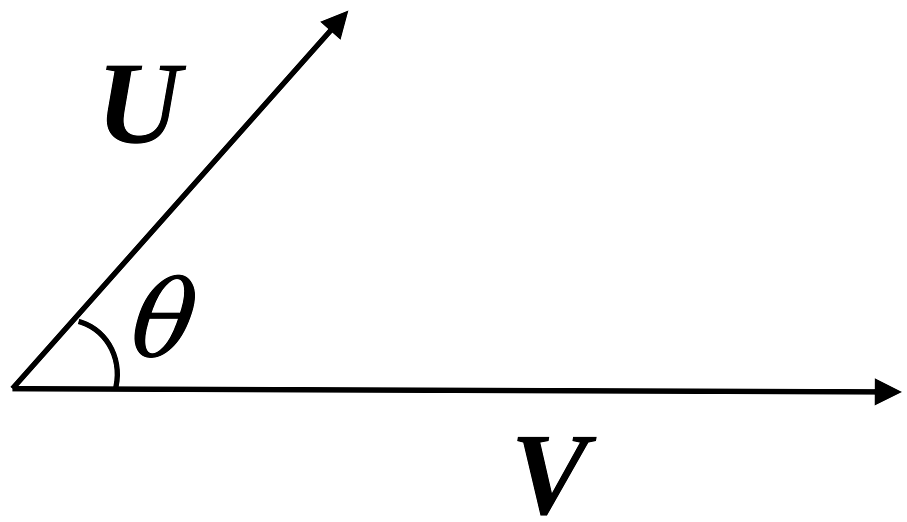
向量的向量積的概念

$$\boldsymbol{U} \otimes \mathbf{V} \neq \mathbf{V} \otimes \boldsymbol{U}$$



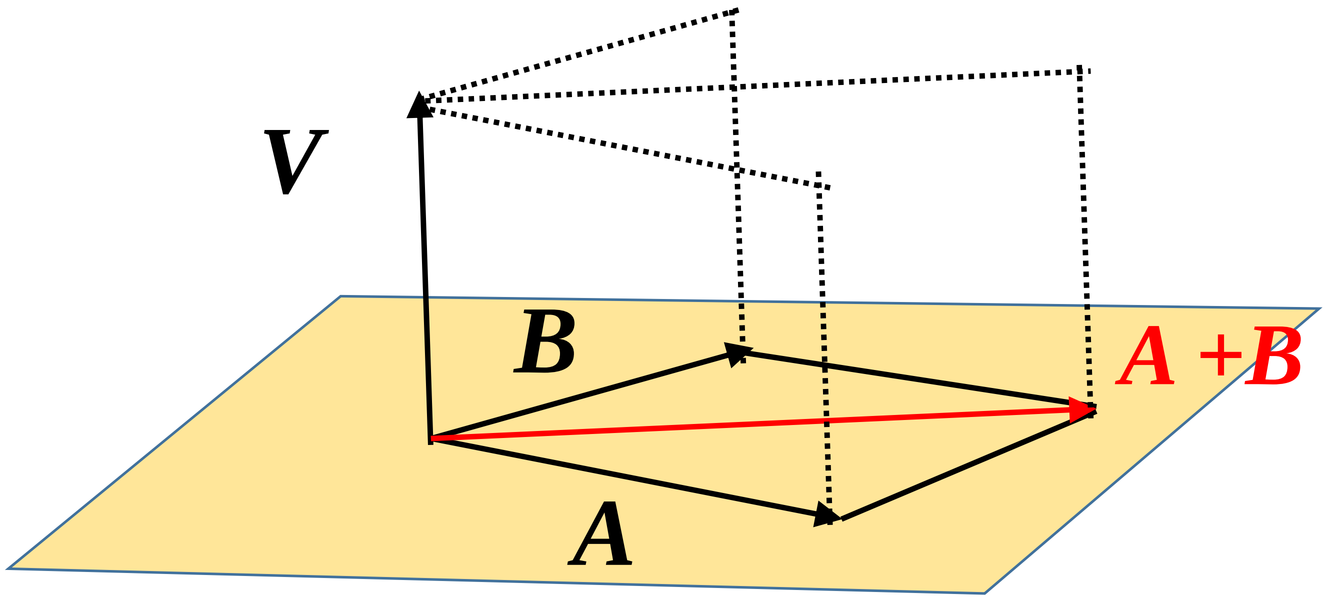
向量的向量積的概念

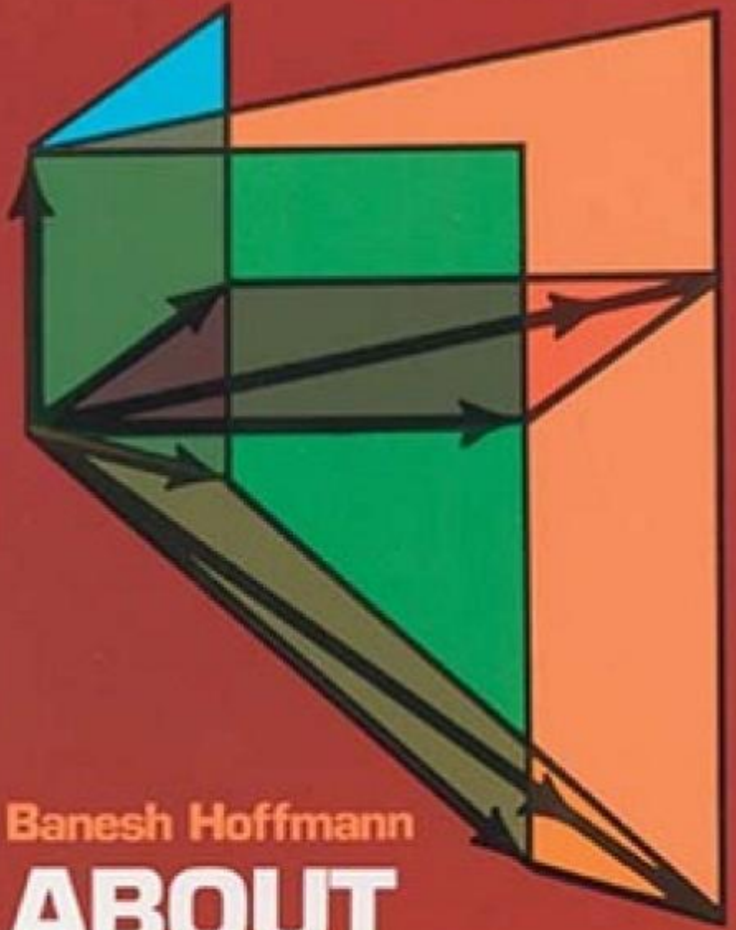
$$\boldsymbol{U} \times \mathbf{V} = |\boldsymbol{U}||\mathbf{V}| \sin\theta \ \boldsymbol{n}$$



向量的純量積的概念

$$\mathbf{V} \times (\mathbf{A} + \mathbf{B}) = \mathbf{V} \times \mathbf{A} + \mathbf{V} \times \mathbf{B}$$





Banesh Hoffmann

ABOUT VECTORS

- 比、率
- 弧度法
- 切線
- 方程
- 概率
- 函數
- 概率分佈
- 置信區間
- .....