Explanatory Notes to Senior Secondary Mathematics Curriculum — Module 1 (Calculus and Statistics)

Mathematics Education Section
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Contents

Foreword

The *Mathematics Curriculum and Assessment Guide (Secondary 4 – 6)* (2007) (abbreviated as "C&A Guide" in this booklet) has been prepared to support the new academic structure implemented in September 2009. The Senior Secondary Mathematics Curriculum consists of a Compulsory Part and an Extended Part. The Extended Part has two optional modules, namely Module 1 (Calculus and Statistics) and Module 2 (Algebra and Calculus).

In the C&A Guide, the Learning Objectives of Module 1 are grouped under different learning units in the form of a table. The notes in the "Remarks" column of the table in the C&A Guide provide supplementary information about the Learning Objectives. The explanatory notes in this booklet aim at further explicating:

1. the requirements of the Learning Objectives of Module 1;
2. the strategies suggested for the teaching of Module 1;
3. the connections and structures among different learning units of Module 1; and
4. the curriculum articulation between the Compulsory Part and Module 1.

The explanatory notes in this booklet together with the "Remarks" column and the suggested lesson time of each learning unit in the C&A Guide are to indicate the breadth and depth of treatment required. Teachers are advised to teach the contents of the Compulsory Part and Module 1 as a connected body of mathematical knowledge and develop in students the capability to use mathematics to solve problems, reason and communicate. Furthermore, it should be noted that the ordering of the Learning Units and Learning Objectives in the C&A Guide does not represent a prescribed sequence of learning and teaching. Teachers may arrange the learning content in any logical sequence which takes account of the needs of their students.

Comments and suggestions on this booklet are most welcomed. They should be sent to:

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Foundation Knowledge Area

- The content of Foundation Knowledge Area consists of two Learning Units. The first Learning Unit “Binomial Expansion” forms the basis of the binomial distribution. The second Learning Unit is “Exponential and Logarithmic functions”. Many mathematical models related to natural phenomena involve the exponential function. The probability function of the normal distribution also involves the exponential function.

- It should be noted that the content of Foundation Knowledge Area is considered as pre-requisite knowledge for Calculus Area and Statistics Area of Module 1. Rigorous treatment of the topics in Foundation Knowledge should be avoided.

Learning Unit	Learning Objective	Time
Foundation Knowledge Area		
1. Binomial expansion	1.1 recognise the expansion of $(a + b)^n$, where n is a positive integer	3

Explanatory Notes:

- In order to facilitate students to express a binomial expansion concisely, teachers may introduce the summation notation (Σ). For example, $\sum_{k=1}^{7} k^3$ and $\sum_{i=0}^{n} 4^i$.

- The relationship should be noted: $\sum_{r=1}^{n}(ax_r \pm by_r) = a\sum_{r=1}^{n} x_r \pm b\sum_{r=1}^{n} y_r$.

- If there are no ambiguities, Σx_i or even Σx could be used in discussions.

- The concept and notation of C_r^n have already been discussed in the Compulsory Part. Students should be able to recognise $(a + b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r$, where n is a positive integer.

- The formal proof of the binomial expansion is not required.

- Common notations such as ${}_nC_r$ or $\binom{n}{r}$ in the expansion are acceptable.

- There are several ways to introduce the concept of the binomial expansion. For example, teachers can ask students to expand $(a + b)^n$ for n = 0, 1, 2, 3, 4 and put the coefficients of the terms into the boxes in the diagram below. By observing the pattern of the coefficients, students may try to write down the expansion of $(a+b)^5$ and $(a + b)^6$.

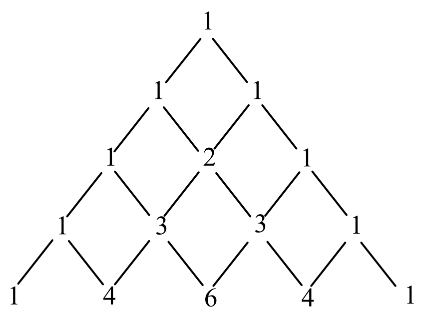
Expansions	Pascal's triangle*
$(a + b)^0 = 1$	$\boxed{1}$
$(a + b)^1 = a + b$	$\boxed{1}\ \boxed{1}$
$(a + b)^2 = a^2 + 2ab + b^2$	$\boxed{1}\ \boxed{2}\ \boxed{1}$
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$\boxed{1}\ \boxed{3}\ \boxed{3}\ \boxed{1}$
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	$\boxed{1}\ \boxed{4}\ \boxed{6}\ \boxed{4}\ \boxed{1}$

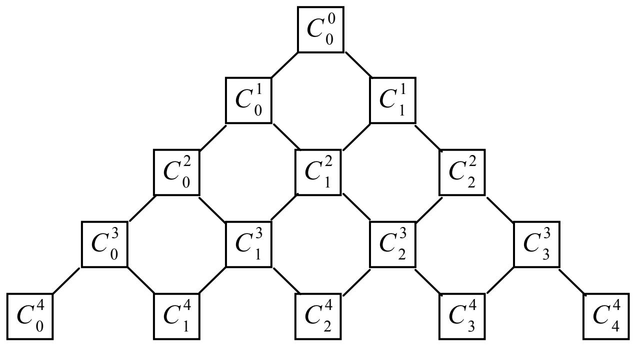
Alternatively, teachers may use the concept of combination to explain the binomial expansion. For example, in the expansion of $(a + b)^3$, the coefficient of the term ab^2 can be viewed as the number of different combinations in choosing 2 b's out of 3,

$$\left.\begin{array}{ll} (\boxed{a} + b)\,(a + \boxed{b})\,(a + \boxed{b}) & \longrightarrow\ a\cdot b\cdot b = ab^2 \\ (a + \boxed{b})\,(\boxed{a} + b)\,(a + \boxed{b}) & \longrightarrow\ b\cdot a\cdot b = ab^2 \\ (a + \boxed{b})\,(a + \boxed{b})\,(\boxed{a} + b) & \longrightarrow\ b\cdot b\cdot a = ab^2 \end{array}\right\}\ C_2^3\,ab^2 = 3\,ab^2$$

Similarly, we can find the coefficients of other terms and obtain
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

- Teachers can ask students to find the coefficient of each term in the expansion $(a + b)^n$ by combination and compare the numerical values with those in Pascal's triangle on the left.





In general, $(a + b)^n = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \ldots + C_{n-1}^n ab^{n-1} + C_n^n b^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r$

* The arrangement of the binomial coefficients in a triangle is named after mathematician Blaise Pascal as he included this triangle with many of its application in his treatise, *Traité du triangle arithmétique* (1653). In fact, in 13th Century, Chinese mathematician Yang Hui (楊輝) presented the triangle in his book 《詳解九章算法》(1261) and pointed out that Jia Xian (賈憲) had used the triangle to solve problems. Thus, the triangle is also named Yang Hui's Triangle (楊輝三角) or Jia Xian's Triangle (賈憲三角).

Learning Unit	Learning Objective	Time
Foundation Knowledge Area		
2. Exponential and logarithmic functions	2.1 recognise the definition of the number e and the exponential series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ 2.2 recognise exponential functions and logarithmic functions 2.3 use exponential functions and logarithmic functions to solve problems 2.4 transform $y = kx^n$ and $y = ka^x$ to linear relations, where a, n and k are real numbers, $a > 0$ and $a \neq 1$	7

Explanatory Notes:

- Some approaches to introduce the concept of e are:

  (a) Compound Interest

  The amount will be greater if interest is compounded monthly instead of annually, and the amount will be even greater if it is compounded daily. What will happen to the amount if it is compounded hourly?

  Teachers can start by asking students to find the amount of a sum of money, for example, \$10000 at a rate of 12% p.a. for 1 year if the interest is compounded (i) monthly, (ii) daily and (iii) hourly.

  (b) Define $e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$. Teachers may ask student to find, as n increases, the value of $\left(1 + \frac{1}{n}\right)^n$ using a calculator.

n	10	100	1 000	10 000	100 000	1000 000
$\left(1 + \frac{1}{n}\right)^n$	2.59374	2.70481	2.71692	2.71815	2.71827	2.71828

The student can observe that $e \approx 2.71828\ldots$

Teachers can also introduce the definition of the exponential function

$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, but students are not required to master the technique involved.

$$\left(1+\frac{x}{n}\right)^n = C_0^n(1)^n + C_1^n(1)^{n-1}\left(\frac{x}{n}\right) + C_2^n(1)^{n-2}\left(\frac{x}{n}\right)^2 + C_3^n(1)^{n-3}\left(\frac{x}{n}\right)^3 + C_4^n(1)^{n-4}\left(\frac{x}{n}\right)^4 + \ldots$$

$$= 1+n\left(\frac{x}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{x}{n}\right)^4 + \ldots$$

$$= 1+x+\left(1-\frac{1}{n}\right)\frac{x^2}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{x^4}{4!}+\ldots$$

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$= \lim_{n\to\infty}\left[1+x+\left(1-\frac{1}{n}\right)\frac{x^2}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{x^4}{4!}+\ldots\right]$$

$$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

By putting $x = 1$, we have the irrational number

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \ldots$$

and, by using a calculator, it is obvious that the value of e converges approximately to 2.71828...

- Students should also know how to expand the exponential function such as e^{-x}, e^{kx}, e^{-x^2} and e^{x+k} into a power series, where k is a constant.

- Students have learnt the properties of the exponential function $y = a^x$ ($a > 0$, $a \neq 1$), the logarithmic function $y = \log_a x$ ($a > 0$, $a \neq 1$) and the characteristics of their graphs in the Compulsory Part. These concepts can be treated as the prerequisite knowledge to this Learning Unit. Meanwhile, teachers may point out that the exponential function $y = e^x$ and the natural logarithm function $y = \ln x$ ($x > 0$) are special cases of $y = a^x$ and $y = \log_a x$. As a consolidation to the meaning of $y = e^x$, teachers may ask their students to compare the graphs of $y = 2^x$, $y = e^x$, $y = 3^x$, $y = 2^{-x}$, $y = e^{-x}$, $y = 3^{-x}$, $y = \log_2 x$, $y = \ln x$ and $y = \log_3 x$. Teachers can further discuss with their students $e^{\ln x} = x$, $\ln e^x = x$ and $a^x = e^{x\ln a}$.

- The exponential function can be used to model many natural phenomena, for example, the growth of bacteria, the rate of cooling of substances, the loss of heat energy of substances, and the rates of growth and decay of substances. Students are required to know the following formulae:

Compound interest： $A = P_0 e^{\frac{rt}{100}}$

Population growth： $P(t) = P_0 e^{kt}$, $k > 0$

Radioactive decay： $P(t) = P_0 e^{-kt}$, $k > 0$

- If the relation between x and y is in the form of $y = kx^n$ or $y = ka^x$, where n, k and a are constants, the relation can be reduced to a linear relation. The values of the constants can be found from the slope and the y-intercept of the linear relation plotted on a graph paper.

Calculus Area

- Calculus Area consists of two sections, namely Differentiation with Its Applications and Integration with Its Applications. The concept of the derivative of a function involves the concept of the limit of a function. In the section "Differentiation and Its Applications", students should understand the definition of the derivative of a function, the fundamental formulae and the rules of differentiation. They should also be able to use derivatives to find the equation of the tangent to a curve and to investigate the maximum/minimum values of a function.

- Students need to find the function $f(x)$ from its derivative $f'(x)$ in various situations related to science, technology and economics. This reverse process is the idea of the indefinite integral. Teachers need to explain clearly the idea of the definite integral as the numerical limit of a summation. Teacher can lead students to recognise that the Fundamental Theorem of Calculus can link the two apparently different concepts (the indefinite integral and the definite integral) together.

- Notations should be firmly established and well understood by students.

- The approaches adopted should be intuitive but the concepts involved should be correct. In difficult topics such as limits, numerical approaches using calculators (or computer software) can help students understand the related concepts without paying attention to abstract definitions. Teachers can make use of graphing tools (such as Graphmatica, Winplot etc.) to illustrate the concepts.

Learning Unit	Learning Objective	Time
Calculus Area		
Differentiation with Its Applications		
3. Derivative of a function	3.1 recognise the intuitive concept of the limit of a function 3.2 find the limits of algebraic functions, exponential functions and logarithmic functions 3.3 recognise the concept of the derivative of a function from first principles 3.4 recognise the slope of the tangent of the curve $y = f(x)$ at a point $x = x_0$	6

Explanatory Notes:

- Teachers should briefly review the idea and notation of a function before introducing the limit of a function.

- Students need to know that the value of $\lim_{x \to x_0} f(x)$ is affected by the values of x near $x = x_0$ and that $f(x)$ may not even be defined at $x = x_0$. Students should be able to identify "continuous functions" and "discontinuous functions" from their graphs. Teachers may also point out that the limit of a function $f(x)$ at $x = x_0$ is equal to the value of the function at x_0 if and only if the function is continuous at $x = x_0$, but a rigorous treatment of continuity is not required.

- Tables and graphs showing small changes of the functional values close to $x = x_0$ may help to illustrate the meaning of the limit of $f(x)$ as x approaches x_0. The $\varepsilon - \delta$ approach to explain the meaning of the limit of a function at the stage is not required.

- For abler students, the following problems can also be discussed:

  (a) If $f(x) = \begin{cases} x+2, & x \geq 2 \\ x^2, & x < 2 \end{cases}$, find $\lim_{x \to 2} f(x)$.

  (b) If $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$, find $\lim_{x \to 0} f(x)$.

- Students should be able to find the limits of algebraic functions, exponential functions and logarithmic functions by using limits of sum, difference, product, quotient and composition. When x tends to infinity, students should know that $\frac{1}{x}$ tends to zero. Students are also required to find the limits of some simple functions such as $\frac{2x+3}{x^3}$ and $\frac{3}{xe^x}$ as x approaches infinity.

- The derivative of a function $y = f(x)$ with respect to x can be defined as $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ if the limit exists. Teachers may demonstrate how to find the derivative of simple functions like x^2 and $\frac{1}{x-1}$ from first principles. However, **students are not required to find the derivatives of functions from first principles**. Teachers may introduce the geometrical meaning of the difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$. Students should know common notations of derivative such as y', $f'(x)$ and $\frac{dy}{dx}$. They should understand that $\frac{d}{dx}$ is an operator and $\frac{dy}{dx}$ should not be taken as a fraction.

- Students should recognise the notations of $f'(x_0)$ and $\left.\frac{dy}{dx}\right|_{x=x_0}$, where x_0 is a given value. Students should understand that as Δx approaches 0, the limiting value of $\frac{\Delta y}{\Delta x}$ will give the slope of the tangent to the curve at the point $(x_0, f(x_0))$. They should also be able to find the equations of tangents of simple curves.

Learning Unit	Learning Objective	Time
Calculus Area		
Differentiation with Its Applications		
4. Differentiation of a function	4.1 understand the addition rule, product rule, quotient rule and chain rule of differentiation 4.2 find the derivatives of algebraic functions, exponential functions and logarithmic functions	10

Explanatory Notes:

- Teachers may derive the rules of differentiation. Students should grasp these basic rules to find the derivatives of functions.

- Teachers may give some typical examples of composite functions and inverse functions and then introduce the chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. Students do not need to understand the differentiation of inverse functions, but they may use the formula $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ to solve problems. Differentiation of parametric equations is not required.

- Students should learn how to differentiate a polynomial. When rules for differentiating the sum, the product and the quotient of functions are established, students should be able to differentiate the product of polynomials and rational functions such as $(2x + 3)(4x^2 + 5)$ and $\frac{1-2x^2}{2+3x}$.

- Students do not need to learn implicit differentiation, but they should know how to carry out logarithmic differentiation. When it is required to differentiate functions of the form $h(x)k(x)$, $\frac{h(x)}{k(x)}$ or $[h(x)]^{k(x)}$, where $h(x)$ and $k(x)$ are functions of x, it is easier to find their derivatives by logarithmic differentiation. For example, in finding the derivative of the functions $(x+1)(x+2)(x+3)(x+4)$, $\frac{x-1}{x\sqrt{x^3+1}}$ and $y = x^x$.

- Students should know how to use Chain rule to find the derivatives of functions of the form $y = e^{f(x)}$ and $y = \ln f(x)$.

- Students should know how to find the derivatives of composite functions such as e^{x^2+1} and $\ln\sqrt{3x^2-5x+7}$.

Learning Unit	Learning Objective	Time
Calculus Area		
Differentiation with Its Applications		
5. Second derivative	5.1 recognise the concept of the second derivative of a function 5.2 find the second derivative of an explicit function	2

Explanatory Notes:

- The second derivative may be obtained by differentiating the first derivative. If $y = f(x)$, the second derivative may be written as $f''(x)$, y'' or $\frac{d^2y}{dx^2}$.

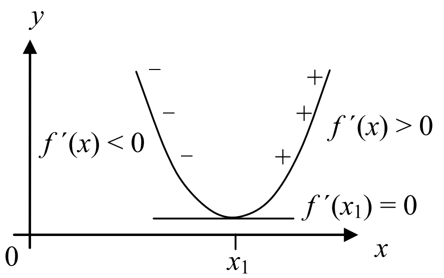
- Teachers may point out that, in general, $\frac{d^2y}{dx^2} \neq \frac{1}{\frac{d^2x}{dy^2}}$ and $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$.

Learning Unit	Learning Objective	Time
Calculus Area		
Differentiation with Its Applications		
6. Applications of differentiation	6.1 use differentiation to solve problems involving tangents, rates of change, maxima and minima	9

Explanatory Notes:

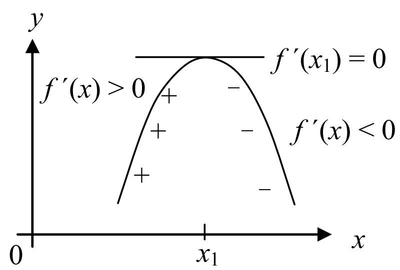
- Maxima and minima:
  (a) First derivative test
  Students should know how to determine the region of increase and the region of decrease of a given function by analysing the changes in the signs of its first derivative.
  (i) If $f'(x_1)=0$ and the sign of $f'(x)$ changes from negative to positive as x increases through x_1, then $f(x)$ attains a local minimum at $x=x_1$.
  (ii) If $f'(x_1)=0$ and the sign of $f'(x)$ changes from positive to negative as x increases through x_1, then $f(x)$ attains a local maximum at $x=x_1$.

$f(x)$ has a local minimum at $x=x_1$



[$f'(x)$ increases (i.e. $f''(x_1) > 0$) as x increases through x_1]

$f(x)$ has a local maximum at $x=x_1$



[$f'(x)$ decreases (i.e. $f''(x_1) < 0$) as x increases through x_1]

  (b) Second derivative test
  Students are required to know the geometrical meaning of the test:
  (i) If $f'(x_1) = 0$ and $f''(x_1) < 0$, then $f(x)$ attains a local maximum at $x=x_1$.
  (ii) If $f'(x_1) = 0$ and $f''(x_1) > 0$, then $f(x)$ attains a local minimum at $x=x_1$.

- The local extremum is not necessarily the global extremum. In finding the global extremum in an optimization problem, students should also consider the values at the end points of the interval(s) concerned.

- Students should be able to apply either the first derivative test or the second derivative test to find the extrema of a function. When $f''(x_1) = 0$, the second derivative test is not applicable to find the local extrema. In this case, students have to revert to the first derivative test.

- Students should be able to use the second derivative to determine the concavity and convexity of a function. They are not required to sketch the graph of the function. Students are not required to learn the concept of the point of inflexion of a curve. Local extrema at $x = x_1$ related to the non-existence of $f'(x_1)$ are not required.

Learning Unit	Learning Objective	Time
Calculus Area		
Integration with Its Applications		
7. Indefinite integrals and their applications	7.1 recognise the concept of indefinite integration 7.2 understand the basic properties of indefinite integrals and basic integration formulae 7.3 use basic integration formulae to find the indefinite integrals of algebraic functions and exponential functions 7.4 use integration by substitution to find indefinite integrals 7.5 use indefinite integration to solve problems	10

Explanatory Notes:

- The terms "integrand" and "constant of integration" should be introduced. Students should note that indefinite integration is the reverse process of differentiation.

- Students are required to master the following rules:
  - $\int kf(x)dx = k\int f(x)dx$, where k is a constant
  - $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
  - $\int k\,dx = kx + C$, where k and C are constants
  - $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, where C is a constant, n is real and $n \neq -1$ ($n = 0$ should also be discussed)
  - $\int \frac{1}{x}\,dx = \ln|x| + C,\ x \neq 0$
  - $\int e^x\,dx = e^x + C$

- In order to change the integrand into one of the forms of the basic integration formulae, students need to substitute $x = \phi(t)$ and hence $\int f(x)dx = \int f[\phi(t)]\phi'(t)dt$. Teachers may introduce integration by substitution through examples, such as $\int (2x + 1)^5 dx$ and $\int 2x\sqrt{x^2 + 1}\, dx$.

- Using integration by parts to find indefinite integral is not required.

Learning Unit	Learning Objectives	Time
Calculus Area		
Integration with Its Applications		
8. Definite integrals and their applications	8.1 recognise the concept of definite integration 8.2 recognise the Fundamental Theorem of Calculus and understand the properties of definite integrals 8.3 find the definite integrals of algebraic functions and exponential functions 8.4 use integration by substitution to find definite integrals 8.5 use definite integration to find the areas of plane figures 8.6 use definite integration to solve problems	15

Explanatory Notes:

- The definite integral can be introduced by considering the area under the curve to distinguish its concept from that of the indefinite integral.

- Some properties of the definite integral are useful. Their geometrical meaning should be explored.
  - $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
  - $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is a constant
  - $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

- Students should know that the above formulae can only be applied when the functions under discussion are continuous in the interval $[a, b]$.

- When the method of substitution is used to evaluate a definite integral, students should be reminded that the upper and lower limits of the definite integral should be changed accordingly.

Learning Unit	Learning Objective	Time
Calculus Area		
Integration with Its Applications		
9. Approximation of definite integrals using the trapezoidal rule	9.1 understand the trapezoidal rule and use it to estimate the values of definite integrals	4

Explanatory Notes:

- In practice, it is sometimes hard or even not possible to evaluate some definite integrals such as $\int_1^2 e^{x^2} dx$ by simple methods. The trapezoidal rule is one of the methods to approximate the values of definite integrals. In applying the rule, the widths of all strips should be the same and a better approximation of the definite integral can be obtained from more strips.

- Students are not required to understand the error estimation in the application of the trapezoidal rule. However, they should be able to tell whether an approximation is an under-estimate or over-estimate by considering the second derivative of the function and the concavity. If a curve concaves upwards, the trapezoidal rule will over-estimate the required area. If a curve is concave downwards, the trapezoidal rule will under-estimate the required area.

<u>Example</u> Given the curve $y = -x^2 + 4$. As $\frac{d^2y}{dx^2} = -2 < 0$, we know that the curve concaves downwards. Thus, the trapezoidal rule under-estimates the area bounded by the curve and the x-axis.

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Statistics Area

- Statistics Area consists of four sections, namely Further Probability; Binomial, Geometric and Poisson Distributions and Their Applications; Normal Distribution and Its Applications; and Point and Interval Estimation.

- Probability is considered elementary and important in this Area. The concept of a random variable is new to students. Binomial, geometric, Poisson and normal distributions serve to widen students' knowledge on probability distributions. Discussions of statistical inference are also included.

- A study of population parameters and sample statistics depicts the relationship between populations and samples. Point estimation and interval estimation are included.

- Point estimation involves the use of sample data to calculate a statistic which is to serve as a guess for an unknown population parameter. A confidence interval (CI) is an interval estimate of a population parameter. Confidence intervals are used to indicate the reliability of an estimate. How likely the interval is to contain the parameter is determined by the confidence level. When the desired confidence level is increased, the corresponding confidence interval will be widened.

Learning Unit	Learning Objective	Time
Statistics Area		
Further Probability		
10. Conditional probability and independence	10.1 Understand the concepts of conditional probability and independent events 10.2 use the laws $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B \mid A)$ and $\mathrm{P}(D \mid C) = \mathrm{P}(D)$ for independent events C and D to solve problems	3

Explanatory Notes:

- The addition law and multiplication law of probability, the concepts of exclusive events, complementary events, independent events and conditional probability are discussed in Learning Unit 15 in the Compulsory Part. Students need to further investigate conditional probability in this Learning Unit.

- Venn diagrams can be used to illustrate the meaning of conditional probabilities before the introduction of $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$, where A may be considered as a reduced sample space. It is straightforward to arrive at the law $P(A \cap B) = P(A)P(B \mid A)$.

- Students may have confusions in the concepts related to $P(B \mid A)$ and $P(A \cap B)$.

- Students should note the following points about mutually independent events:

  - If two events A and B are mutually independent, then the occurrence of A (or B) does not affect the probability of the occurrence of B (or A).
  - If two events A and B are mutually independent, then $\overline{A}$ and B, $\overline{B}$ and A, $\overline{A}$ and $\overline{B}$ are also mutually independent.

- The difference between “mutually independent events” and “mutually exclusive events” should be discussed. Two events are mutually independent if the occurrence of one event does not affect the probability of the occurrence of another event. Two events are mutually exclusive if they cannot happen at the same time.

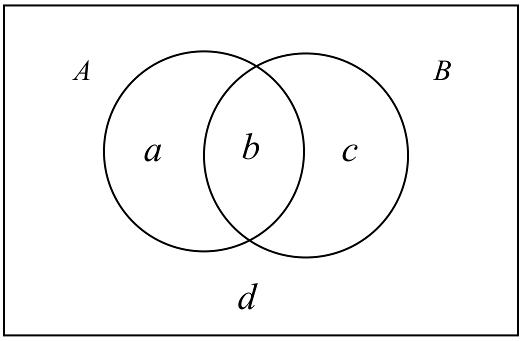
- In handling problems involved a finite number of outcomes, drawing a tree diagram is an efficient way to consider all the outcomes.

- Appropriate examples should be chosen to demonstrate the use of the laws in Learning Objective 10.2. The following are some examples:

<u>Example 1</u> A fair dice is thrown. Show that the event "the score is odd" and the event "the score is prime" are not independent.

<u>Example 2</u> A biased dice is thrown, with $P(1)=P(2)=P(3)=a$ and $P(4)=P(5)=P(6)=b$. Find the values of a and b when the event "the score is odd" and the event "the score is prime" are independent. (Answer: $a = \frac{\sqrt{3}-1}{3}$ and $b = \frac{2-\sqrt{3}}{3}$)

<u>Example 3</u> Let A and B be two events. In the Venn diagram below, a, b, c and d represent the numbers of elements in the corresponding parts.



(a) Find out the condition that events A and B are independent. (Answer: $ac = bd$)

(b) Now we have four numbers 10, 15, 30 and 45. If we assign them randomly to a, b, c and d in the Venn diagram, what is the probability that events A and B are independent? (Answer: $\frac{1}{3}$)

Learning Unit	Learning Objective	Time
Statistics Area		
Further Probability		
11. Bayes' theorem	11.1 use Bayes' theorem to solve simple problems	4

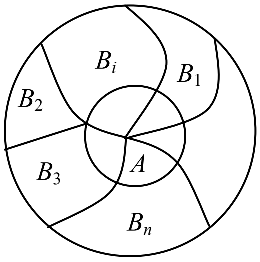
Explanatory Notes:

- Teachers can introduce Bayes' theorem from the definition of conditional probability. In general, $P(B \mid A)$ and $P(A \mid B)$ might not be equal. As $P(A \cap B) = P(B \mid A)\,P(A)$ and $P(A \cap B) = P(A \mid B)P(B)$, so $P(A \mid B)P(B) = P(B \mid A)P(A)$, that is, $P(B \mid A) = \dfrac{P(A \mid B)\,P(B)}{P(A)}$.

  This is the simplest case of Bayes' theorem. $P(B)$ and $P(B \mid A)$ are the prior probability and posterior probability respectively. Bayes' theorem states that if B_1, B_2, … , B_n are mutually exclusive and exhaustive events, then

  $$P(B_j \mid A) = \frac{P(A \cap B_j)}{\sum_{i=1}^{n} P(A \cap B_i)} = \frac{P(A \mid B_j)P(B_j)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}, \quad i = 1, 2, \ldots, n.$$

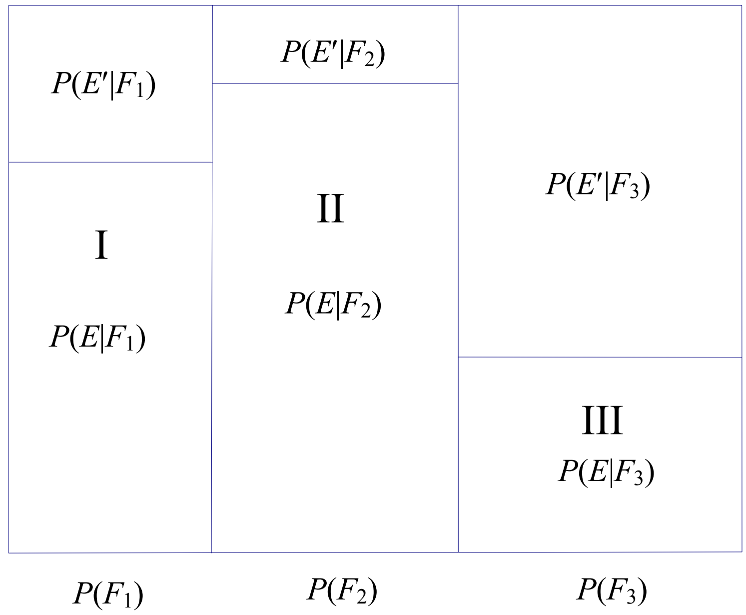
  Teachers can use the following diagram as an illustration.



- After students are acquainted with the concept of conditional probability, teachers can go further to Bayes' theorem. Before the teacher derives Bayes′ theorem, students may be guided to calculate the conditional probability from definition with the aid of a tree diagram and/or Venn diagram.

- Students are not expected to derive the theorem. In problems involving the applications of Bayes' theorem, tree diagrams are usually used.

- Instead of relying on tree diagrams, students may visualise Bayes' Theorem as a ratio of areas of rectangles by using an area model, which is basically a Venn diagram. For example, the case of $n = 3$ can be represented by the area model on next page:



where $P\left(F_j|E\right)=\dfrac{P\left(E\mid F_j\right)P\left(F_j\right)}{\sum_{i=1}^{3}P\left(E\mid F_i\right)P\left(F_i\right)}$

Learning Unit	Learning Objective	Time
Statistics Area		
Binomial, Geometric and Poisson Distributions		
12. Discrete random variables	12.1 recognise the concept of a discrete random variable	1

Explanatory Notes:

- Teachers may explain the concept of random experiments before introducing random variables. An experiment can be considered as random if it satisfies the following conditions:
  - (i) The experiment can be performed repeatedly under the same condition(s)
  - (ii) All possible outcomes of the experiment are obtainable and there are more than one possible outcome
  - (iii) The outcome of the experiment is unknown before the experiment

- Teachers can introduce the preliminary idea of a random variable by using simple examples such as tossing of coins (for discrete random variables) and life times of electric bulbs (for continuous random variables).

Learning Unit	Learning Objective	Time
Statistics Area		
Binomial, Geometric and Poisson Distributions		
13. Probability distribution, expectation and variance	13.1 recognise the concept of discrete probability distribution and its representation in the form of tables, graphs and mathematical formulae 13.2 recognise the concepts of expectation $E(X)$ and variance $\mathrm{Var}(X)$ and use them to solve simple problems 13.3 use the formulae $E(aX + b) = aE(X) + b$ and $\mathrm{Var}(aX + b) = a^2\ \mathrm{Var}(X)$ to solve simple problems	5

Explanatory Notes:

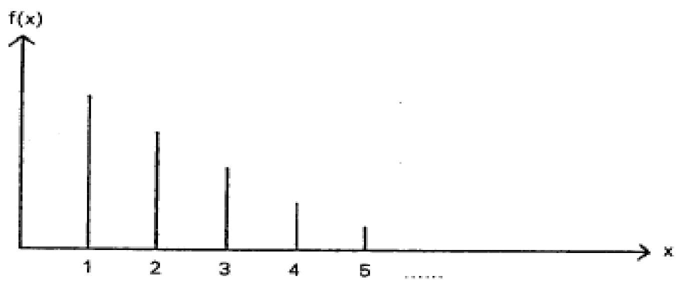
- The values of a random variable X and the corresponding probability of $P(X = x_i)$ are tabulated as follows:

X	x_1	x_2	$\dots$	x_i	$\dots$	x_n
$P(X = x_i)$	p_1	p_2	$\dots$	p_i	$\dots$	p_n

The above table is the probability distribution of the random variable X, where $0 \le p_i \le 1$, $i = 1, 2, \dots, n$ and $\sum_{i=1}^{n} p_i = 1$.

- Students should recognise that capital letters (e.g. X) are used to denote random variables and small letters (e.g. x) are used to denote the values of random variables.

- The discrete probability distribution can be represented by a bar chart.



- If we are given the probability distribution of a random variable, most properties of the random variable are immediate. However, it is hard to find the probability distributions of most random variables.

- In the Compulsory Part, students have learnt the meaning and applications of the mean and standard deviation. Teachers may briefly review these concepts before introducing the expectation and variance of a discrete random variable. Students should also be able to evaluate $E(X^2)$ and $E[X(X-1)]$.

- The variance of a random variable X is a measure of the stability and dispersion of X and is denoted by Var(X). Var(X) = $E[(X - \mu)^2] = \sum(x-\mu)^2 P(X=x) = \sigma^2$, where μ is the expectation of X.

- Students should understand and be able to apply the following properties of expectation and variance, where a and b are constants:
  (i) $E(a) = a$
  (ii) $E(aX) = a\,E(X)$
  (iii) $E(aX + b) = a\,E(X) + b$
  (iv) $\mathrm{Var}(a) = 0$
  (vi) $\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$
  (vi) $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$
  Students should be able to prove that $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$.

- To familiarise students with the properties of expectation or variance, teachers may ask students to conduct the following two experiments:
  Experiment 1 Roll two fair dice and the sum of the outcomes is denoted by the random variable X. Repeat the process n times.
  Experiment 2 Roll one dice. The scores obtained are doubled. The result is denoted by a random variable Y. Repeat the process n times.
  Suppose the data generated from Experiment 1 are $x_1, x_2, \ldots x_n$ and those from Experiment 2 are $y_1, y_2, \ldots y_n$. Students can calculate the expectation and the variance of the two sets of data. The probability distribution tables for the discrete random variables X and Y can then be drawn up.
  Teachers may ask the students to guess the relationship between the expectation of X and the expectation of Y, and the relationship between the variance of X and the variance of Y.

- To ensure that students know how to find the expectation for discrete random variables, teachers may ask the students to fill the following table with the numbers 0, $\frac{1}{9}$, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$ in some order (without repetition of the numbers) such that a probability distribution can be formed.

X			
$P(X=x)$			

  Teachers may ask the students to count the number of possible distributions and find out whether any of these distributions have equal expectations.

Learning Unit	Learning Objective	Time
Statistics Area		
Binomial, Geometric and Poisson Distributions		
14. Binomial distribution	14.1 recognise the concept and properties of the binomial distribution 14.2 calculate probabilities involving the binomial distribution	5

Explanatory Notes:

- Students should recognise that a binomial experiment has the following properties:
  - (i) There are n identical trials or observations.
  - (ii) There are only 2 possible outcomes for each trial, say S (for success) and F (for failure).
  - (iii) The probability of success (p) and the probability of failure ($1 - p$) remain the same for all trials.
  - (iv) The trials are independent.

- A binomial random variable, say X, is the number of successes in n trials. Students should note that $E(X) = np$ and $\mathrm{Var}(X) = np(1 - p)$, but the proofs of these 2 formulae are not required. The use of the binomial distribution table to find corresponding probabilities is not required.

- The following formula in EXCEL may be used to find the probability in a binomial distribution.

  BINOMDIST(r, n, p, T)

  <u>Example</u> $T = 0$: $X \sim B(10, 0.5)$

  BINOMDIST (2, 10, 0.5, 0) $\Rightarrow P(X = 2)$

  $T = 1$: (cumulative)

  BINOMDIST (2, 10, 0.5, 1) $\Rightarrow P(X \leq 2)$

Learning Unit	Learning Objective	Time
Statistics Area		
Binomial, Geometric and Poisson Distributions		
15. Geometric distribution	15.1 recognise the concept and properties of the geometric distribution 15.2 calculate probabilities involving the geometric distribution	4

Explanatory Notes:

- Students should be able to distinguish the geometric distribution and the binomial distribution. In a binomial distribution, the random variable is the number of successes in n trials (n is fixed beforehand). If the number of trials is not fixed and the experiment continues until a "success" occurs, the number of trials is a random variable. In this case, the only success is in the last trial. This probability distribution is called a geometric distribution.

- If X follows a geometric distribution with probability of success p for each trial, students should know that $E(X) = \frac{1}{p}$ and $\mathrm{Var}(X) = \frac{1-p}{p^2}$, but the proofs of these 2 formulae are not required.

- We may use the following formula in EXCEL to find the probability in the geometric distribution.

  NEGBINOMDIST(x, 1, p)

  Example NEGBINOMDIST (3, 1, 0.6)
  is the probability of having 3 failures prior to the first success in independent Bernoulli trials with probability of success 0.6.

Learning Unit	Learning Objective	Time
Statistics Area		
Binomial, Geometric and Poisson Distributions		
16. Poisson distribution	16.1 recognise the concept and properties of the Poisson distribution 16.2 calculate probabilities involving the Poisson distribution	4

Explanatory Notes:

- When $n \to \infty$ and $p \to 0$ with $np = \lambda$ = constant, the Poisson distribution can be approximated by a binomial distribution. The idea of the approximation can be introduced to more able students as an enrichment topic. However, this idea is beyond the scope of the curriculum.

- Students should know that a Poisson experiment has the following properties :
  (i) The number of successes in an interval is independent of the number of successes in other non-overlapping intervals.
  (ii) The probability of a single success occurring during an interval is proportional to the length of the time interval and does not depend on the number of successes occurring outside this time interval.
  (iii) The probability of more than one success in a very small interval is negligible.

- If X follows a Poisson distribution with λ as the mean number of occurrence in the interval, students should recognise that $E(X) = \lambda$ and $\mathrm{Var}(X) = \lambda$. The proofs of these formulae are not required. The use of the Poisson distribution table to find the corresponding probabilities is also not required.

- We may use the following formula in EXCEL to find the probability in Poisson distribution.

  POISSON (x, n, T).

  Example $T = 0$: $X \sim Po(4)$
  POISSON (2, 4, 0) $\Rightarrow P(X = 2)$
  $T = 1$: (cumulative)
  POISSON (2, 4, 1) $\Rightarrow P(X \leq 2)$

Learning Unit	Learning Objective	Time
Statistics Area		
Binomial, Geometric and Poisson Distributions		
17. Applications of binomial, geometric and Poisson distributions	17.1 use binomial, geometric and Poisson distributions to solve problems	5

Explanatory Notes:

- This Learning Unit focuses on the applications of different discrete probability distributions. It is not easy for students to identify the probability distribution of a random variable. Students should therefore have a good understanding of the characteristics of binomial, geometric and Poisson distributions.

- In the binomial distribution, the variance is less than the mean. In the Poisson distribution, the variance is equal to the mean. These facts provide clues for students on the identification of the two distributions. If several random samples are collected, an appropriate distribution may be chosen by comparing the mean and variance of each sample and comparing their local values.

Learning Unit	Learning Objective	Time
Statistics Area		
Normal Distribution		
18. Basic definition and properties	18.1 recognise the concepts of continuous random variables and continuous probability distributions, with reference to the normal distribution 18.2 recognise the concept and properties of the normal distribution	3

Explanatory Notes:

- Students should know the difference between the probability distribution of a discrete random variable and the probability distribution of a continuous random variable.

- The proofs of $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \mu$ and $\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \sigma^2$ are not required. However, students should know that the formulae $E(aX+b) = aE(X) + b$ and $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$ also apply to continuous random variables.

Learning Unit	Learning Objective	Time
Statistics Area		
Normal Distribution		
19. Standardisation of a normal variable and use of the standard normal table	19.1 standardise a normal variable and use the standard normal table to find probabilities involving the normal distribution.	2

Explanatory Notes:

- The statement "X follows a normal distribution with mean μ and variance σ^2" may be represented by $X \sim N(\mu, \sigma^2)$

- The standard normal distribution is a special case of the normal distribution with $\mu = 0$ and $\sigma = 1$ and is denoted by $X \sim N(0,1)$.

- Students should know that if $X \sim N(\mu, \sigma^2)$ and $Z = \dfrac{X - \mu}{\sigma}$, then

  (i) $Z \sim N(0, 1)$

  (ii) $E(Z) = 0$ and $\mathrm{Var}(Z) = 1$

  (iii) $P(a < X < b) = P(\dfrac{a-\mu}{\sigma} < \dfrac{X-\mu}{\sigma} < \dfrac{b-\mu}{\sigma}) = P(z_1 < Z < z_2)$

- Students are expected to use the standard normal table to find values like $P(Z > a)$, $P(Z \le b)$ and $P(a \le Z \le b)$.

- The following normal distribution formulae are available in EXCEL:
  - NORMDIST (x, μ, σ, T): For $X \sim N(\mu, \sigma^2)$, when $T = 1$, we get $P(X \le x)$
  - NORMINV(p, μ, σ): For $X \sim N(\mu, \sigma^2)$, we get the value of x such that $P(X \le x) = p$
  - NORMSDIST(z): For $Z \sim N(0,1)$, we get $P(Z \le z)$
  - NORMSINV(p): We get z such that $P(Z < z) = p$
  - STANDARDIZE (x, μ, σ): We get $Z = \dfrac{x - \mu}{\sigma}$

Learning Unit	Learning Objective	Time
Statistics Area		
Normal Distribution		
20. Applications of the normal distribution	20.1 find the values of $P(X > x_1)$, $P(X < x_2)$, $P(x_1 < X < x_2)$ and related probabilities, given the values of x_1, x_2, μ and σ, where $X \sim N(\mu, \sigma^2)$ 20.2 find the values of x, given the values of $P(X > x)$, $P(X < x)$, $P(a < X < x)$, $P(x < X < b)$ or a related probability, where $X \sim N(\mu, \sigma^2)$ 20.3 use the normal distribution to solve problems	7

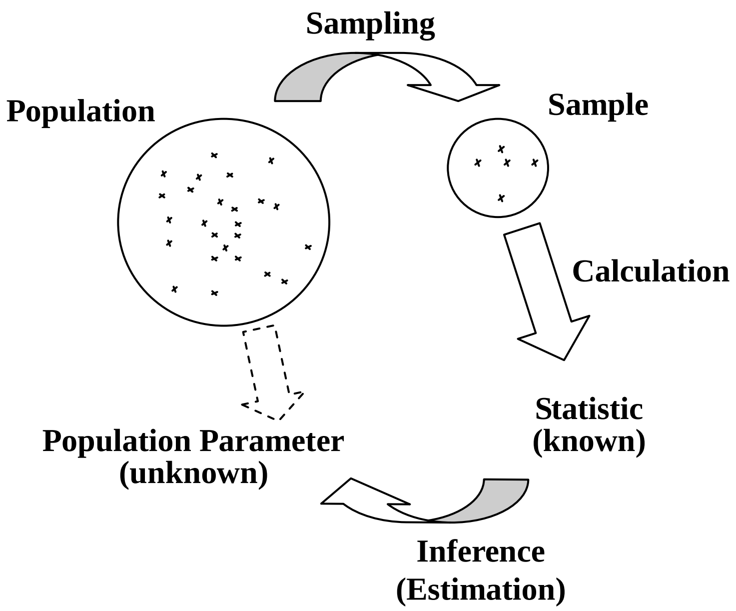
Explanatory Notes:

- Students do not need to recognise that the sum of scalar multiples of independent normal variables is also normal. For example, if $X_1 \sim N(8, 3^2)$, $X_2 \sim N(12, 4^2)$ and X_1 and X_2 are independent, students are not expected to know that the distribution of $Y = X_1 + X_2$ is normal and $Y \sim N(20, 5^2)$.

Learning Unit	Learning Objective	Time
Statistics Area		
Point and Interval Estimation		
21. Sampling distribution and point estimates	21.1 recognise the concepts of sample statistics and population parameters 21.2 recognise the sampling distribution of the sample mean from a random sample of size *n* 21.3 recognise the concept of point estimates including the sample mean, sample variance and sample proportion 21.4 recognise Central Limit Theorem	7

Explanatory Notes:

- Students should have learnt the concepts of "population" and "sample" in the Compulsory Part. Terms like population, sample, sampling, statistical inference, population parameter, statistic, sample mean, sample variance and sampling distribution of the sample mean should be introduced.



- Students are expected to know the following formulae:

  (i) sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{k} f_i x_i$, where $n = \sum_{i=1}^{k} f_i$

  (ii) sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{k} f_i (x_i - \bar{x})^2$, where $n = \sum_{i=1}^{k} f_i$

  (iii) The values of the sample mean $\bar{x}$ and sample variance s^2 will be respectively close to the population mean μ and population variance σ^2 when the sample size is sufficiently large.

  (iv) For a finite population, population variance $\sigma^2 = \frac{1}{N}\sum_{i=1}^{k} f_i (x_i - \mu)^2$, where N is the population size.

- Teachers may conduct some sampling activities and discuss with students:

  (i) The meaning of the mean of the sample means and the variance of the sample means.

  (ii) If the population is normally distributed with mean μ and variance σ^2, the mean of the sample means is normally distributed with mean μ and variance $\frac{\sigma^2}{n}$.

  (iii) The sampling distribution of the sample means approaches a normal distribution when n is sufficiently large. The population concerned is not necessarily normal.

- The following points might be highlighted in class:

  (i) A sample statistic is not necessarily the same as the corresponding population parameter, but it can provide good information about that parameter.

  (ii) Most sample statistics are close to the population parameters. Few are extremely larger or smaller than the corresponding population value.

  (iii) The goodness of a particular estimate is directly dependent on the size of the sample. In general, samples that are larger produce statistics that vary less from the population value.

- Point estimation is one of the methods of parameter estimation. It is worthwhile, at this stage, for teachers to introduce the concept of estimation of an unknown population parameter from a sample statistic. Examples like estimating a population mean μ by using a sample mean $\bar{x}$ can be used for illustration. Teachers should indicate to students that there may be several sample statistics which can be used as estimators. For example, the sample mean, sample median and sample mode could also be used to estimate the population mean μ.

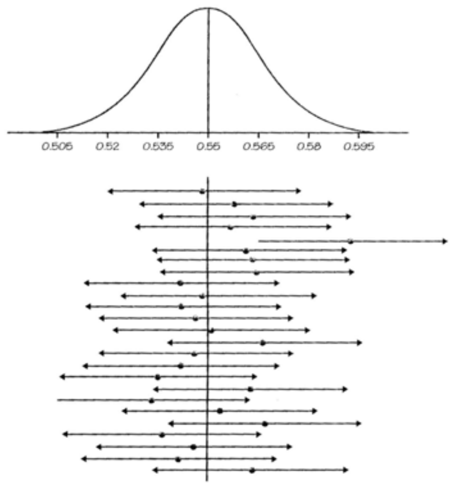
- In the process of sampling, different estimates are obtained from different samples. It is difficult to determine which estimator is the most suitable one. We may use the unbiased estimator to estimate the unknown parameter. It is expected that, in the long run, the average value of our estimates taken over a large number of samples should equal the population value: E(sample estimator) = population parameter. Teachers may show that $\overline{X}$ is an unbiased estimator of the population mean μ, but $\sigma_{\bar{x}}{}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ is not an unbiased estimator of the population variance σ^2. Hence, students should use $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$, as an unbiased estimator of the population variance.

- Central Limit Theorem is one of the most important and useful concepts in Statistics. It states that, given a distribution with a mean μ and a variance σ^2, the sampling distribution of the mean will be approximately normally distributed with mean μ and variance $\frac{\sigma^2}{n}$ when n (the sample size) is sufficiently large. However, the concept is too abstract for students. Teachers may use the interactive and simulation programmes on the Internet to illustrate the theorem.

- By simulation programmes, students may note that:
  (i) no matter what the shape of the original distribution is, the sampling distribution of the mean approaches a normal distribution as the sample size increases
  (ii) Most distributions approach a normal distribution very quickly as the sample size increases
  (iii) the number of samples is assumed to be infinite in a sampling distribution
  (iv) the spread of the distributions decreases as the sample size increases

Learning Unit	Learning Objective	Time
Statistics Area		
Point and Interval Estimation		
22. Confidence interval for a population mean	22.1 recognise the concept of confidence interval 22.2 find the confidence interval for a population mean	6

Explanatory Notes:



- A confidence interval for a population mean is an interval estimate of an unknown population parameter θ, based on a random sample from the population. The confidence interval is an abstract concept. Teachers may illustrate its meaning by computer simulation programmes or statistical software such as Winstat.

- Teachers should point out that the confidence interval is derived from random samples. 95% confidence interval and the 99% confidence interval are most commonly used.

- Before constructing a confidence interval for μ , it is essential to ask the following questions:
  - Are the random samples taken from a normal population?
  - Is the population variance known?
  - Is the sample size large enough?

- For a normal population, based on a sample of size n, the 95% confidence interval for the population mean μ with known variance σ^2 is ($\bar{x}-1.96\frac{\sigma}{\sqrt{n}}$) ， $\bar{x}+1.96\frac{\sigma}{\sqrt{n}}$).

  If $X \sim N(\mu, \sigma^2)$, then $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ and $0.95 = P(-1.96 \le Z \le 1.96) =$

  $P(-1.96 \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le 1.96)$. It should be noted that the results are true for samples of any sizes.

- For a non-normal population, the 95% confidence interval for the population mean μ with a known variance σ^2 and large n (say, $n \geq 30$) is $\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$. As the sample size is large, Central Limit Theorem can be used. $\overline{X}$ is approximately normal and $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.

- For a normal or non-normal population, the 95% confidence interval for the population mean μ with unknown variance σ^2 and large n (say, $n \geq 30$) is $\left(\bar{x} - 1.96\frac{s}{\sqrt{n}},\ \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$, where s^2 represents the sample variance and $P\left(-1.96 \leq \frac{\overline{X} - \mu}{s/\sqrt{n}} \leq 1.96\right) \approx 0.95$, where $\frac{s}{\sqrt{n}}$ is called the standard error of the sample and is denoted by $SE(\bar{x})$.

- Students should be able to evaluate the confidence interval for the population mean μ under the following situations:

Conditions	95% confidence interval for μ	99% confidence interval for μ
Normal population ▪ with known variance σ^2 ▪ large or small sample size n ▪ sample mean $\bar{x}$	$(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}})$	$(\bar{x} - 2.575\frac{\sigma}{\sqrt{n}},\ \bar{x} + 2.575\frac{\sigma}{\sqrt{n}})$
Non-normal population ▪ with known variance σ^2 ▪ large sample size n ($n \geq 30$) ▪ sample mean $\bar{x}$	$(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}})$	$(\bar{x} - 2.575\frac{\sigma}{\sqrt{n}},\ \bar{x} + 2.575\frac{\sigma}{\sqrt{n}})$
Non-normal population ▪ with unknown variance σ^2 ▪ large sample size n ($n \geq 30$) ▪ sample mean $\bar{x}$ ▪ sample variance s^2	$(\bar{x} - 1.96\frac{s}{\sqrt{n}},\ \bar{x} + 1.96\frac{s}{\sqrt{n}})$	$(\bar{x} - 2.575\frac{s}{\sqrt{n}},\ \bar{x} + 2.575\frac{s}{\sqrt{n}})$

- Students should know that the width of the confidence interval can be reduced:
  - by increasing the sample size
  - by decreasing the confidence level (e.g. choosing a confidence level of 95% instead of 99%)

- In constructing a confidence interval, it is desirable to have a narrow width (for a more precise estimate) with a high level of confidence, but in most cases, we cannot attain these two conditions at the same time.

- If random samples are independently taken from a population and 95% confidence intervals constructed for each sample, students may expect about 5% of the intervals do not include the population parameter. When students calculate their confidence intervals, they will not know whether the parameter is included in these intervals or not.

Learning Unit	Learning Objective	Time
Statistics Area		
Point and Interval Estimation		
23. Confidence interval for a population proportion	23.1 find an approximate confidence interval for a population proportion	3

Explanatory Notes:

- The following key points should be stressed during the lesson:
  A number of random samples, each of size n, are drawn from a parent population. If the proportion of successes for each sample is p_s, these proportions form a distribution called the sampling distribution of proportions P_s. When n is sufficiently large, the distribution of P_s is approximately normal and $P_s \sim N\left(p, \frac{p(1-p)}{n}\right)$, where $\sqrt{\frac{p(1-p)}{n}}$ is the standard error of the proportions. The larger the sample size n is, the better is the approximation. Since p is not known, we use p_s to approximate p.

- Students should be able to evaluate the approximate confidence interval for the population proportion p:

	Conditions
	large sample size n and sample proportion p_s
95% confidence interval	$\left(p_s - 1.96\sqrt{\frac{p_s(1-p_s)}{n}}, p_s + 1.96\sqrt{\frac{p_s(1-p_s)}{n}}\right)$
99% confidence interval	$\left(p_s - 2.575\sqrt{\frac{p_s(1-p_s)}{n}}, p_s + 2.575\sqrt{\frac{p_s(1-p_s)}{n}}\right)$

Learning Unit	Learning Objective	Time
Further Learning Unit		
24. Inquiry and investigation	Through various learning activities, discover and construct knowledge, further improve the ability to inquire, communicate, reason and conceptualise mathematical concepts	10

Explanatory Notes:

This Learning Unit aims at providing students with more opportunities to engage in the activities that avail themselves of discovering and constructing knowledge, further improving their abilities to inquire, communicate, reason and conceptualise mathematical concepts when studying other Learning Units. In other words, this is not an independent and isolated Learning Unit and the activities may be conducted in different stages of a lesson, such as motivation, development, consolidation or assessment.

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CDC Committee on Mathematics Education

CDC-HKEAA Committee on Mathematics Education (Senior Secondary)

CDC-HKEAA Working Group on Senior Secondary Mathematics Curriculum (Module 1)