Solutions to Challenge Corner, 1st Issue of IMOment

1. In this problem, a positive integer is said to be *strange* if it has an odd number of distinct positive divisors. For example, 4 is strange because it has 3 distinct positive divisors, namely 1, 2 and 4, while 10 is not strange because it has 4 distinct positive divisors, namely 1, 2, 5 and 10. How many strange numbers are there among 1, 2, …, 2016?

   Solution: If a is a positive divisor of a positive integer n, then so as $b=\frac{n}{a}$, and $\frac{n}{b}=a$. Thus the positive divisors of n can be paired up, so n always has an even number of positive divisors, unless for some positive divisor a, it turns out that $\frac{n}{a}=a$, i.e. $n=a^2$ is a perfect square. Among 1, 2, …, 2016, there are 44 perfect squares, and hence there are 44 strange numbers.

2. Given that the expression below represents a real number, find it.

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$$

   Solution: Let $x=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$. Then $x=\sqrt{2+x}$, so $x^2-x-2=0$. Solving, $x=2$ or $x=-1$ (rejected).

3. Assume that on a certain social networking website, there is no limit a user's number of friends, and friendship is mutual (i.e. if A is a friend of B, then B is also a friend of A). Prove that there are two Mathbook users with the same number of Mathbook friends.

Solution: Suppose there are a total of n Mathbook users. Then each user's number of friends is between 0 and $n-1$ (inclusive). If no two users have the same number of friends, the number of friends of the n users must be 0, 1, …, $n-1$. Now, the user with 0 friends does not have any friends while the user with $n-1$ friends is friends with every other user, a contradiction.

4. Let P be a point in the plane of ΔABC. Let X be a point in the plane such that the midpoint of PX is the midpoint of BC. Let Y be a point in the plane such that the midpoint of PY is the midpoint of CA. Let Z be a point in the plane such that the midpoint of PZ is the midpoint of AB. Show that AX, BY and CZ meet at a point. (Hint: Try the method introduced in the article on coordinate geometry in this issue!)

Solution: We use coordinate geometry. Let $A=(a_1,a_2)$, $B=(b_1,b_2)$, $C=(c_1,c_2)$ and $P=(p_1,p_2)$ 。Then $Z=\left(\frac{a_1+b_1-p_1}{2},\frac{a_2+b_2-p_2}{2}\right)$, $X=\left(\frac{b_1+c_1-p_1}{2},\frac{b_2+c_2-p_2}{2}\right)$ and $Y=\left(\frac{c_1+a_1-p_1}{2},\frac{c_2+a_2-p_2}{2}\right)$ (prove it!).

Hence the midpoint of CZ is $\left(\frac{a_1+b_1+c_1-p_1}{2},\frac{a_2+b_2+c_2-p_2}{2}\right)$.

Similarly, the midpoint of AX and the midpoint of BY are also this point.

List of awardees:

Name of student	Name of school
Cheung Kai Hei Trevor	St. Paul’s Co-educational College
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Lai Chun Ho	Po Leung Kuk Mrs. Ma Kam Ming Cheung Fook Sien College
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