Modelling the spread of a disease

Key Stage: 4 (Compulsory Part and Module 1)

Learning Unit: Further Application
Inquiry and investigation

Objectives:
(i) To help students relate STEM education with the real life
(ii) To let students recognise the mathematics behind everyday life and apply information technology to solve problems
(iii) To let students recognise mathematics as a powerful tool for planning

Prerequisite Knowledge:
(i) "Exponential and logarithmic functions" and "Geometric Sequence" in Compulsory Part
(ii) "Simple idea of probability" in junior secondary
(iii) Topics on calculus in Extended Part Module 1

Modelling the spread of a disease

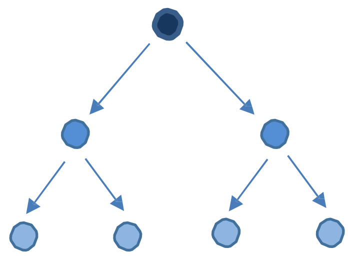
Background information: Bird flu, Coronavirus disease and Ebola were examples of fatal epidemics that emerged in a large scale in past two decades. They badly threatened human lives in the world.

Basic Assumptions of the two mathematical models on the spread of diseases:

1. The population is in a closed area.
2. The total number of the population is fixed.
3. The disease is transmitted once only from infected person by direct individual contact.
4. The number of recovered or dead person would not be considered.

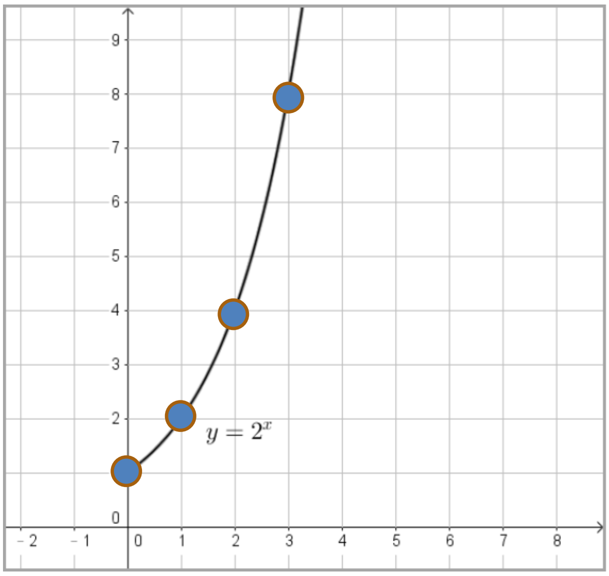
Model 1:

1. Suppose at the beginning, there is one infected person.
2. The person directly spreads the disease to two susceptible persons by individual contact.
3. The two newly infected persons then each transmit the disease to two more persons
4. The process continued until everyone is infected, as shown in the following figure:



Construction of model 1:

According to the above situation, we get $y = 2^x$, where x represents which step of contact, y represents the number of newly infected person at that step.



Discussion questions on this model:

1. How many steps are needed to infect all the people in the classroom? How about the whole school?
2. How about the result if 3 persons were infected at each step?
3. Suppose the population of Hong Kong are 7.8 million. How many steps do we need to infect all the people in this model?
4. Besides basic assumptions, what are the restrictions of this model?

Note for teachers:

It is the simplest way to model the spread of disease but too simple. For example, the model has not taken into account the people who recover, die or continue to infect other people after consultation. The model also has not considered the isolation of the infected person and the precaution policy of non-infected persons, etc.

Model 2:

We toss a fair dice to simulate the variability of the number of infected people in the model. The number of infected person is not fixed. We consider two situations:

(i) The expected number of infected people is greater than 1

(ii) The expected number of infected people is smaller than 1

Case 1:

Number on dice	Number of newly infected people
1	0
2	0
3	1
4	2
5	2
6	3

The expected number of infected person = $\frac{1}{6}\times 0+\frac{1}{6}\times 0+\frac{1}{6}\times 1+\frac{1}{6}\times 2+\frac{1}{6}\times 2+\frac{1}{6}\times 3=\frac{4}{3}$, which is greater than 1. **Epidemics take off.**

Case 2:

Number on dice	Number of newly infected people
1	2
2	1
3	1
4	0
5	0
6	0

The expected number of infected person = $\frac{1}{6}\times 2+\frac{1}{6}\times 1+\frac{1}{6}\times 1+\frac{1}{6}\times 0+\frac{1}{6}\times 0+\frac{1}{6}\times 0=\frac{2}{3}$, which is less than 1. **Epidemics die out.**

Classroom Activity:
Teachers require students to form groups. Each group receive a fair dice and 30 counters, conduct the following activity:

1. Choose Case 1 or Case 2.
2. Place one counter on the table top to represent one infected person, i.e. the origin of the infection.
3. Toss the dice once. Suppose the dice shows 4. In Case 1, this means that two people are infected. Two more counters have to be placed on the desk to indicate the number of infected people. In Case 2, no new person is infected. The disease dies out.
4. Repeat tossing the dice for EACH newly infected person and place the relevant number of counter on the desk.
5. Repeat tossing the dice until either the epidemic dies out or the counters are out of stock.
6. Record the process of the spread of the epidemic on a graph.
7. Run the simulations several times for the two cases and compare the graphs.

Discussion questions on this model:

1. Does the epidemic take off or die out in each case?
2. On average, how many steps does the epidemic run in each case?
3. What aspects of the model help us to understand how epidemics progress?
4. Are there important factors which the model does not include? Can you think of ways to improve it?

Challenging Problem:

Suppose a population is threatened by an infectious disease. The population can be divided into two groups, the healthy one and the infected one. Let p be the probability that a healthy person gets the disease, r be the probability that an infected person recovers in each month. Suppose there are 8 healthy people and 2 infected people initially.

Mathematical Model:

Students can model what happens to the 10 people over a period of 10 months by tossing a fair 6-sided dice. Assume no one dies in 10 months. Students toss the fair dice for each person in each month to measure the changes of their health status.

1. If we get number 1, 2 or 3 for a healthy person, then he becomes infected at that month (i.e. $p=\frac{1}{2}$);
2. If we get number 4 or 5 for an infected person, then he recovers that month (i.e. $r=\frac{1}{3}$).
3. For other outcomes, their status remain unchanged.

Suggested solution:

Let x be the proportion of population that are infected, its initial value is 0.2. We can present the rate of change of x by the following expression:

$$\frac{dx}{dt}=p(1-x)-rx$$

$$\frac{dx}{dt}=\frac{1}{2}(1-x)-\frac{1}{3}x$$

$$\boxed{\frac{dx}{dt}=\frac{3-5x}{6}}$$

Notes for Teacher:

(i) From the given condition, students are able to set up the expression for rate of change of x. Students are not required to solve differential equation in our curriculum.

Teachers may demonstrate the technique of solving the problem for the more able students.

$$\frac{dx}{dt}=\frac{3-5x}{6}$$

$$\int \frac{6}{3-5x}dx=\int dt$$

$-\frac{6}{5}\ln(3-5x)=t+k$ ，where k is a constant

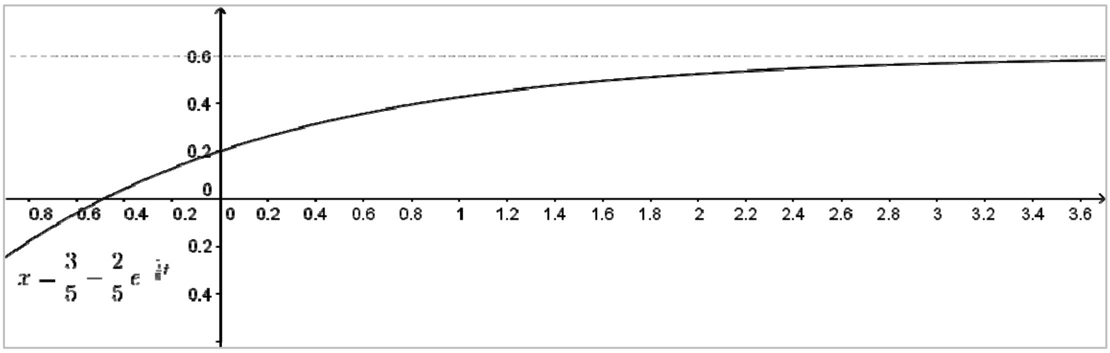
When $t = 0$, $x = 0.2$; so $k=-\frac{6}{5}\ln(2)$, hence

$$-\frac{6}{5}\ln(3-5x)=t-\frac{6}{5}\ln(2)$$

$$-t=\frac{6}{5}\ln\left(\frac{3-5x}{2}\right)$$

$$\frac{3-5x}{2}=e^{-\frac{5}{6}t}$$

Hence, $x=\frac{3}{5}-\frac{2}{5}e^{-\frac{5}{6}t}$. When t tends to infinity，x tends to $\frac{3}{5}$. The graph of the mathematical model is as follows:



What happens to the solution if the value of p and r is changed?

(ii) Another way of dealing with this problem is to give the suggested solution $x=\frac{3}{5}-\frac{2}{5}e^{-\frac{5}{6}t}$ to students and ask them to verify that this solution satisfies the differential equation.

Reference:

Website about the modelling of the spread of disease:

https://motivate.maths.org/content/DiseaseDynamics/Activities/StandingDisease

This exemplar mainly involves the following generic skills:

1. Critical Thinking Skills
   - Compare mathematical model and real situation, analyse the shortcomings of the model

2. Problem Solving Skills
   - Formulate a mathematical solution when tackling a real-life problem
   - Use real objects to simulate abstract mathematical context