The Shortest path and reflection

Key stage: 3

Strand: Measures, Shape and Space

Learning Unit: Congruent triangles

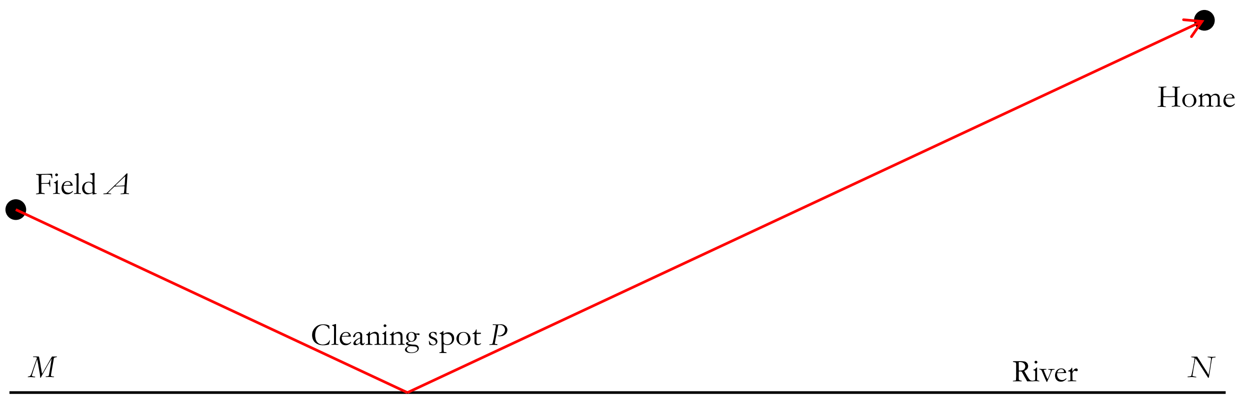
Objective: Use mathematical knowledge to solve the shortest path problem and appreciate the relationship between this problem and phenomena in optics.

Pre-requisite Knowledge: Knowledge of geometric construction and proofs.

Relationship with other KLA(s) in STEM Education:
"Light, Colours and Beyond" in *Supplement to the Science Education Key Learning Area Curriculum Guide: Science (Secondary 1 – 3)* (The Curriculum Development Council, 2017) of Science Education KLA.

Resources Required: GeoGebra files (ShortestDistance1_en.ggb, ShortestDistance2_en.ggb, ShortestDistance3_en.ggb, ShortestDistance4_en.ggb)

Background information:
According to legend, a farmer in ancient Greece asked the mathematician Hero of Alexandria (about 10 – 70 AD) [1]:
"I have to go to the riverbank to wash the farm tools every day after farming in the field, and then go home. How should I choose the best cleaning spot *P* from the straight riverbank *MN*, so that the distance I go from the field *A* to *P*, then from *P* to home *B*, is the shortest? "



[1] 傅海倫（2004）:〈物理原理在數學中的應用〉。《數學傳播》28 卷 1 期，頁 63-69。 Retrieved from http://web.math.sinica.edu.tw/math_media/d281/28107.pdf

Description of the activities:

<u>Activity 1: The Shortest path</u>

1. The teacher first introduces the shortest distance problem.

2. The teacher guides students to use the provided GeoGebra files to explore the best location of the cleaning spot P and complete the worksheet 1.

<u>Activity 2a: Draw the shortest path</u>

1. Students use the knowledge of plane geometry on Worksheet 2 to answer the following questions:
   (a) Let B' be a point in the figure such that $BB' \perp MN$ and $BC = B'C$, where C is the intersection point of BB' and MN. For any point P' on MN, prove that $\Delta BP'C \cong \Delta B'P'C$ if P' does not coincide with C.
   (b) In the GeoGebra exploratory activity, point out the best cleaning spot P and its geometric properties. Thus, prove that for any point P', $AP' + P'B \geq AP + PB$.
   (c) For any points A and B, find the best location P and draw the shortest path by using compasses and a straightedge.
   (d) Prove that $\angle APM = \angle BPN$.

2. The teacher may guide students to compare the similarities between the aforementioned path and the reflection of light on a plane mirror.

<u>Activity 2b: Extended activity</u>

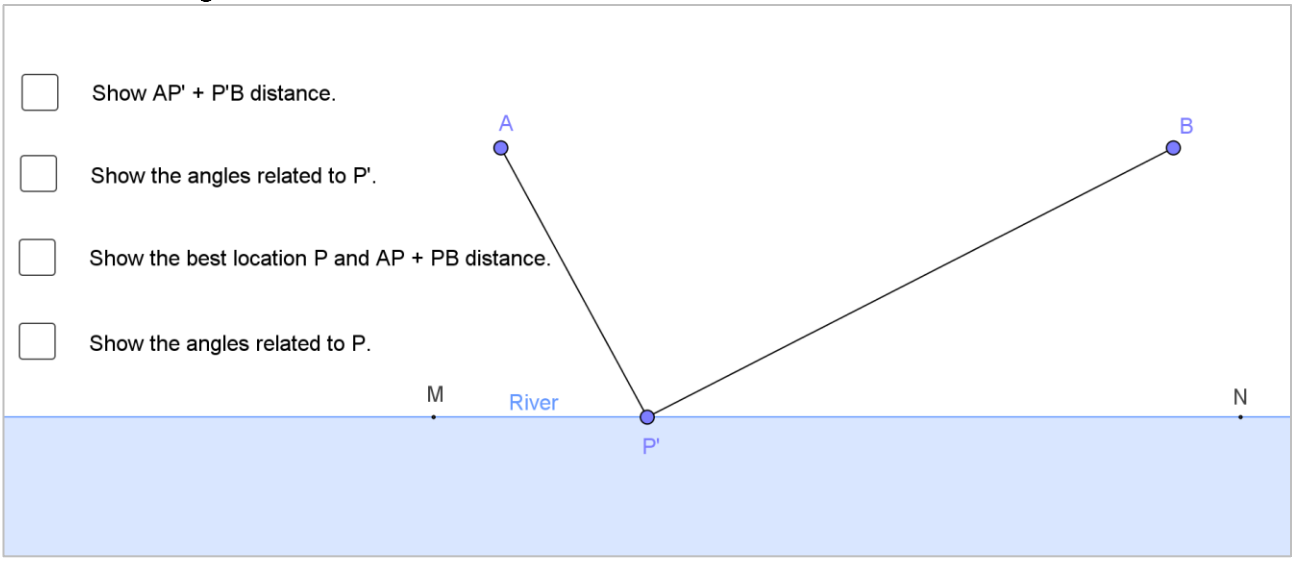
1. Teacher may guide the students to use the optical reflection principle and knowledge in geometry to design a puzzle game by using reflection in worksheet 2.

2. Teacher may collaborate with Science teachers to introduce to students the achievements of Prof. Sir Charles Kuen KAO in his researches in optical fibre and its real-life applicationss.

Explore the shortest path

Using the GeoGebra file provided, move P' to explore the best location P such that distance $AP + PB$ is minimised.

(1) (a) From the positions of A and B below, guess the best location P and colour the location in the figure.



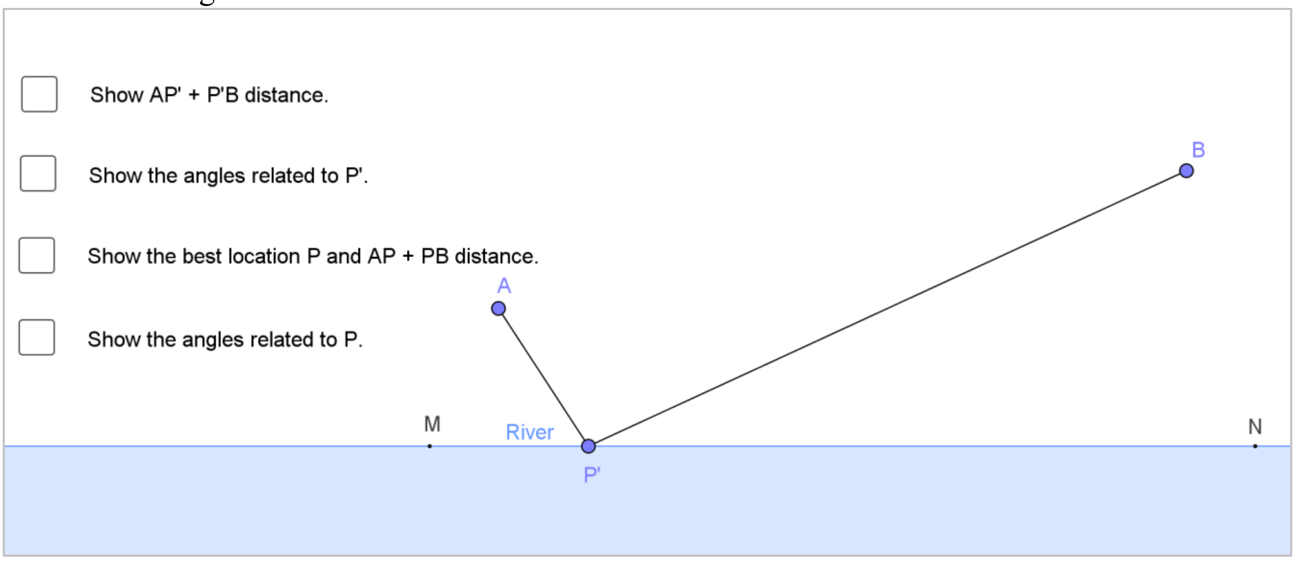
(b) Open the file "ShortestDistance1_en.ggb" and click on "Show $AP' + P'B$ distance.". Find the best location P in the figure by moving point P'.

(c) Click on "Show the angles related to P'." to observe and write down the measures of the following angles:

$\angle AP'M =$ __________, $\angle BP'N =$ __________.

(d) Is $\angle APM$ equal to $\angle BPN$ at point P? Answer: __________
Click "Show the best location P and $AP + PB$ distance." and "Show the angles related to P." for verification.

(2) (a) From the positions of A and B below, guess the best location P and colour the location in the figure.



(b) Open the file “ShortestDistance2_en.ggb” and click on "Show $AP' + P'B$ distance.". Find the best location P in the figure by moving point P'.

(c) If the distances of A and B from the river are reversed (as shown below), describe the change in the location of point P.

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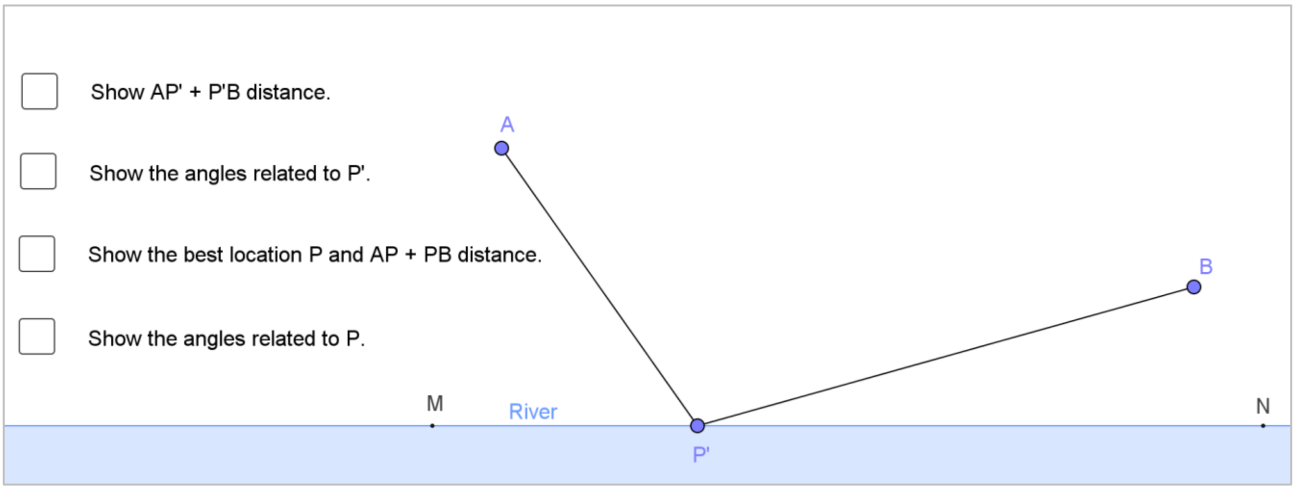

(d) Click on "Show the angles related to P'." to observe and write down the measures of the following angles:

$\angle AP'M =$ __________, $\angle BP'N =$ __________.

(e) Is $\angle APM$ equal to $\angle BPN$ at point P? Answer: __________
Click "Show the best location P and $AP + PB$ distance." and "Show the angles related to P." for verification.

(f) In the first two exploratory activities, is the best location from the GeoGebra file close to your estimate? (If necessary, open the file “ShortestDistance3_en.ggb” to verify your estimate.) Try to explain your estimate and reflect on whether your estimate is reasonable.

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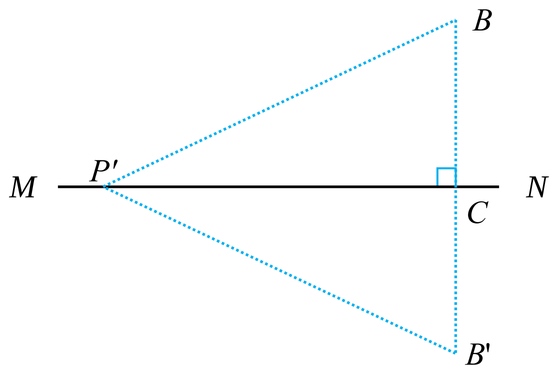
(3) (a) From the above scenarios, what geometric characteristics of point P did you observe?

(b) Open the file “ShortestDistance4_en.ggb”, move points A and B to different positions and then find the best location P to verify your observations.

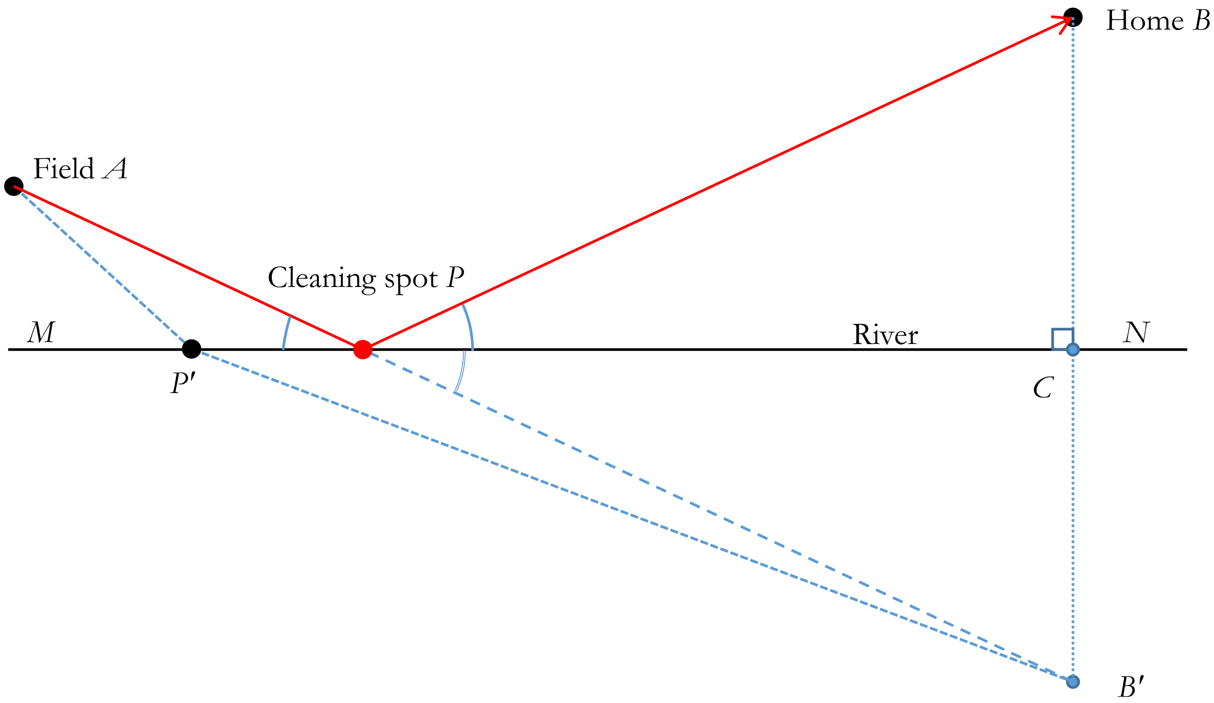
Using straightedge and compass construction to find the shortest path

(1) Using the properties of congruent triangles, prove that if point P is the best location for the above problem, the angle formed by the path AP and the river is equal to the angle formed by the path BP and the river.

(a) Let B' be a point in the figure such that $BB' \perp MN$ and $BC = B'C$, where C is the intersection of BB' and MN. For any point P' on MN, if P' does not coincide with C, prove that $\Delta BP'C \cong \Delta B'P'C$.



(b) Open the file "ShortestDistance4_en.ggb" and click on "Show the best location P and $AP + PB$ distance." and "Show B' and PB'.". Try to point out the condition that the point P is the best location, and prove that $AP' + P'B \geq AP + PB$ regardless of where P' is located.



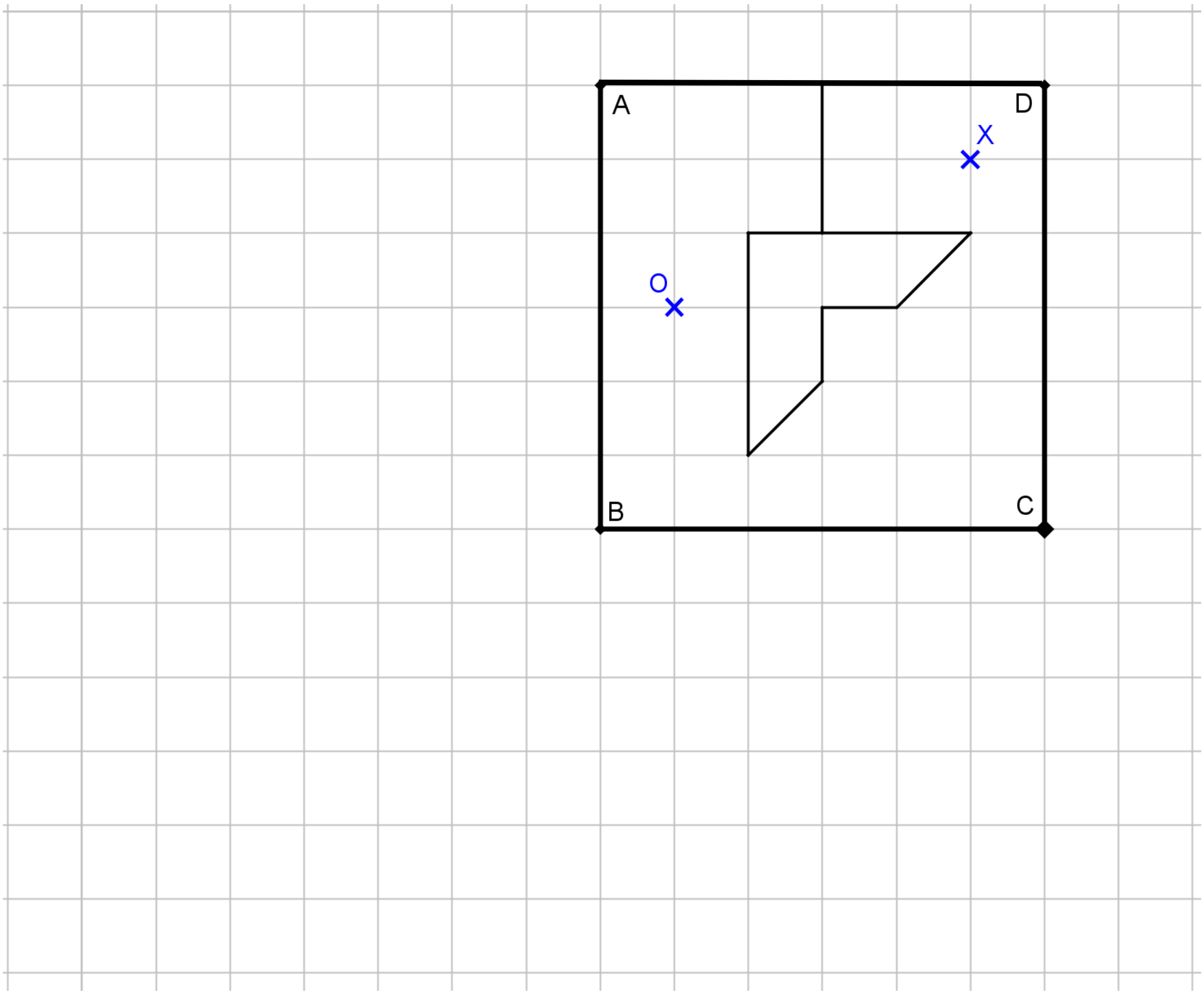
The condition for the best location P: ______________________

(c) For any point A and point B, find the best location P and draw the shortest path by straightedge and compass construction.

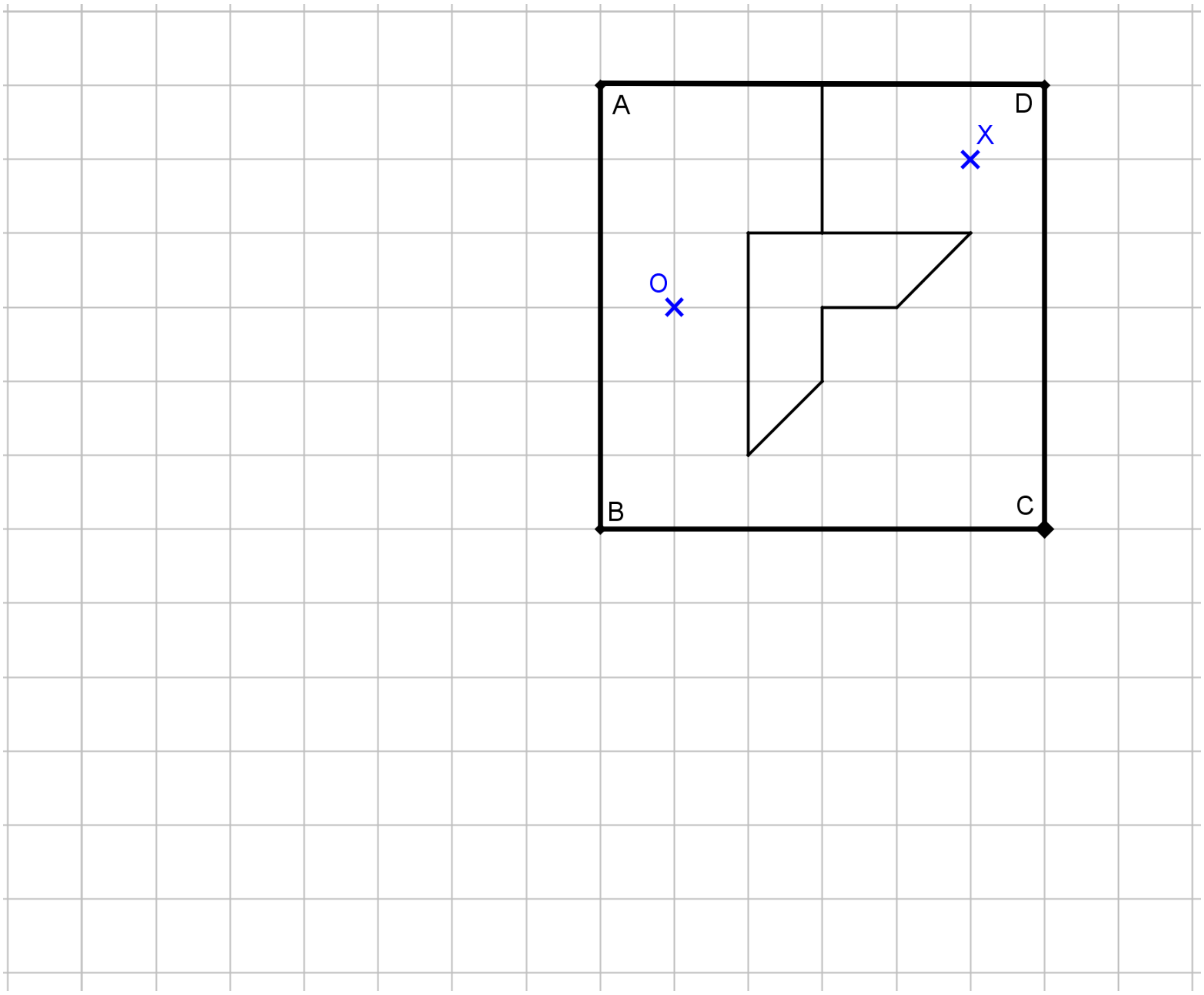
(d) Prove that $\angle APM = \angle BPN$.

(2) Extended exercises:

(a) In the figure, *ABCD* is a sealed room and all solid lines represent opaque walls. Point *O* and point *X* in the room are the locations of the security guard and the monitored object respectively. Try to set up some plane mirrors in appropriate position(s) in the room so that the security guard can monitor point *X* through the mirror(s) at point *O*. Also, draw the reflection route.



(b) Can you suggest another way to set up the mirrors? Try to describe your suggestion in the figure and draw the reflection route.



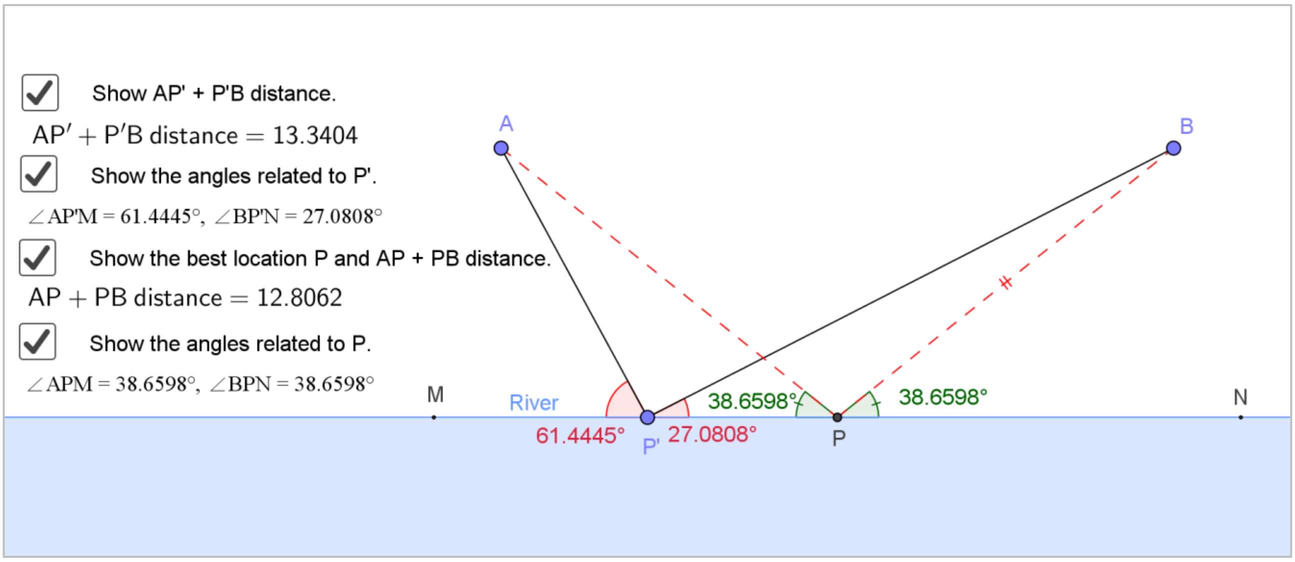
Notes for Teachers:

1. Similar problems arise from time to time in different recreational mathematics books. For example, "build a water tower by the river and use it to supply water to two villages A and B. Where should the water tower be to minimise the total length of the water pipe?" [2]

2. Suggested solution (Worksheet 1)

(1) (a) The question on colouring the best location is an open-ended question.

(b) The best location P may be referred to the GeoGebra file provided.



(c) The measures of $\angle AP'M$ and $\angle BP'N$ may be referred to the GeoGebra file provided.

(d) Is $\angle APM$ equal to $\angle BPN$ at point P? Answer: Yes

(2) (a)

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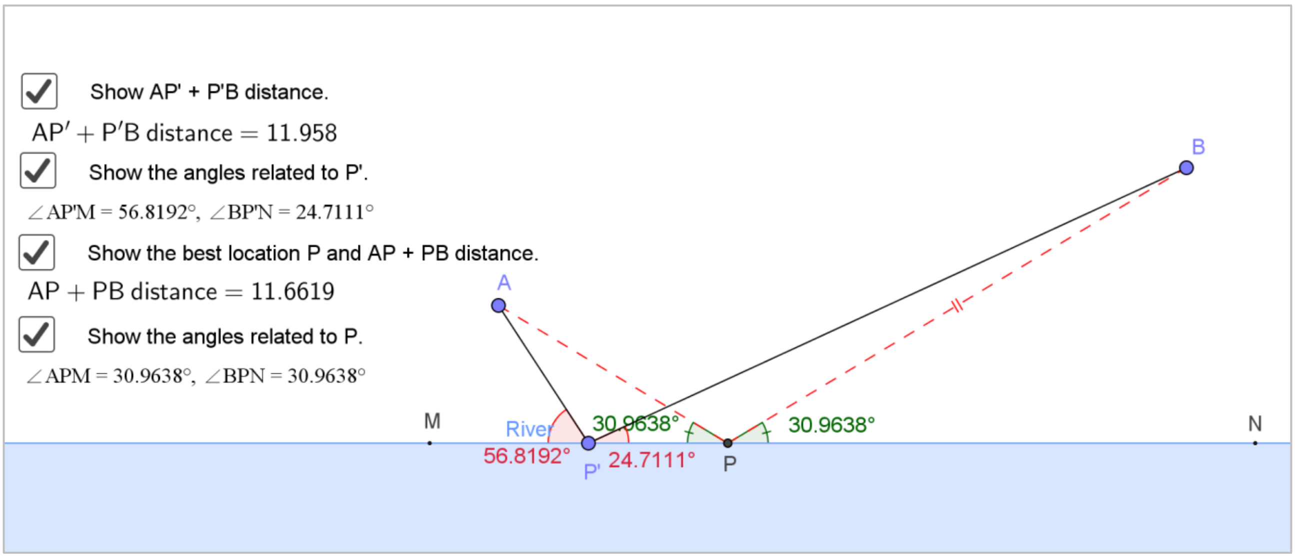
(b) The best location P may be referred to the GeoGebra file provided.



(c) (Suggested solution) If the distances of A and B from the river are reversed, the point P will move from the side closer to point A to the side closer to point B, and the original

[2] Perlman, Y（原著），戴中器譯（2000）：《趣味幾何學》，臺北：九章出版社，頁 282-283。

distance between point P and point A is equal to the distance between point P and point B after the reversion.

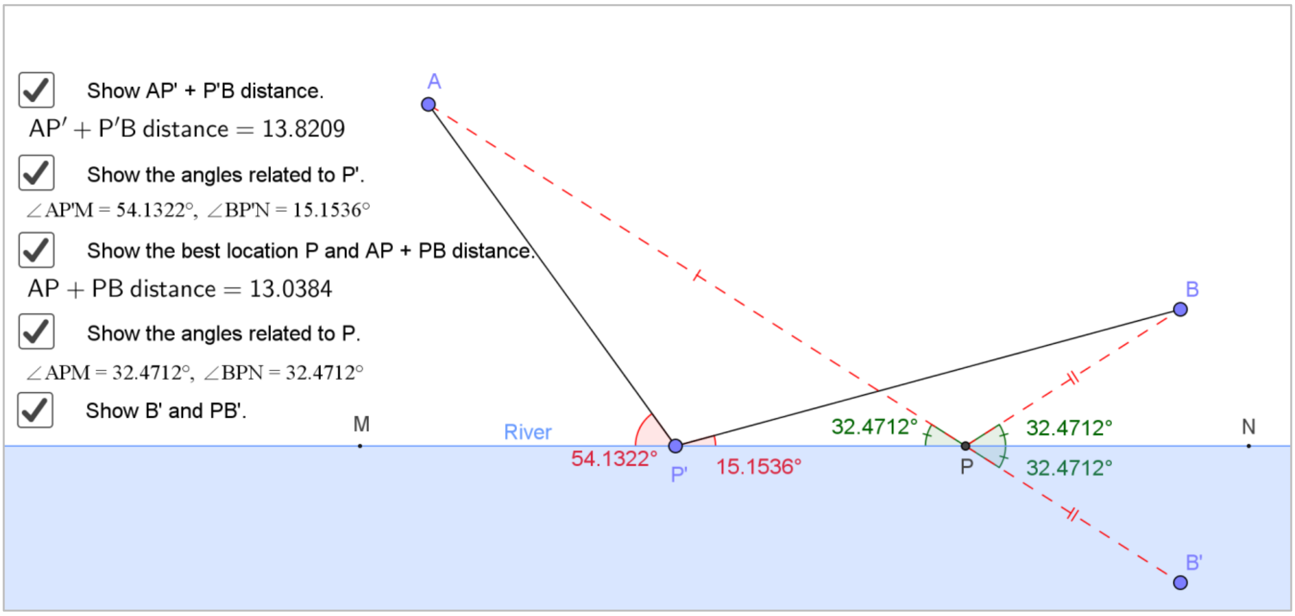
(d) The measures of $\angle AP'M$ and $\angle BP'N$ may be referred to the GeoGebra file provided.

(e) Is $\angle APM$ equal to $\angle BPN$ at point P? Answer: Yes

(f) **(Open-ended question, suggested direction for solution)** By the exploratory activities of guessing the best location according to different distances of point A, B from the river, students are expected to point out that the location of point P is closer to the point which is closer to the river relatively. In case (1), since the distances of points A and B from the river are equal, point P should be approximately in the middle of A and B; and in case (2), since point A is closer to the river, so the location of point P should be closer to A. Otherwise, if the distances are reversed, the location of point P will be adjusted to be closer to B.

(3) (a) **(Open-ended question, suggested direction for solution)** Students should be able to observe that $\angle APM$ is equal to $\angle BPN$, so as to guess that for any point A and point B, the geometric characteristic for the best location P is $\angle APM$ being equal to $\angle BPN$.

(b) The best location P may be referred to the GeoGebra file provided.



3. Suggested solution (Worksheet 2)

(1) (a) Let B' be a point in the figure such that $BB' \perp MN$ and $BC = B'C$, where C is the intersection point of BB' and MN.

For any point P' on MN, if P' does not coincide with C,

$P'C = P'C$ (common side)

$\angle P'CB = \angle P'CB'$ (given)

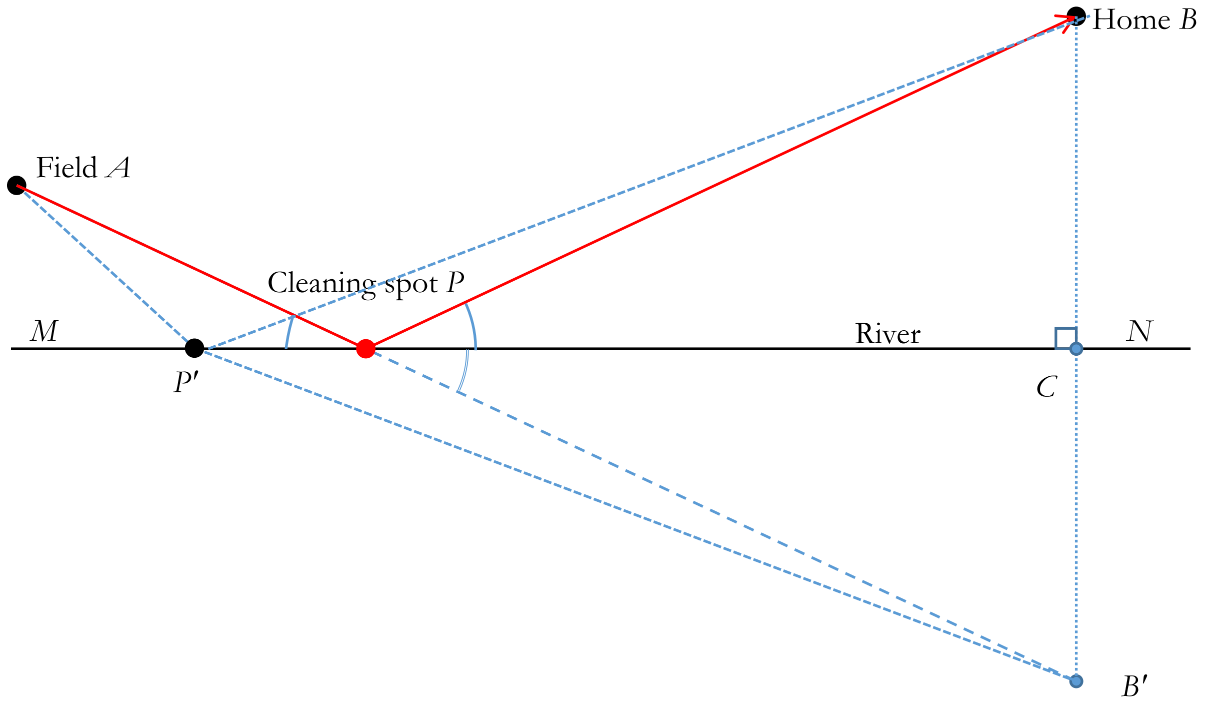
$BC = B'C$ (given)

$\therefore$ $\Delta BP'C \cong \Delta B'P'C$ (S.A.S.)

(b) From the GeoGebra file, point P is the best location if A, P and B' are collinear.

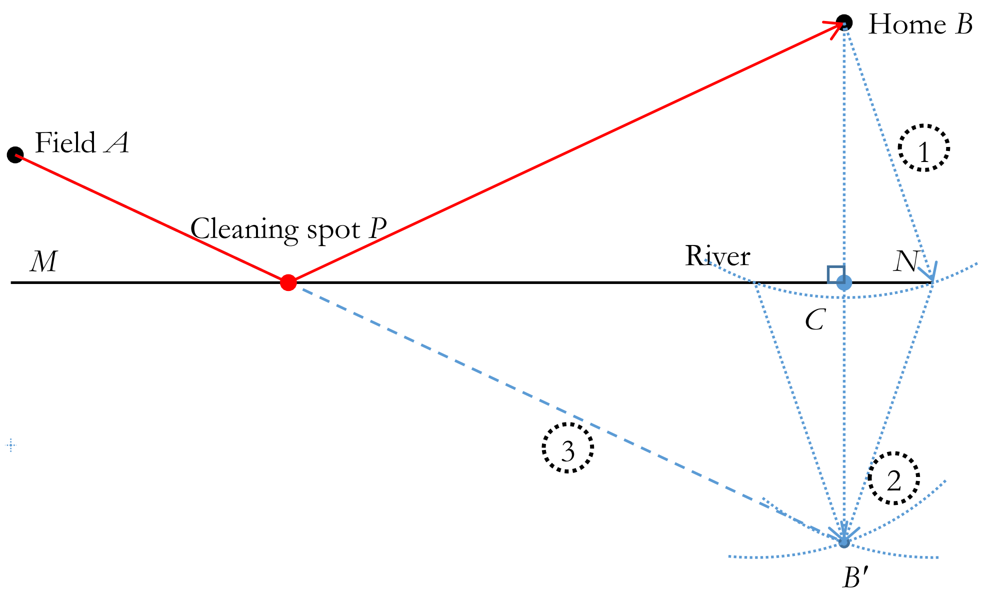
Let P' be any point on the MN.

From (a), $PB = PB'$ and $P'B = P'B'$ (corresponding sides of congruent triangles)
By triangle inequality, $AP' + P'B' \geq AB' = AP + PB'$
Besides, $AP' + P'B = AP' + P'B'$ and $AP + PB' = AP + PB$
Hence, $AP' + P'B \geq AP + PB$, meaning that $AP + PB$ is the shortest path.



The condition for the best location P: _A, P and B' are collinear._

(c) For any point A and point B, the following steps in the figure demonstrate how to draw the best location P by straightedge and compass construction:



① Construct an arc by using B as the centre and taking a distance larger than the distance between B and MN as the radius. Mark the two intersection points of the arc and MN (or its extension).

② On the other side of MN, mark the other intersection point B' of two arcs by using the same radius in Step ① and the two intersection points obtained in step ① as the centre.

③ Mark the intersection point P by joining AB' and MN. Drawing completed.

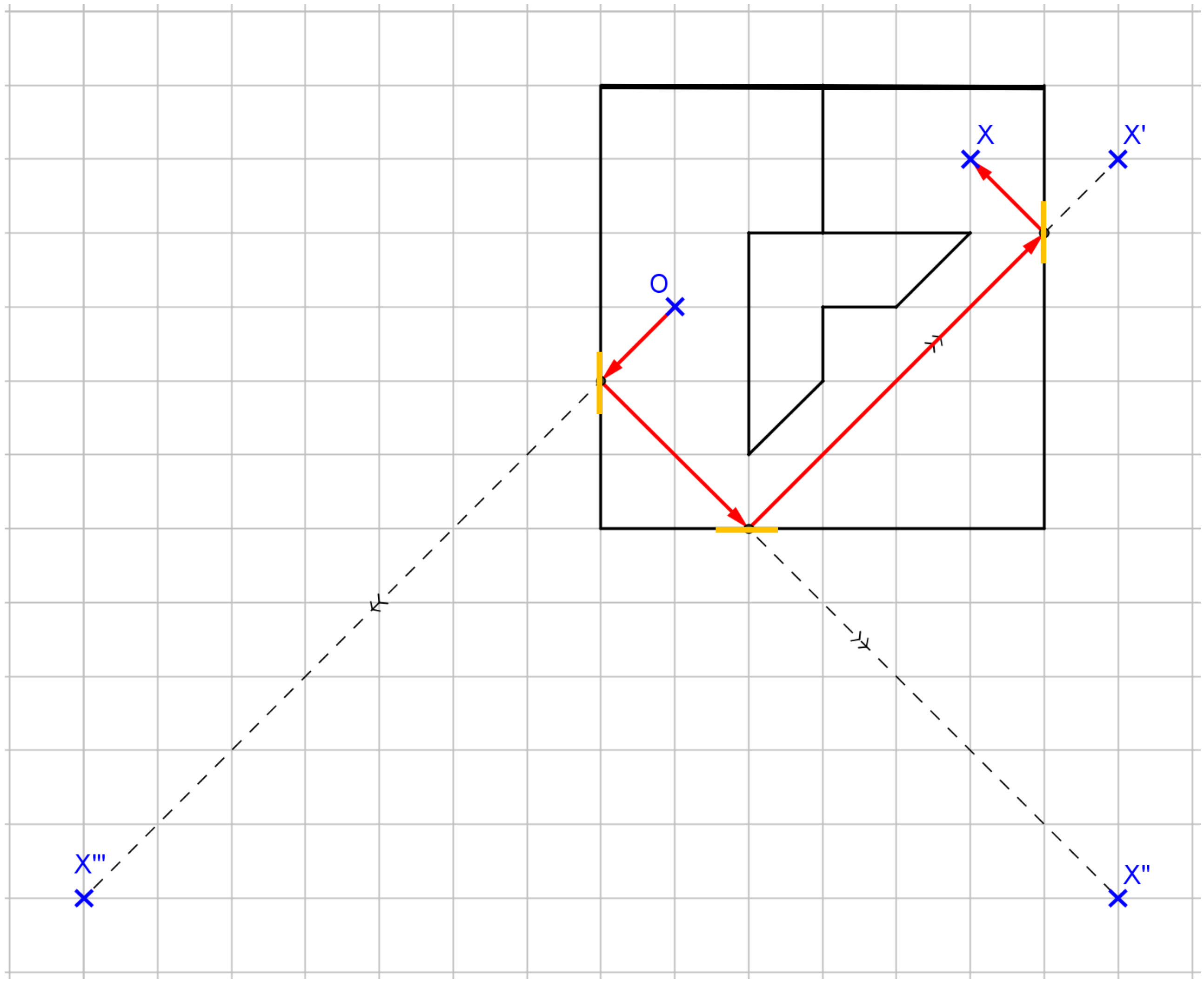
(d) Since $\angle APM = \angle B'PC$ (vertically opposite angles)

and $\angle BPC = \angle B'PC$ (corresponding angles of congruent triangles)

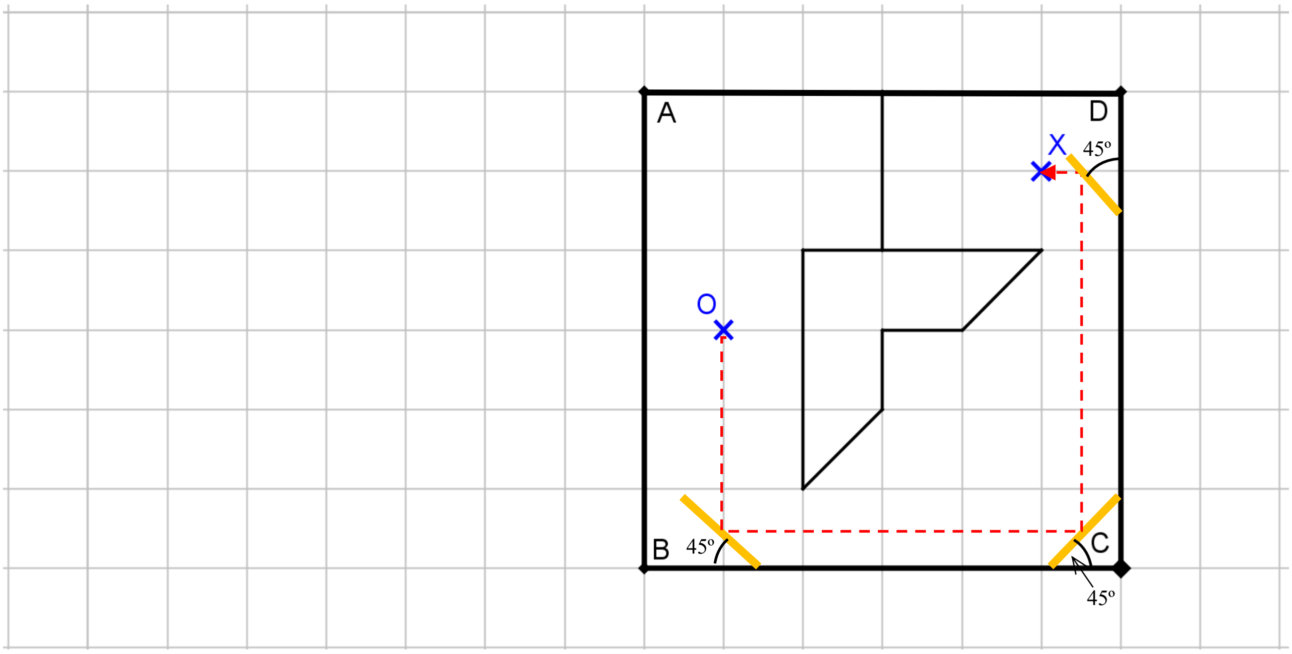
$\therefore$ $\angle APM = \angle BPC = \angle BPN$

(2) Extended exercises:

(a) (The orange line segments indicate the position of the plane mirrors)



(b) (The following example is for reference only and students may suggest other possible solutions.)



4. Teachers may discuss with students that the ancient Greek mathematicians such as Euclid and Ptolemy had already observed the shortest path and the path of the light through specular reflection were same. Thus using the shortest path in geometry, and the angle of incidence is equal to the angle of reflection, the principle of reflection in optics may be described.

5. Teachers may introduce Prof. Sir Charles Kuen KAO's research in optical fibre and its applications in real life (for example, "Success stories: Prof. Charles K KAO - Father of Fiber Optic Communications" http://www.rthk.hk/tv/dtt31/programme/successstories2000). Students may also be asked to gather some applications of optical fibre in telecommunication and medical fields (for example, endoscopy).