Solutions to Challenge Corner of 4th issue of IMOment

1. On the social networking website Mathbook, friendship is mutual (i.e. if A is a friend of B, then B is also a friend of A). Is it possible to have 5 users such that no 3 of them are ***all*** friends with each other (i.e. each of these 3 users is friends with the other two) and no 3 of them are ***all*** strangers with each other (no two of these 3 users are friends)?

   Solution: It is possible. For example, let the 5 users be A, B, C, D and E and the only pairs of friends be (A,B), (B,C), (C,D), (D,E) and (E,A).

2. If x is a positive real number, find the minimum value of $x+\frac{1}{x}$.

   Solution: Note that

$$x+\frac{1}{x}=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2+2\geq 2$$

   and equality is attained when $x=1$. So the answer is 2.

3. Let $ABCD$ be a parallelogram and P, Q, R, and S be points on the sides AB, BC, CD, DA respectively such that $PR//BC$ and $QS//AB$. If no two of the lines AC, PQ and RS are parallel, prove that these three lines meet at one point.

   Proof: Without loss of generality assume that the intersection of PQ and RS is closer to P than to Q (and, hence, closer to S than to R). Let T be the intersection of PQ and AC and U be the intersection of RS and AC. Applying Menelaus's theorem to triangle ABC and line TPQ, we have $1=\frac{AT}{TC}\cdot\frac{CQ}{QB}\cdot\frac{BP}{PA}=\frac{AT}{TC}\cdot\frac{SD}{AS}\cdot\frac{PB}{AP}$. Similarly $1=\frac{AU}{UC}\cdot\frac{SD}{AS}\cdot\frac{PB}{AP}$. Hence we see that T and U are the same point.

4. Evaluate the following expression:

$$\frac{1}{1\times 2\times 4}+\frac{1}{2\times 3\times 5}+...+\frac{1}{97\times 98\times 100}$$

Solution:
Since

$$\frac{1}{n(n+1)(n+3)}=\frac{1}{3}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+3)}\right)$$

$$=\frac{1}{3}\left[\frac{1}{n}-\frac{1}{(n+1)}-\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)\right]$$

$$=\frac{1}{3}\cdot\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n+1}+\frac{1}{6}\cdot\frac{1}{n+3}$$

we have

$$\frac{1}{1\times 2\times 4}+\frac{1}{2\times 3\times 5}+...+\frac{1}{97\times 98\times 100}$$

$$=\frac{1}{3}\cdot\frac{1}{1}-\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{6}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{5}+...+\frac{1}{3}\cdot\frac{1}{97}-\frac{1}{2}\cdot\frac{1}{98}+\frac{1}{6}\cdot\frac{1}{100}$$

$$=\frac{1}{3}\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{97}\right)-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}+...+\frac{1}{98}\right)+\frac{1}{6}\left(\frac{1}{4}+\frac{1}{5}+...+\frac{1}{100}\right)$$

$$=\frac{1}{3}\cdot\frac{1}{1}+\left(\frac{1}{3}-\frac{1}{2}\right)\cdot\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{6}\right)\left(\frac{1}{4}+\frac{1}{5}+...+\frac{1}{97}\right)+\left(-\frac{1}{2}+\frac{1}{6}\right)\cdot\frac{1}{98}+\frac{1}{6}\cdot\left(\frac{1}{99}+\frac{1}{100}\right)$$

$$=\frac{1}{3}\cdot\frac{1}{1}+\left(-\frac{1}{6}\right)\cdot\left(\frac{1}{2}+\frac{1}{3}\right)+0\cdot\left(\frac{1}{4}+\frac{1}{5}+...+\frac{1}{97}\right)+\left(-\frac{1}{3}\right)\cdot\frac{1}{98}+\frac{1}{6}\cdot\left(\frac{1}{99}+\frac{1}{100}\right)$$

$$=\frac{565801}{2910600}$$

List of awardees:

Name of student	Name of school
CHEUNG Kai Hei Trevor	St Paul's Co-educational College
CHENG Yan Yau	Discovery College
FONG Tsz Lo	SKH Lam Woo Memorial Secondary School

Awardees will also be notified with a separate e-mail.