Solutions to Challenge Corner of 8th issue of IMOment

1. Let *ABCD* be a convex quadrilateral that is not a parallelogram. Let *E* be the midpoint of *BD* and *F* be the midpoint of *AC*. Prove that for any point *L* on the line segment *EF*, the sum of the areas of triangles *ALB* and *CLD* is half the area of the quadrilateral *ABCD*.

   Solution: Let [*XYZ*] denote the area of triangle *XYZ*. Noting that $[ALF]=[CLF]$ and $[BLF]=[DLF]$ (why?), we have

   $$\begin{aligned}[ALB]+[CLD] &= [AFB]-[ALF]-[BLF]+[CFD]+[CLF]+[DLF] \\ &= [AFB]+[CFD].\end{aligned}$$

   It remains to prove that $[AFB]+[CFD]$ is half the area of *ABCD*. Indeed, since *F* is the midpoint of *AC*, we have $[AFB]=[CFB]$ and $[CFD]=[AFD]$.

2. Prove that for any integers *a*, *b*, *c* and *d*, the following expression is divisible by 12:

   $$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

   Solution: Since two of *a*, *b*, *c*, *d* must have the same remainder when divided by 3, it follows that at least one of the multiplicands in the expression is divisible by 3. It is easy to check that there must also be at least two multiplicands that are divisible by 2.

3. There are 1000 students in a school and they are numbered 1 to 1000. Each student must choose to study one of French or German, with the additional restriction that any group of 20 students whose numbers form an arithmetic sequence cannot all choose the same language. Is it possible to satisfy the requirement? (You may refer to the article *Probability and the IMO* in this issue.)

   Solution: Suppose every student chooses randomly between the two languages, and consider the events of the form "the students whose numbers are in S all choose the same language," where S is any set of 20 integers from 1 to 1000 that form an arithmetic sequence. Since each arithmetic sequence is determined by its first two terms, there are at most $k = \frac{1000 \times 999}{2} = 499500$ such events. Each event happens with probability $p = \frac{2}{2^{20}} = \frac{1}{2^{19}}$. Thus, the probability that at least one of them happens is no greater than $kp = 0.95... < 1$. In other words, there is a positive probability that none of these events happens, meaning that it is possible to satisfy the requirement stated in the question.

   Alternatively, the proof can also be done with the Lovász local lemma. Again, consider the events of the form described above. Note that each event is correlated with at most $d = 20 \times 20 \times 52 = 20800$ others, because:

   - each event involves 20 students;
   - each student's number is in at most 20×52 arithmetic sequences of length 20 since it can be in one of 20 places (1st, 2nd, …, 20th) of the sequence and there are 52 possible common differences for the sequence (namely, 1, 2, …, 52).

   Checking that $epd = 0.10... < 1$, we are done by the Lovász local lemma. It should be remarked that this method is stronger in the sense that it will work even if the number of students is changed from 1000 to some larger numbers.

4. Refer to the article *The Art of the Impossible* in this issue. Assume there is an odd number of voters. The Condorcet method for single-winner elections is defined as follows: Each voter votes by reporting a ranking of all the choices. If there exists a choice P such that for any other choice Q, more than half of the voters rank P higher than Q, then P is the winner. Prove that if a winner exists in the case of all voters voting honestly, then no single voter will have a unilateral incentive to vote dishonestly.

   Solution: Consider any voter V. If the winner P in the case of all voters voting honestly is V's favorite choice, then V has no incentive to change anything. On the other hand, if P is not V's favorite choice, can V make a more preferred choice R win by voting dishonestly? In particular, V must make sure that more than half of the voters rank R above P. But V is already ranking R above P in the honest case, and yet it still turns out that more than half of the voters rank P above R. V cannot help this.

List of awardees:

Name of Student	Name of School
Jeffrey TSE	Pui Ching Middle School
Jasmine CHEUNG	Heep Yunn Primary School / Heep Yunn School
FAN Man-hon	HKUGA College

Awardees will also be notified with a separate e-mail.