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MATHEMATICS SECTION

EDUCATION DEPARTMENT

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PREFACE

The School Mathematics Newsletter (S.M.N.) is for teachers. The articles are contributed mainly by teachers. Your keen support by sending in articles relating to mathematics teaching etc is vital to the success of this publication.

School teachers are therefore requested to give their full support to S.M.N. In particular, we welcome views, opinions, experience, critiques on a number of issues such as the JSEA Scaling Test, the examination syllabuses and the new subject "Computer Studies".

Articles need not be typed. All contributions should be sent to the Editor, School Mathematics Newsletter, Mathematics Section, Advisory Inspectorate, Education Department, Lee Gardens, Hong Kong.

May I take this opportunity to thank all who have contributed to this issue of S.M.N.

S.B. Teng
Principal Inspector
(Mathematics)

Please ensure that every member of your mathematics staff has an opportunity to read this Newsletter.

Personal Reflections on Computer Education in Secondary School

delivered by Sr. Margaret Wong at a seminar organized by ONFLO INTERNATIONAL at Grantham College on 27th February, 1982.

Good morning, Ladies and gentlemen,

I am very happy here today to share with you some of the practical experiences and personal reflections on Computer teaching in secondary schools. Before I begin, I must make it clear that I am not a computer expert myself; in fact, I know only very little about computers. The only reason why I am here today is because I am very keen and interested in introducing computer studies in secondary schools. Perhaps it is just this little spark of interest and enthusiasm that has got me standing up here to share with you some of the reflections and observations I have made on this subject. I have no doubt that many of you present here may have greater insight and experience in this subject, I hope that after my introductory sharing, you will also be kind and generous enough to pop up and share with us your insight and experience.

Today, my sharing will consist of 3 parts. First of all, I am going to share with you how we have introduced this subject into our school, this will be followed by a sharing on what I have observed early this year in England with regard to Computer Education there. Then finally, I will assess the H.K. position of Computer Education in Secondary Schools.

This Computer adventure began in May, 1980 when the Chief Inspector of Mathematics of the Education Department, Mr. Parkin, visited our school, inviting us to join in the Pilot Scheme of Computer Studies. To get me interested in the project, he brought me to his office and demonstrated to me all the things a computer could do. In an hour or so, he proved to me the power and efficiency of the machine as well as the entertainment and challenges one can get out of it. I was greatly impressed and fascinated by it all and agreed to consider joining the scheme.

Of course, it is no small issue for a school like ours, with 20 years of history to introduce a new technological subject into the curriculum. There are many factors to be considered. First and foremost; I must justify to myself whether the introduction of this particular subject into the curriculum would increase the worth-whileness and relevancy of the educational experiences of the students. As literacy and numbracy were required in the past, some form of computeracy will definitely be needed for the future. Like it or not, cybernetics will definitely dominate many areas of man's life. A good way to break through its domination is to have sufficient basic knowledge of it so as to pierce through its mystification and subordinate it to its true status as a good and faithful servant. To have knowledge of a machine is a way of gaining control over it, demanding it to do what we want done. In a fast expanding society like H.K., where commercial and technological skills are the only living resources available to us, it is important that we educate our next generation to be computerate to some degree, so as to make some sense of the rapidly changing computerized world around them.

Having considered the profitableness of including this subject into the curriculum, there are still many other more subtle and practical problems to be solved. There is the personnel problem, the financial problem, space problem, time-tabling, equipment and fixture problems, problems of syllabus and so on.

It happens that I am a very fortunate Principal who has a large group of very enthusiastic, efficient, co-operative, dedicated and experienced professional teachers in the staff. A number of them have had computer experience before, so I have no problem in finding teachers to teach the subject. Even those who were not appointed to teach the subject attended a preliminary course on computers organized by the Education Department during the Summer Vacation, 1980.

As the whole school was being re-wired and re-decorated in the Summer Vacation 1980, I took the opportunity and made a quick decision of transforming a large storeroom on the Lower Ground of the building into a Computer Demonstration Room with a number of 13 Amp-plugs and fixed side benches against the walls for the computers. To begin with, a set of local-made micro-computer costing around $3,500 was bought in Sept., 1980. Shortly after the opening of the term, the Prefect Board donated two sets of micro-computers and a printer to the school in honour of its 20th Anniversary. Consequently the initial financial cost turned out to be almost minimal for the School. We therefore started off very humbly with 3 sets of micro-computers and a printer. We did not want to invest any more than that on the project because money to be spent is always limited and hard to come by and also because we have great hopes that the Education Department will eventually take us as one of the pilot schools and provide us with the necessary equipments to run the curriculum properly.

Since the pupils of the school are exceptionally strong in the Sciences and in Mathematics, after due consultation with the various panel heads, it was decided that two out of the four periods of Additional Mathematics for the two Science Streams would be allocated to the study of Computer Studies at Form Four Level. As there were no examination nor teaching syllabus, the teacher felt quite free to experiment and to explore with the pupils. The response of the girls was tremendous. They loved it and were very good at it. They were not only able to grasp the basic concepts but were also able to apply them in their daily lives. The decision of the Prefect Board to donate two micro-computers and a printer to the school showed in action the pupil's response and interest in this subject.

The computer language we use is BASIC and most of the lessons are in lecture form with demonstration by the teacher. Each student, apart from attending the normal lesson, is allocated a Computer time of approximately one hour outside school hours once every fortnight to do her homework or practical work. At the end of the year, a group of three to four pupils must hand in a project for assessment beside the formal written examination of two-hours duration. The project topics that they chose were very varied. They ranged from simple telephone bills, pay rolls, taxation to space landing, traffic light control systems, graphic designs, songs, and calendars.

In June and August, 1981, short intensive courses of one month duration were run for the F.5 and F.7 graduates who had just completed their public examinations. Past students who were interested could also join in the course. So we were really making good use of the very limited resources that we have. F.5 and F.7 pupils were debarred from the Computer Room before their public examinations because I am quite convinced that computer could be highly addictive and that it would be wiser not to introduce it to pupils who have to sit for public examinations.

Beginning in Sept., 1981 some students from an Arts Stream joined in the Computer Studies course by choice. This is because there is a basic assumption that this discipline is suitable for all pupils, Arts and Science alike, provided that they are capable of logical thinking and have a good command of English. I hope in Sept., 1982 this subject will be introduced into the common core curriculum of Form Three so that every pupil will have a fore-taste of what computer studies is like before choosing to study it as an examination subject at F.4-5 level. The only contact the F.3 pupils has at the moment is through the Computer Club which offers them weekly computer games to play with. This is not very ideal because playing computer games and writing out computer programmes demand very different abilities and aptitudes. By including Computer Studies into the Form 3 curriculum, each pupil will come to know what a computer can do and what it cannot do; hence eliminate the fear of having computers dominati one's life. With some basic theory and practice, computers can in fact become a very useful tool doing whatever we want them to, accurately and faithfully in an untiring manner. There basic theory and practice, I think, are very simply presented in a Thomas Nelson publication "Introducing Computers" (Author Peter Bishop). I therefore have every intention of adopting this book for use at F.3 level.

The most and most difficult problem to be solved is to decide which subject Computer Studies is to replace and what sort of combination of subjects would provide a most worthwhile and profitable educational experience for the students. At the moment, each student in our school is already taking nine examination subjects which is the maximum number allowed; so one has to decide which subject to give up in order to give place to this new examination subject. Considerations on whether this new subject will interrupt the continuity of one's post secondary education need also be weighed carefully because there is as yet no Form Six syllabus worked out in this subject. I think that a lot of dialogues and discussions are needed among the staff before a wise decision could be reached. One must be careful that the creation of a new department will not engender too much unhappiness for some members of staff and that those responsible for the subject being replaced will not feel being slighted or threatened by redundancy. A certain amount of internal politics of the school might be involved as this new subject may expand into a most powerful and prestigious department. An awareness of the climate of thinking and feelings among both pupils and staff may help one to gear the speed of development of this new department. I envisage that as more teachers get trained in the use of computers and when more computers and soft ware materials will be available for use in the school, this new technology will surely boom. At such time, this will not only be limited to being studied as a subject but will be used in the administration and as a tool for the teaching of the Sciences and Languages as well as other subjects; particularly as an aid for remedial work in the form of programmed learning which we are badly in need at the moment.

After having run the curriculum for 2 years, I am glad to say that the experience has been a most rewarding one for all concerned, Principal, Staff and students alike. The only dissatisfaction I have is with the maintenance of the micro-computers. I want the firm to draw up a maintenance contract with the school so that we can get immediate and proper service when the machines break down. But the firm does not seem to be in a hurry to collect our maintenance fees, even up till now, the contract has not yet been drawn up. So when a machine does break down, we have to cart our computers back to the firm to be fixed. They simply cannot manage to send a staff down to repair it for us. The only consolation we have is that the machines do not break down that often.

Since most of the times the girls were entirely on their own when they work in the Computer Room and that the printer and printing paper are lying there freely for use, we made it a point that they have to drop in a ten cents coin into a box after having printed a sheet so that they would not waste paper thoughtlessly. This method seems to work very well, no wastage was found and surprisingly enough, the amount of paper used is often equivalent to the coins collected. So indirectly, we help to instill the moral virtues of thrift and honesty into the students through the use of computers. Charity, understanding, tolerance, self-discipline and co-operativeness are also demanded of the students in the use of the computers. It is not easy to share out the computer time to over one hundred students with only 3 sets of computers. However, they did it and there had never been a row or fight over the use of computers, (as far as I know). This in part must be attributed to Mr. S.M. Lai here, who had helped to distribute the booking time to the students.

I can also feel that a sense of responsibility has also been promoted and strengthened through this curriculum. The machines, lights and fans had not yet been found left unattended after use and although many students handle the computer room key, not one has yet been lost. So cross fingers, so far so good on our school experience. Now I turn to the English experience.

Early this year, I've just been to England to attend a Conference on Science Education in which the topic of Computer teaching also popped up and in which I had some very good and enlightening sharing with the educators in England. While being there, I also managed to visit some secondary schools, both state and independent, that offer Computer Science to the students.

Computer Science had been introduced into secondary schools in England for over ten years but it is still not very widespread. Not too many schools are offering this subject to the students. The main reason may be due to a lack of teachers who are ready to teach it and also for those who are qualified to teach, they are extremely critical over the present O level exam syllabus. Teachers claim that the syllabus is such that students can easily obtain a pass on Computer Science without ever touching a computer i.e. with no computer experience at all. This to them is absolutely ridiculous and unacceptable. However, they are being blackmailed to take the syllabus because this is the only way in which one can get a computer for the school. Generally, the teachers there are not too keen on taking Computer Science as a subject but rather prefer to use it as a tool. They also suggest that the teaching of electronics should go hand in hand with Computer Science. When I heard of this, I was a bit stunted, knowing that our H.K. syllabus took shape from the U.K. syllabus, and wondered if we had not again taken on something which others had discarded.

My fear dispelled as I showed them our proposed syllabus. They recommended it very highly and commented that it is very well done and thorough; so a word of congratulations to all those who have helped to draft the syllabus. They were absolutely impressed as they learned a bit more on what has been done in our school and what we will be doing in H.K. and therefore requested me to contribute an article to the Journal of MUSE "Computers in schools" on Computer Education in Hong Kong, which you can find among the notes in your folder in the article "Introducing Computer Teaching in a Secondary School in H .K.".

As I have said, I had visited some of their schools. I found great discrepancies of provision and teaching standard between a State and an Independent School. Most of the State schools possess only a BBC Research Machine and a printer. Nearly all the teachers were not originally trained for teaching this subject. They are mainly drawn from the Maths. panel. Teachers chosen to teach computers just pick up bits and pieces of computer knowledge and experience from some in-service workshops or courses. Their knowledge of computers, if I may say so, is more or less equivalent to that of mine only. No wonder every one of them told me that within a very short time they found the students doing much better than they. At first I thought that the Computer Room we had in St. Paul's is rather primitive, but after seeing theirs, I realize that ours is in fact rather posh; a sort of middle class arrangement. However, as I stepped into an independent school I have a totally different picture. The independent school I went to had nearly a whole wing designated to the Computer Department. The Computer Room was so large that after having the benches all fixed around it, there still left enough room in the middle for a class of over 30 to take place there.

There are 8 sets of Commodore machines and two sets of Apple in that room. Next to this, there's a room which serves as a kind of computer library with programmes, magazines and books. Further on, there's another room for the CPU. The teacher there was a real expert in computer studies. He was a Computer Science graduate himself and had had two years of industrial experience before he joined the staff not long ago. In comparison with the teachers serving in the State school, he was most confident and knowledgeable in the subject matter; as a result computer teaching flourishes in that school. This school offers computer studies right from Form One. The teacher there noted that there are 8 times more boys than girls taking Computer Studies as an examination subject. I wonder will the same trend exist in H.K. since girls are often thought less adept than boys in Science and Technology.

Having seen the set up in England, I feel rather happy and optimistic on what is going on in Hong Kong with regard to Computer Education. I must say that our Education Department is much more generous in equipping the pilot schools than the Education Authority in England. This, of course, may also be due to the large difference of numbers of secondary schools in the two places.

Coming back to H.K., I think it is in fact fairly cheap to introduce this subject when compared with other practical or science subjects. If I remember correctly, I spent only around HK$6,000.00 to transform a storeroom into a Computer Room, while I spent over HK$30,000.00 to transform a classroom into an Art Room; and over HK$77,000.00 to set up a Junior Science Room and its store. Those who are teaching Physics will know that just a rectifier nowwould have cost HK$12,000.00, which in actual fact, is comparable to what one needs to get started with computer studies in a small scale. So all in all, computer studies is cheap, useful and relevant.

It is cheap in comparison with other practical or science subjects. It is no doubt useful because computer helps us to create new opportunities, extend our capabilities by releasing us from monotonous jobs and increasing our efficiencies. It is relevant because computers will be increasingly used in more and more applications and affect more and more people.

In order to be successful in introducing Computer Education in Secondary Schools, it is important that there be adequate provision of facilities and enough trained teachers ready to teach the subject. I am sure student's interest will not be lacking as most youngsters in H.K. are very keen and capable in learning, particularly in something as new and as challenging as the computer. With regard to facilities, I presume that E.D. will take upon themselves to provide for the schools whatever is necessary to run the curriculum. The details of help which will be given to schools will be delineated by Mr. Joseph Shin from the Education Department in a short while in Computer Studies Pilot Scheme.

With regard to the training of teachers I think much thought must be given to it if computer education is to succeed in Hong Kong. During the past two years, the Education Department has run a number of short in-service courses for teachers in secondary schools. The course content ranges from very simple concepts such as Assembly Language, Basic, Fortran, Data Processing to Advanced Programming. The Colleges of Education have not yet introduced Computer Studies as an elective subject for the students although they spend 40% of their time on computer work with the Mathematics students. This, to me, is not satisfactory as it gives the impression that Computer Studies and Mathematics require similar abilities and aptitudes in its students. While the School of Education of the Hong Kong University had already introduced Computer Studies as a minor subject for the teachers, the Chinese University simply offered Computer Studies as a Major subject for a B.Sc. Degree. Moreover, the Chinese University offers highly commended intensive one-year extra-mural courses to graduate teachers of secondary schools. The curriculum of this course is equivalent in level to a B.Sc. Minor in Computer Studies and in practice is specially tailored to the teaching of the Hong Kong School Certificate Level Syllabus. Hence, this is very useful for intending teachers. Polytechnic graduates trained in Computers normally go into industry or private enterprises, therefore they do not contribute directly to Computer Education. Owners of Private Computer Firms and the H.K. Productivity Centre also produce many training courses for beginners in Computer work and for the Banking and Commercial field. They play their part in Computer Education by keeping interest alive through seminars, exhibitions and talks for both students and teachers as well as for the general public.

Much efforts have been made by interested parties to help popularize Computer Education in schools. There is the formation of the Hong Kong Computer Education whose aim is to help develop Computer Education in schools; to allow sharing of experiences among its members, to build up a system of exchange of programmes and softwares and to alert members to the kind of services one should expect from the computer firms. The detailed description of this Association and its constitutions will be dealt with by Mr. Lai Suk Ming later in the morning. So to end, I must say that with a united effort from different quarters to promote Computer Education in Secondary Schools, I am confident and certain that it will expand and grow from strength to strength. If we but have the courage to follow and go where our immagination and vision will take us, Computer Education will inevitably flourish and leave its indelible impact in schools and in society at large. Thank you.

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一九八二年五月

微積分的故事

——數學發展史的一個範例

蕭文强*

1. 前言

微積分是一門比較年青的數學學科（它只有三百多年的歷史！），然而它的基本思想早已孕育於古代東西方數學家的先驅工作中。微積分是數學史上，甚至是人類思想史上的一頁輝煌成就，對後世的科學、技術、哲學思想都有影響。同時，它的發展過程也反映了數學發展過程的一些規律。本文是筆者由市政局科學館及香港科學協進會合辦的一次普及科學講座的講稿整理而成，試圖以淺明的語言，從微積分的浩瀚發展史裏面執取一些有趣的片斷來說明以上那一點，並且也勾劃出這門學科的一個粗略的輪廓。

讓我們先講一個小故事：十九世紀英國數學家棣摩甘（DeMorgan）有一天向朋友解釋怎樣估計一羣人當中在若干年後有多少人依然活着，他寫下了一條數學公式。他的朋友看了，指着公式裏一個符號問他那代表什麼，他回答說："這是π，即是一個圓的圓周和它的直徑的比率。"他的朋友聽罷，一臉不相信的神色，對他說："看在老朋友份上，不要捉弄我吧，有多少人活着，又與圓何干？"

這個小故事是眞是假並不重要，重要的是它說明了一點，即是數學的用途很廣，很多數學上的概念，往往在意想不到的地方出現。正是這個一直困擾着哲學家、數學家、科學家的現象，使數學成爲人類從古至今用來瞭解他們所處的世界裏各種現象的利器。微積分就是其中一件這樣的利器。我們在低年級的時候，聽見"微積分"便都"肅然起敬"，覺得那是十分高深的學問，有些人甚至以爲那是高等數學的頂峯了。其實，它只不過是高等數學的開始。從有文獻記載以迄今天，數學已經有四千多年的歷史，微積分作爲一門具備體系的學科，卻只有三百多年的歷史。然而在過去三百多年間，數學發展之迅猛蓬勃，卻遠遠超乎在這之前那三千七百多年！難怪有人說過："數學的黃金時代——那可不是歐幾里得的時代，那正是我們的時代。"

2. 微積分的名稱

微積分的英文詞 Calculus，拉丁文原意是"石子"，意指計算，因爲古代歐洲人用石子來幫助計算的。以下我們會再談爲什麼用了這樣的一個名字，固然，它也有它恰當的意思，但它也會引起誤解，使人以爲微積分就是機械化的計算而已。

至於"微積分"這個中文詞，最早見諸清代數學家李善蘭和英國人偉烈亞力（Alexander Wylie）在1859年合譯的《代微積拾級》（英文原本是美國數學家羅密士（Elias Loomis）在1850年著的"Analytical Geometry and Calculus"）。李善蘭在譯序中說："是書。先代數。次微分。次積分。由易而難。若階級之漸升。譯既竣。即名之曰代微積拾級。"又說："我朝康熙時。西國來本之奈端（註：今譯萊布尼茲和牛頓）二家又創微分積分二術。……其理大要。凡線面體皆設由小漸大。一刹那中所增之積即微分也。其全積即積分也。"這就是我國微積分名稱的來由。

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3. 微積分的應用

微積分在很多學科上都找着了應用，不只在物理科學、材料科學、工程科學上有用，在生物科學、社會科學、企業管理上也開始大展身手。在它發展的初期，曾有兩項重大的勝利，雖然大家或者已經對之耳熟能詳，但重提一下也是值得的。

第一項是在十七世紀中葉牛頓（Newton）利用他剛建立起來的微積分去闡述他的力學體系，解釋了天體運行的規律，爲後世科學家對大自然規律之探索打開了一個新局面。十八世紀初英國詩人蒲柏（Alexander Pope）曾寫下這樣的句子：Nature and Nature's laws lay hid in night; God said, 'Let Newton be', and all was light.

第二項是在十九世紀中葉麥克斯韋（Maxwell）把當時所知的電磁現象通過數學處理總結爲著名的麥克斯韋微分方程，從而在數學理論上推斷有電磁波的存在，促使科學家在實驗室裏找尋它。隔了二十四年後，赫兹(Hertz)終於找到了電磁波，再過七年後，馬可尼(Marconi)還利用它實現了無線電通訊的夢想。赫兹曾經有如下的贊嘆："我們有個感覺，這些數學方程有它們獨立的存在和自身的理解力。它們比我們聰明，甚至比發現它們的人還要聰明，因爲我們從它們所取的比放進去的還要多。"

4. "天才的時代"？

提起微積分，很多人便想起牛頓和萊布尼兹（Leibniz），因爲通常書本上都說他們兩人是微積分的發明人。牛頓是英國人，萊布尼兹是德國人，都是十七世紀的卓越博學之士。兩人各自在差不多同時期建立了微積分的體系，正因爲兩人在差不多同時期作出同樣重大的貢獻，便引致一場"誰抄襲誰"的爭論，甚至因而導致其後幾乎一百年間英國數學家和歐陸數學家不相往還，使英國的數學水平在整整一個世紀中沉寂不前！這是數學史上一件非常不幸的事，也是一個發人深省的教訓。更叫人嘆息的，這場爭論還不是由"當事人"發起，而是由一位對萊布尼兹懷有敵意的瑞士數學家挑起，演變爲狹隘民族主義者各擁一方的罵戰。但兩位"當事人"無可避免地給捲進了漩渦，他們的心情也不會好過。事實上，正所謂"識英雄重英雄"，萊布尼兹對牛頓曾給予很高的推崇，他說："自天地初開以至牛頓活着的時代，全部數學中牛頓佔了一大半功勞。"

其實，這種"爭認第一"的爭論，是十分無謂的。根本上，十七世紀湧現了一大批卓越博學之士，他們當中不少人對微積分都或多或少作了重要的貢獻，牛頓和萊布尼兹乃總其大成，畫龍點睛吧。我這樣說，絕對沒有絲毫看輕這兩位偉大數學家的貢獻，我只想說，我們也不應該忘記他們所處的時代。我們將在下文看到，微積分到了十七世紀中葉，基本概念都差不多具備，已經到了"呼之欲出"的地步了！

有人把十七世紀稱爲"天才的時代"。其實，所謂"天才"，只不過指他們比一般人有更敏銳的洞察力，有更高遠的見識，也比一般人下更多的苦功，比一般人更專心致志於熱切求眞知。於是，他們把那個時代提供給他們的條件盡量發揮，爲後世作出他們所能作出的最大的貢獻。牛頓便曾說過："倘若我比別人看得更遠一些，那是因爲我站在巨人肩膊之上。"又當別人問及他怎樣有所發現時，他簡要地答道："不斷地思考。"的確，牛頓經常每天工作十八至十九個小時，有時還長期守在實驗室裏徹夜不眠！

5. 亞基米得的工作

以下我不能再避免使用一些數學上的討論了，但爲了顧及讀者的數學程度不一，我只好取其要義棄其支節。幸好數學上的重要基本概念，都是簡單的，若不講求細節，即使用日常語言

也可以勾劃出一個粗疏的輪廓來。

在幾千年前的東西方古代數學文獻界，已經出現不少面積體積的計算，但直至古希臘的數學家，才給這些公式加以証明。其中最卓越的貢獻來自公元前三世紀的亞基米得(Archimedes)。他不只以即使用今天尺度也稱得上嚴謹的演譯推理証明公式的正確性，更叫人佩服的，是他運用靈活的直覺想像、類比推測、甚至借助其他學科之長來發現很多深刻的結果。

亞基米得是用一種叫做"窮竭法"(Method of exhaustion)的辦法來証明他的公式，這辦法是基於一個由公元前四世紀的希臘數學家歐多克斯(Eudoxus)提出來的原理：如果從一個數量減去至少一半，再從剩下的數量又減去至少一半，這樣做下去，經過足夠多次後，剩下來的數量可以任意地小。為了簡單地表達"窮竭法"的中心思想，讓我講述一個較簡單的定理，那不是由亞基米得發現的公式，但也相當有名，就是歐幾里得(Euclid)著的《原本》("Elements")第十二卷第二條定理：兩圓面積之比等於它們直徑平方之比。用現代數學語言，就是說兩圓的面積 a 和 A 及直徑 d 和 D 滿足等式 $a/A = d^2/D^2$。假設等式不成立，便有 a/A 大於 d^2/D^2 或 a/A 小於 d^2/D^2，我們將要証明每一個情況也導致謬誤，所以只能等式成立。因為兩個情況的推理是完全相似的，讓我只解釋前者。假如 a/A 大於 d^2/D^2，便必有小於 a 的 a1 使 $a1/A = d^2/D^2$，把 a-a1 叫做 e。考慮每個圓的內接正N邊形(譬如從內接三角形開始)，每次把邊的數目加一倍，得出內接正 2N 邊形。當內接正多邊形有N條邊時，它們的面積分別是 p(N)和P(N)，已知 p(N)/P(N) $= d^2/D^2$，這是一個比較容易証明的幾何上的結果，所以 p(N)/P(N) $= a1/A$。從N邊形擴大至 2N 邊形的過程中，圓與內接正多邊形的面積之差減去至少一半，所以按照歐多克斯原理，經過足夠多次後，a-p(N)便小於 e= a-a1 了，即是 p(N)大於 a1，所以P(N)也大於A，但內接正N邊形在圓的裏面，又怎麼可以有更大的面積呢？這就是謬誤了。

其實，歐多克斯原理已經蘊含了後世微積分的"無窮小"(infinitesimal)和"極限"(limit)的概念，但它卻以有窮的形式來描述。這樣做雖是嚴謹，卻掩蓋了"無窮"這個關鍵的本質。微積分不能在更早的時候誕生，其中一個原因，就是古代數學家這種廻避"無窮"的心態。不過，二千多年前的人有這樣嚴密的推理，卻不能不叫人佩服。亞基米得就是用這種辦法証明了許多面積體積的公式，但有一點使人大惑不解的，就是這種辦法不能用來發現公式是什麼，而若事前不知公式是什麼，這種辦法卻"無所施其技"！那麼，難道亞基米得是未卜先知的神仙嗎？這個謎直到1906年才被打破，那一年有位專研究古希臘數學的德國學者海堡(Heiberg)在君士坦丁堡(即今之伊斯坦布爾)一所寺院裏找到一張羊皮紙，上面有在十三世紀手抄的經文，但經文下面卻隱約可見別的字跡。原來那是在十世紀手抄的另一份文獻，收藏至十三世紀時，僧侶認為沒用便把它洗去，用那羊皮紙來抄經文。海堡是一位非常細心而勤懇的學者，他花了很多工夫仔細審閱那被洗去的文字，果然"皇天不負有心人"，那竟然是亞基米得失傳之作，是他寫給另一位數學家厄拉多塞(Eratosthenes)的信，解釋他如何發現面積體積公式的方法，故後世稱這份文獻為《方法》("The method")。原來亞基米得把圖形(物體)掛在一個假想天平的一端，他把圖形(物體)看成由線(面)組成，把每條線(每塊面)從天平的一端移到另一端，巧妙地掛在適當的地方，使兩頭平衡，然後利用力矩原理，他計算出原來圖形(物體)的面(體)積。

亞基米得自己最欣賞的一條定理，是關於一個恰巧套着圓球的圓柱，他說圓球的表面積是圓柱的表面積(連面和底)的三份二，圓球的體積也是圓柱的體積的三份二。他甚至吩咐親人，在他死後，把這個圖形刻在他的墓碑上。亞基米得之死，是人類文化史上一個極富寓意的悲劇。相傳羅馬大將馬塞流斯(Marcellus)攻陷叙拉古(Syracuse，在今意大利南部，當時亞基

米得居住的城市），士兵衝入一所房子，看見一位老人正在沙盆上畫圖沉思，士兵弄亂了圖。老人怒曰："不要弄亂我的圖！"士兵凶性頓發，手起刀落殺死了老人，老人就是亞基米得。另一說馬塞流斯久慕亞基米得的才華，着士兵找他帶見。士兵找着亞基米得時，他正在思考幾何難題，不想出不能休，不肯馬上隨士兵走，士兵怒火頓起，手起刀落殺死了他。二千多年過去了，大家還記得亞基米得，但有多少人記得那位征服者馬塞流斯呢？這故事還有一節尾聲，據云馬塞流斯事後感到十分懊悔，吩咐把亞基米得厚葬。爲它立碑，碑上刻上他生前最喜歡的圓柱盛圓球的圖形。隨着時日消逝，後人已不復知亞基米得之墓落在何方了。直到1965年在敍拉古一處快將興建酒店的工地上，挖地基時挖出了一塊墓碑，上面刻了一個圓柱盛着一個圓球，於是亞基米得之墓重現人間！

6. 劉徽的工作

說了一個西方的故事，該輪到說一個東方的故事了。我國最早一部有完整體系的數學著述叫做《九章算術》，記載了至秦漢爲止的數學成就。書裏有不少面積體積的公式，但都沒有証明。三國時劉徽爲《九章算術》作註，補充了很多解釋。《九章算術》的第五章叫做"商功"，是關於工程的計算，主要是計算體積，其中一題是求"陽馬"的體積。"陽馬"是當時的建築名詞，指方錐體。劉徽註："斜解立方得兩塹堵。斜解塹堵。其一爲陽馬。一爲鼈臑。陽馬居二。鼈臑居一 不易之率也。""塹堵"是護城河的牆，指三稜柱體。"鼈臑"是龜的臂骨，指某種特殊的四面體。這段話的意思，就是說"陽馬"體積是"鼈臑"體積的兩倍，而兩個"陽馬"和兩個"鼈臑"合成一個立方，所以"鼈臑"體積是立方體積的六份一，"陽馬"體積是立方體積的三份一，就是底面積乘高乘三份一了（見圖1）。

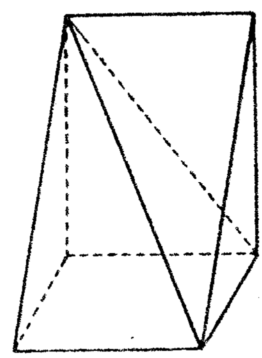

圖 1

過了一千六百多年後，德國數學家希耳伯特（Hilbert）在1900年提出著名的"二十三個問題"，對本世紀的數學研究產生了重大的影響。其中最先被解答的是第三個問題，就在那一年裏被狄恩（Dehn）答覆了。希耳伯特的第三個問題是什麼呢？如果把一個多邊形分割成若干（有窮數目）份，然後把這些碎片拼湊成另一個多邊形，顯然它的面積和以前那個多邊形的面積相等。在1832年數學家証明了反過來說也對，即是如果兩個多邊形的面積相等，把其中一個適當地分割成若干份，可以拼湊成另外那一個。希耳伯特的第三個問題就是問：對多面體來說，類似的事情對不對？如果兩個多面體的體積相等，能否把其中一個適當地分割成若干份，把碎片拼湊成另外那一個？狄恩的答覆是：不一定可能！它的數學意義是深遠的，它說明了多面體的體積理論和多邊形的面積理論，有本質上的區別。粗略地說，計算多面體的體積，必須倚靠微積分。讓我們以"事後孔明"的眼光重讀劉徽的註解，他計算了方錐體的體積，豈不是說他用了微積分嗎？

是的，他的確用了微積分，但只隱約出現於字裏行間，而且連他自己也摸不透這回事。以當時的數學水平，他是不容易理解這回事的，能點出這個意思，已屬難能可貴。原來在"陽馬居二。鼈臑居一。不易之率也。"後面還有一大段註解，大概是這麼做：把"陽馬"分成兩個小"陽馬"和四個"塹堵"，把"鼈臑"也分成兩個小"鼈臑"和兩個"塹堵"（見圖2）。除卻小"陽馬"和小"鼈臑"不計，一個的體積便是另一個的兩倍。把每個小"陽馬"照樣分

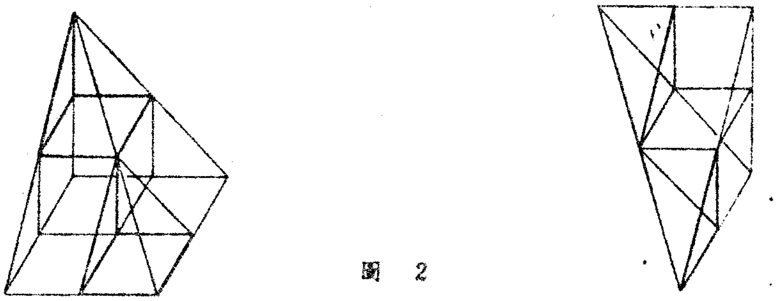

圖 2

成兩個更小的“陽馬”和四個“塹堵”，把每個小“鼈臑”照樣分成兩個更小的“鼈臑”和兩個“塹堵”。除卻更小的“陽馬”和更小的“鼈臑”不計，一個的體積又是另一個的兩倍。這樣照做下去，劉徽註說：“半之彌少。其餘彌細。至細曰微。微則無形。由是言之。安取餘哉。”以今天的“無窮小”概念看，“微則無形”（結果等於零）是不妥當的話，但不容否認，劉徽已經捕捉了基本精神。

亞基米得把面（體）看成由線（面）組成，也含有“無窮小”的想法。劉徽也有相同的想法，他經常使用一個這樣的原理：如果兩個物體（圖形）在等高的截面面積（截線長度）有一個固定的比率，這兩個物體（圖形）的體（面）積也有這個比率。最有趣的例子是他註第四章“少廣”時指出原書的一處錯誤，原書說圓球體積和它的外接立方體積的比率是 $\pi^2:4^2$（原書以三作 π 的值）。大概前人知道圓的面積和它的外接正方的面積的比率是 $\pi:4$，所以圓柱體積和它的外接立方體積的比率也是 $\pi:4$，他們便以爲圓球體積和它的外接圓柱體積的比率也是 $\pi:4$，由此得出 $\pi^2:4^2$ 這個答案。劉徽指出 $\pi:4$ 並非是圓球體積和它的外接圓柱體積的比率，而是圓球體積和另一件物體體積的比率。他把這件物體叫做“牟合方蓋”，想像圓球由一疊圓組成，由小至大，又由大至小，每個圓有它的外接正方，也是由小至大，又由大至小，這疊正方便組成“牟合方蓋”了。要計算圓球體積，只用計算“牟合方蓋”體積便成。劉徽嘗試過，卻沒有成功，但留下這樣一段話：“觀立方之內，合蓋之外。雖衰殺有漸。而多少不掩。判合總結。方圓相纏。濃纖詭互。不可等正。欲陋形措意。懼失正理。敢不闕疑。以俟能言者。”這是何等踏實的作風和謙虛的襟懷！過了二百多年後，“能言者”出現了，就是公元五世紀南北朝的祖冲之祖暅父子，他們循劉徽的思路，以巧妙的辦法計算了“牟合方蓋”的體積，因而得出正確的圓球體積公式。可惜他們的著述《綴術》早在北宋時代已經失傳，我們知道的不多，很可能這是數學史上第一部微積分著作呢！按照唐代李淳風註《九章算術》，寫下他們怎樣運用劉徽的原理，註說：“緣冪勢既同。則積不容異。”“冪”指截面面積，“勢”指高度，這是劉徽原理中當比率是一的特殊情形。在西方這個原理被稱爲“卡瓦列利原理”，因爲意大利數學家卡瓦列利（Cavalieri）在他的1635年著述裏提出這個結果，利用它計算面積體積。其實，即使在西方，這種辦法在卡瓦列利之前已經廣泛地被應用了。例如德國數學家和天文學家刻卜勒（Kepler）在十七世紀初便是這樣求橢圓的面積。他把半長軸爲 a、半短軸爲 b 的橢圓包在半徑爲 a 的圓裏面（見

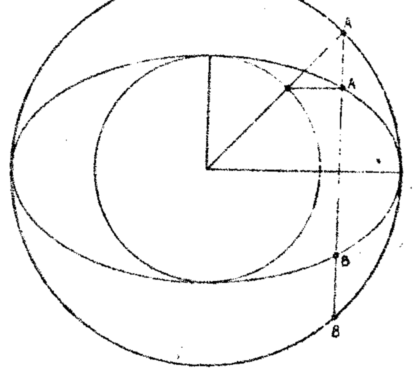

圖 3

圖 3 ），從橢圓的特性他知道橢圓內的線長AB 和圓內的線長A'B'的比率是 b ： a 。圓內的線組成圓，面積是 πa^2，橢圓內的線組成橢圓， 面積便是圓面積乘 b/a，所以是 πab。

7．十七世紀初的數學氣候

爲什麼刻卜勒對橢圓面積這麼感興趣呢？橢圓曲線在早刻卜勒二千年前已經被希臘數學家發現了，但直到刻卜勒之前，那只不過是幾何學研究中的一種曲線而已，刻卜勒把它帶進了實際世界裏！刻卜勒爲了探討星空的奧秘，特地跑到當時最著名的天文學家第谷(Tycho Brahe)那兒當一名助手。過了兩年，第谷忽然去世，刻卜勒便接替了他的職位，並且承繼了當時全歐洲最詳盡最豐富的天文數據資料。刻卜勒苦心孤詣，立志要從這大量的數據中尋求天體運行的規律。他作了不下百次的構想，但每一次只要由構想推斷出來的計算結果和第谷的觀測數據相差少許（關鍵性的一次不過相差八分而已！），他便放棄舊構想重頭又再來，這是科學家尊重客觀事實的高度表現！終於在1609年他發表了兩條著名的定律：（一）行星繞日運行的軌道是個橢圓，以日爲一個焦點。（二）行星繞日運行時，從日至行星的線段在相等的時間內掃過相等的面積。再過了十年後，他發表第三條定律：（三）行星繞日運行一周的周期之平方和橢圓軌道的半長軸之立方成正比。這理論是1543年波蘭天文學家哥白尼（Copernicus）的日心說的深化結果，雖然哥白尼提出行星繞日運行這個在當時來說是石破天驚的事實，但他仍舊擺脫不了傳統古希臘學說的影響，所以他認爲行星循圓形軌道以均勻速度運行。刻卜勒的發現，不只顯示了有需要研究非均勻速度和研究圓以外的曲線，它還直接地促成了微積分的發展，牛頓就是爲了要進一步解釋刻卜勒三大定律才建立了他的力學體系，而在這個過程中奠下了微積分的基礎。

與刻卜勒同時期的意大利數學家和物理學家伽利略（Galileo Galilei）在促成微積分的發展中也扮演了一個重要的角色。根本上，他開展了科學數學化的方向。他認爲必須從大自然錯綜複雜的現象中抓緊"物質"和"運動"這兩個基本概念來研究。科學研究是從觀察實際世界得出基本原理，然後按照數學研究的辦法，從這些基本原理出發，演繹推理得出新的結果，再從實驗中驗証這些新結果是否正確。爲了尋求定量的描述，數學便成爲科學研究中不可或缺的部份。伽利略在1610年說了一句有名的話："大自然的奧秘都寫在這部永遠展開在我們面前的偉大書本上，如果我們不先學會它所用的語言，就不能了解它。……這部書是用數學的語言寫的。"正是這種信念，爲後世科學研究指出新的路向。

整個十七世紀初葉，便是充滿這樣一種對大自然熱切求知的生氣，這是自十五世紀開始的"文藝復興運動"的必然的延續發展，它拓廣了人們的視野，激發了人們的思想。同時，當時的社會發展，帶來工商業的發達，連帶引起一連串的變化，也帶來不少新的科研問題，亟待解決。特別地，這類問題涉及變動的數量，是傳統的"靜"的數學沒有考慮過的。已有的數學顯得不敷應用了，很自然地，當時的第一流的數學家的努力，差不多都集中在那一方面了。

8．微分法和積分法

讓我們先看一個重要的概念，就是變化率（rate of change）。爲了方便敍述起見，假設考慮的是所走距離對時間來說的變化率，也就是我們通常所謂"速度"。首先，必須分清楚兩種速度，就是"平均速度"和"瞬時速度"。平均速度是個十分簡單的概念，把在某段時間內所走的距離，以所需的時間除它便是。但使人感興趣的卻是瞬時速度。例如從槍發射出來的子彈擊中人的時候，平均速度是多少說明不了什麼，但擊中人的一刹那的瞬時速度卻不可不知！怎樣理解這個所謂"瞬時速度"呢？假設子彈在前進，從某瞬時T秒起計，在T＋0.5秒它走了207.5

米，平均速度便是每秒 415 米。把觀察時間縮短一點，在 T +0.1 秒它只走了 41.9 米，平均速度便是每秒 419 米。再把觀察時間縮短一點，在 T +0.01 秒它只走了 4.199 米，平均速度便是每秒 419.9 米。這樣把觀察時間越縮越短，得出來的平均速度便越來越接近某個數量，有理由把這個數量叫做子彈在T秒的瞬時速度。或者有人會說，既是瞬時，何來速度？時間既不變，位置當然也不變，何來什麼平均速度呢？這種爭辯，古已有之，在公元前五世紀希臘哲學家提出過，後來在十八世紀上半期也有人提出過。我不想在這兒給糾纏在這個問題上，不如讓我們相信大自然吧，任何人若曾試過走路不小心碰得鼻青唇腫的話，一定不否認有瞬時速度這回事的！

伽利略發現了一條這樣的定律：物體由靜止狀態自由下跌時，下跌的距離和所需時間之平方成正比。凡是一個數量隨着另一個數量變更而變更的話，在數學上便把這種變更的關係寫成一個叫做"函數"（ function ）的東西。在這裏以S代表下跌的距離，以T代表下跌的時間，我們說S是T的函數。如果以米量度S，以秒量度T，伽利略的定律可以定量地表示爲 $S=5T^2$（5是約數，敘述上較方便，它應該是4.9……）。我們問五秒後物體的瞬時速度是什麼？（五秒後的平均速度是每秒25米，因爲在5秒內物體下跌了125米。）我們這樣想，5秒後S是125，5 + h 秒後S是 $5(5+h)^2=125+50h+5h^2$，所以在這h秒內物體實際下跌了 $k=50h+5h^2$ 米，平均速度便是每秒 $k/h=50+5h$ 米。如果h越來越小，平均速度便越來越接近每秒50米。按照剛才對瞬時速度的理解，5秒之後物體的瞬時速度是每秒50米。要注意一點，這個答案，好像只要在 $k/h=50+5h$ 中置 $h=0$ 便得到，其實在思路方面卻是截然兩回事。如果真的置 $h=0$，那麼也會有 $k=0$，所以 k/h 變成沒有意思了！

以上的解釋，其實就是微分法（ differentiation ）的基本思想。更普遍的情形，是不把對象局限於速度。考慮 y 是 x 的函數，即是 y 隨 x 變更而變更（例如 y 代表氣壓，x 代表高度，氣壓隨高度而變），對 x 來說 y 的瞬時變化率便叫做 y 的導函數，如何求這個導函數的方法便叫做微分法。用幾何語言來敘述，y 和 x 的關係可以用一條曲線表示，曲線上的點的橫座標表示 x 的值，縱座標表示相應的 y 的值。當 x 變更時，點便在曲線上走動，y 對 x 來說的瞬時變化率，即是曲線在 (x, y) 這點上的切線的斜度。在十七世紀初，切線斜度的研究是一個重要的題目。法國數學家費馬（ Fermat ）在1630年左右提出一個有系統的解法，基本上有如我們剛解釋過的辦法。費馬計算了 $y=x^2$，$y=x^3$，$y=x^4$ 等等曲線在橫座標是 a 那點上的切線斜度，發現答案分別是 $2a$，$3a^2$，$4a^3$ 等等。費馬也繼承了前人對求面積的興趣，但他在前人基礎上比前人又邁進了一步。他想到先計算由很多很多很窄很窄的在曲線底下的長方條構成的面積，這只是真正面積的近似值吧（見圖4）。就如同理解瞬時變化率一樣，當這些長方條越變越窄但也就越來越多時，這近似值便會越來越接近某個數量，有理由把這個數量稱爲真正的面積。用這種想法，費馬也計算了 $y=x^2$，$y=x^3$，$y=x^4$ 等等曲線底下由原點至橫座標是 a 的點的面積，發現答案分別是 $a^3/3$，$a^4/4$，$a^5/5$ 等等。他的方法已經包含積分法（ integration ）的基本思想，至此微積分的

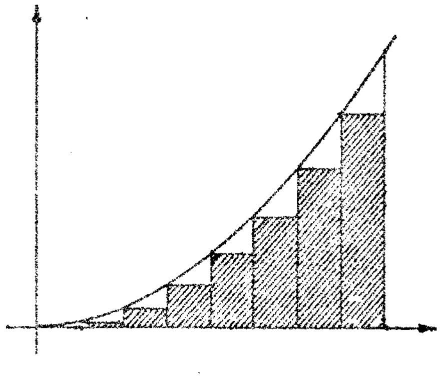

圖 4

兩種基本方法都具備了。

大家見到，微分法是個無窮的相減操作，積分法是個無窮的相加操作，再看費馬的特例，很難相信以他那種敏銳的頭腦沒注意到微分法和積分法兩者之間的關係。但在所有已知的文獻上，他都未曾提及這回事。最先意識到微分法和積分法的關係的人，似乎是牛頓在劍橋大學時的老師巴魯（Barrow）。牛頓和萊布尼茲後來更清晰地指出了微分法和積分法的關係，就是今天所謂"微積分基本定理"。以極其粗疏的敍述，這定理說：把一個函數先求積分，再求微分，便得回那個函數（用正確的數學語言說，一個已經很有用的情形是：若 f 是閉區間〔a，b〕上的連續函數，而 $F(x)=\int_a^x f(t)dt$，則 f 是F的導函數。）這定理解釋了爲什麼某些積分問題可以化爲機械化的計算，初學微積分的人已經懂得，先計算不定積分（所謂不定積分，即是一個經微分法後給出原先函數的東西，計算時有很多公式可借助的），然後把上限和下限代入，兩者相減便成。但如果你以爲這個辦法便叫做求積分，便是完全誤解了積分的要義！應該把積分看成是某個級數和的極限值，計算積分即是計算這個極限值。你可以"不擇手段"地計算它，只要捕捉了它便成，利用不定積分來計算只不過是手段之一（全憑"微積分基本定理"才知道這是一種手段！）。其實，除了簡單的情況（和課本上"斧鑿痕跡"很深的習題）外，這辦法通常是行不通的，在十九世紀上半期，甚至有人証明了對某些函數這辦法是行不通的！把積分看成是一個級數和的極限值，才使它的應用廣泛起來，而不是局限於計算面積體積。就如同把微分法看成是求函數的變化率，才使它沒有局限於計算運動的速度而已。

9. 牛頓和萊布尼茲的工作

講微積分而不講牛頓和萊布尼茲，便有如唱"空城計"沒有了諸葛亮！但既然微積分的基本概念在他們兩人之前差不多全具備了，他們兩人的貢獻又在那裏呢？

首先，他們確立了微積分的體系。據牛頓自己說，他在1665年至1666年間發明了微積分，但他有個不喜歡發表著述的習慣，使他的發現很遲才見諸文獻。主要文獻是 1669 年的"De Analysi……"（遲至1711年才印行），1671年的"Methods fluxionum et serierum infinitorum"（遲至1742年才印行），和1676年的"De quadratura curvarum"（遲至1693年才印行）。他最有名的著述，就是"Philosophiae naturalis principia mathematica"，裏面除載有他的力學理論外，還載有他的微積分學說。牛頓寫作這部著述的日期，比前三部還遲，但卻是他第一部面世之作，那多虧得他的好友數學家哈雷（Halley）大力推動，甚至掏腰包給他出版這部著述，使在1687年公諸於世。至於萊布尼茲，大概在1676年左右獨立地發明了微積分，但他比牛頓發表得早，第一篇關於微分法的論文刊登於1684年的德國學術雜誌"Acta Eruditorum Lipsiensium"上，接着，第二篇關於積分法的論文刊登於1686年的同一部雜誌上。他把微分法叫做"calculus differentialis"，積分法叫做"calculus summatorius"（後來瑞士數學家伯努利（James Bernoulli）改稱之爲"calculus integralis"），這就是微積分英文名稱的來源。在牛頓和萊布尼茲以前，微積分是一些巧妙的方法，散見於各家的工作中。牛頓和萊布尼茲把這些方法歸結於一個普遍的有系統而可以用符號操作運算的體系裏（所以稱之爲"calculus"亦不無箇中道理的）。我們今天只要稍懂微積分，便可以用同樣的方法處理不同的問題，但在十七世紀中葉之前，每一個這樣的問題都花卻不少一流數學家的心血，而且每個問題有每個問題的獨特解法，雖云巧妙卻未免予人"臨急周章"之感！關於這一點，我們能不感激牛頓和萊布尼茲嗎？

第二點貢獻，是他們（尤其是牛頓）把"無窮"帶進了數學。牛頓指出無窮級數的運算，可以看成如同一般代數式的運算。例如他那著名的二項式定理，便是代數上次數是正整數的二

項式的推廣。當N並非是正整數時，牛頓証明了$(A+B)^N=A^N+NA^{N-1}B+(N(N-1)/(1.2))A^{N-2}B^2+(N(N-1)(N-2)/(1.2.3))A^{N-3}B^3+\cdots\cdots$。以今天的眼光看，要找出二項式展開時的系數，只要把原來次數是正整數時的系數 $\binom{N}{r}$ 寫作 $N(N-1)(N-2)\cdots\cdots(N-r+1)/(1.2.3\cdots r)$，即可馬上想到怎樣推廣了。但在牛頓的時代，這是絕對不明顯的，事實上牛頓發現這條定理，也不是用這種類比猜測，而是由於他研究某些積分所致。反過來說，這條定理在他發展微積分的過程中也起了極大的作用，例如他利用它來計算 $y=x^n$ 的導函數，從而悟出“微積分基本定理”來。這種把無窮級數的運算看成是有窮級數的運算的推廣的想法，是有毛病的，一不小心便出紕漏。例如把二項式定理用於 $(1+x)^{-1}=1/(1+x)$ 時，得到 $1/(1+x)=1-x+x^2-x^3+x^4-x^5+\cdots\cdots$。若置 $x=-½$，左面是2，右面是 $1+½+½^2+½^3+\cdots\cdots$。的確，把右面越來越多項相加，答案越來越接近2。但若置 $x=1$，左面是 ½，右面是 $1-1+1-1+1-1+\cdots\cdots$，對不對呢？這曾經是數學史上一個叫人大惑不解的問題，有人認爲左面是 $(1-1)+(1-1)+(1-1)+\cdots\cdots=0+0+0+\cdots\cdots=0$，也有人認爲左面是 $1-(1-1)-(1-1)-\cdots=1-0-0-\cdots=1$，也有人認爲左面是 $1-(1-1+1-1+\cdots)$，就是它自己是從1減去自己，所以它的值是 ½，符合以上由二項式定理得到的答案！但怎麼會三個不同的答案也對呢？這個故事的教訓就是：對付無窮級數，切不可掉以輕心！不過，牛頓這種以有窮看無窮的想法，在數學史上卻起了一個非常重要的作用，它打破了從古希臘開始數學家畏避“無窮”的習性，爲數學開闢了一塊新天地。

在接着那二百多年中，數學家勇往直前，發現了一個又一個的新定理，解決了一個又一個的實際問題。但其實微積分的邏輯基礎可以說是薄弱得可憐！這是數學發展的一個使外人感到不解和詫異的現象，通常的人以爲數學是最講求邏輯嚴謹性的學科，怎麼數學發展過程竟是那麼的不合邏輯呢？十八世紀和十九世紀初的數學家，憑着他們的“膽識”，把微積分發展得越來越深刻，越來越有用。在這過程中，犯錯誤是免不了，但奇蹟地在整整二百年間竟然沒有產生什麼重大的錯誤，沒有使微積分整個學科走了歪路。當中的原因，除了當時那批數學家的才智見識是卓越不凡之外，另一個就是當時的數學發展緊密地扣着物理學和天文學的發展。數學家的發現，在實際問題上得到驗証；實際問題的需求，也啟發了數學家的發現。這樣，數學家向前搶攻時，更加信心十足了。

10. 後語

在1734年英國有位主教伯克萊（Bishop Berkeley）寫了一部叫做“The Analyst”的書，攻擊當時微積分理論的弱點，指出它根本毫無邏輯根據，只是“盲打盲着”！到了十九世紀中葉以後，不少重要的問題，帶來越來越多的困惑，連數學家們也强烈地感覺到有需要爲微積分建立嚴謹的邏輯基礎了。爲了建立微積分的邏輯基礎，數學家的工作逐漸形成數學分析（analysis）這一大領域，成爲微積分的深化發展。數學發展是無窮無盡，但文章卻不能如此，而且微積分在十七世紀以後的發展雖然同樣是多姿多采，敍述上卻要求更多的數學知識，不如就把故事說到這爲止吧。我以棣摩甘的小故事開首，讓我也以棣摩甘的說話結束，他說：“數學發現的原動力不是推理，而是想像。”（“The moving power of mathematical invention is not reasoning but imagination.”）

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A Survey of Junior Secondary School Mathematics Teachers and Form Three Pupils on Calculator Usage and Attitudes towards the Use of Calculators by Pupils

There are altogether three surveys on attitudes towards the use of calculators by pupils carried out by the author. The first survey was done in 1980 summer soliciting the views of secondary school mathematics teachers and form six pupils when the Hong Kong Examinations Authority started to allow candidates to use calculators in public examinations in 1980. Results of this survey are given in the article "Attitudes towards calculator usage by pupils : a survey of mathematics teachers and Form six pupils in Hong Kong" (Jointly written by Mr. C.K. Ip and the author and published in the Hong Kong Science Teachers' Journal in June 1981). The second survey was carried out in the months of April and May in 1981 with primary school mathematics teachers. Newly proposed primary school mathematics syllabuses were prepared by the Hong Kong Curriculum Development Committee in 1980. In the preface of those syllabuses, the training in complex computations in the present days of calculating machines was discussed. The article "A survey of primary school mathematics teachers on attitudes towards the use of calculators by pupils" has recently been completed by the author.

The present survey was carried out in the months of June and July in 1981. The target population was junior secondary mathematics teachers and Form 3 pupils. Mathematics teachers who participated in the seminars organized by the Junior Secondary Education Assessment Section of the Education Department were invited to complete and return the questionnaires on the spot. According to the 204 returns, a majority of these teachers were teaching mathematics at the junior secondary levels, with about one

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third of them also teaching Form 4 and 5 classes. In the pupil survey, the sample consists of 1229 Form 3 pupils from nine secondary schools in Hong Kong, Kowloon and the New Territories (two government, two aided, two private independent and three caput secondary schools). These pupils completed and returned the questionnaires under the supervision of invigilators after they had finished the monitoring tests prepared by the Educational Research Establishment.

The items on attitudes towards the use of calculators by pupils in the questionnaires are essentially the same in the three surveys. There are, however, some slight modifications in individual items to suit the target respondents. In the questionnaire used in the present survey, the wording used in Item 7 about homework and Item 11 about the main objectives of using calculators have been refined. The English translation of the questionnaire (original in Chinese), is given in Table 1.

Numbers ranging from 1 to 5 are given at the end of each item, and respondents are required to circle one of them to indicate the extent of their agreement or disagreement to the item. The average responses to individual items, given by the junior secondary mathematics teachers and Form 3 pupils in the present survey, are tabulated in Table 1. Average responses given by senior secondary mathematics teachers, Form 6 pupils and primary school mathematics teachers in the previous two surveys are also tabulated in Table 1 for easy comparison. The average response, if greater than 3, indicates inclination towards agreement, with 5 as the highest; less than 3 indicates disagreement, with 1 as the lowest.

Generally speaking, pupils of different levels and mathematics teachers of different groups had similar attitudes towards the use of calculators by pupils. Both pupils and teachers in the survey accepted that the ability to compute correctly with whole numbers, fractions and decimals was the most important goal of primary school arithmetic, but not that of the junior secondary levels. They also agreed that speed in arithmetic was not as important as understanding how and when to use different arithmetic operation. While mathematics teachers considered that skills with calculators would be essential to pupils' future success in their study, they agreed that the use of calculators should be part of the mathematics curriculum only in the secondary levels, particularly Form 4 and 5. The use of calculators should be taught along with pencil-and-paper calculation. In the learning process, use of calculators could replace pencil-and-paper calculations when proper understanding was achieved. With the exception of primary school mathematics teachers, secondary school pupils and mathematics teachers agreed that pupils should be allowed to use calculators for homework assignments. Both teachers and pupils accepted the use of calculators in tests by secondary school pupils, but disagreed to their use by primary school pupils. As for the main objective of the use of calculators, majority took it to save time in complex calculations. According to the analysis of the responses to the questionnaires, every respondent has, on the average, two calculators at home, and two members in the family use calculators in their work. It is obvious that possessing a calculator and using it at work are very common in the modern days of Hong Kong society.

In order to have a better understanding about the calculator usage, eight additional items were included in the questionnaire for the junior secondary mathematics teachers and Form 3 pupils (Content and analysis are given in Table 2). On the average, the sample of Form 3 pupils had already 2.3 years of experience using calculators, and junior secondary mathematics teachers 6.4 years (32% with more than 8 years). The calculators commonly used by teachers had trigonometric functions and could calculate statistical standard deviations, while those of Form 3 pupils had trigonometric functions and the ability to calculate addition, subtraction, multiplication and division with one memory storage. Due to the limitation of what had been learned, most Form 3 pupils used the sine and $\log_{10}$ keys but not those of $\sin^{-1}$, x^y and e^x. In simple calculations such as 12 + 35, an absolute majority of teachers (98.5%) and Form 3 pupils (95.6%) would do them mentally. For somewhat lengthy calculations such as 35 + 147, majority of teachers (79.4%) and Form 3 pupils (64.8%) would still use mental calculation. The rest would use a pencil and paper, rather than a calculator. Would the use of calculators retard one's ability in mental calculation? About half of the mathematics teachers and one-third of Form 3 pupils in the survey considered that constant use of calculators would weaken one's ability in mental calculation. They also admitted that since the time they started using calculators, their ability in mental calculation had deteriorated. In using a calculator to get an answer to a problem, only half of the teacher and one-third of pupil respondents would often pay attention to whether the answer was reasonable, the rest only occassionally or none at all.

In summary, junior secondary school mathematics teachers and Form 3 pupils agreed that the use of calculators should be part of the secondary school mathematics curriculum, and that secondary school pupils be allowed to use calculators in homework assignments and in tests. They, however, disagreed to such use by primary school pupils. In simple calculation of two-digit addition, a majority of junior secondary school mathematics teachers and Form 3 pupils would use mental calculation and would not rely on calculators. However, they considered constant use of calculators would weaken their ability in mental calculation. It is obvious from the three surveys that it is commonplace having a calculator and using it at work. According to the respondants, the main objective of the use of calculators is to save time in complex calculations. The potentiality of the calculator in instructional and recreational activities would thus need further exploration and wider publicity.

Acknowledgement

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Paul L.M. Lee
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April, 1982.

Table 1

Question	Average response in a 5-point scale (1 = strongly disagree; 5 = strongly agree)				
	Senior Sec. Maths teachers (N=257)	Junior Sec. Maths teachers (N=204)	Primary Sch. Maths teachers (N=251)	Form 6 pupils (N= 435)	Form 3 pupils (N=1229)
1. The most important goal of primary school arithmetic is the ability to compute correctly with whole numbers, fractions and decimals.	3.9	3.6	3.5	3.6	3.6
2. The most important goal of arithmetic in F.1 through F.3 is the ability to compute correctly with whole numbers, fractions and decimals.	2.5	2.3	2.9	2.2	2.4
3. Speed in arithmetic is not as important as understanding how and when to use different arithmetic operations.	3.9	3.7	3.4	3.7	3.3
4 - 6 Calculators should be a part of the mathematics curriculum in P. 1 through P. 3	-	1.5	1.6	-	1.6
Calculators should be a part of the mathematics curriculum in P. 3 through P. 6	1.7	-	-	1.6	-
Calculators should be a part of the mathematics curriculum in P. 4 through P. 6	-	1.8	2.5	-	2.1
Calculators should be a part of the mathematics curriculum in Form 1 through Form 3	3.1	3.1	3.7	2.5	3.7
Calculators should be a part of the mathematics curriculum in Form 4 through Form 6	4.2	-	-	4.0	-
7. Pupils should be allowed to use calculators for homework assignments.	3.6	-	2.6	3.3	-
Form 3 pupils should be allowed to use calculators for homework assignments.	-	4.2	-	-	4.2

Question	Average response in a 5-point scale (1 = strongly disagree; 5 = strongly agree)				
	Senior Sec. Maths teachers (N=257)	Junior Sec. Maths teachers (N=204)	Primary Sch Maths teachers (N=251)	Form 6 pupils (N= 435)	Form 3 pupils (N=1229)
8 - 9 Pupils in P.4 or above should be allowed to use calculators in tests.	-	-	2.1	-	-
Pupils in P.6 or above should be allowed to use calculators in tests.	2.6	2.3	3.1	2.4	3.0
Pupils in F.3 or above should be allowed to use calculators in tests.	4.2	4.0	-	3.8	4.3
10. Skills with calculators will be essential to students' future success in their study.	3.5	3.6	3.4	3.1	3.3
In the situations described in Questions 11, 12 and 13, how would you feel if calculators were used in the school ?					
11. The main objectives of the use of calculators are to					
a. enrich mathematics curriculum	3.7	3.4	3.0	3.2	3.0
b. motivate students in their learning	3.6	3.4	3.1	3.1	3.0
c. provide various mathematical games	3.1	2.9	3.3	2.6	2.5
d. save time in complex calculations to allow for more time in mathematical thinking.	-	4.5	4.1	-	4.2
12. Use of calculators should be taught along with pencil-and-paper calculation so that pupils are competent both ways.	4.1	3.8	3.7	4.1	4.1

Question	Average response in a 5-point scale (1 = strongly disagree; 5 = strongly agree)				
	Senior Sec. Maths teachers (N=257)	Junior Sec. Maths teachers (N=204)	Primary Sch. Maths teachers (N=251)	Form 6 pupils (N= 435)	Forms 3 pupils (N=1229)
13. In the process of learning mathematics, use of calculators can largely replace pencil-and-paper solutions if pupils have proper understanding.	4.2	4.1	4.0	4.1	3.9
	Number of calculators				
14. How many calculators do you have at home ?	2.3	2.3	2.0	2.3	2.3
	Number of family members				
15. How many members in your family use a calculator in their work ?	1.9	2.1	2.0	1.9	2.3

Table 2

Questions	Analysis of responses	
17. How many years of experience do you have in using calculators ?	Junior secondary maths teachers : average 6.4 years (32% with more than 8 years) Form 3 pupils : average 2.3 years	
	Percentage of respondents	
	Junior sec. maths teachers (N = 204)	Form 3 pupils (N = 1229)
18. Which of the following types of calculators do you usually use ?		
a. Can calculate addition, subtraction, multiplication and division only	10.8 %	16.1 %
b. Can calculate addition, subtraction, multiplication and division, with a memory storage only	13.7 %	27.3 %
c. With trigonometric functions	50.0 %	39.4 %
d. Can calculate standard deviations	41.7 %	13.2 %
e. Can be programmed	10.8 %	17.5 %
19. Which of the following keys in the calculator have you used?		
a. sin	96.1 %	83.2 %
b. $\sin^{-1}$	92.6 %	52.8 %
c. x^y	86.8 %	44.7 %
d. $\log^{10}$	93.1 %	77.6 %
e. e^x	66.7 %	22.4 %

20. How would you calculate 12 + 35 ?		
a. by mental calculation	98.5 %	95.6 %
b. by a calculator	1.5 %	1.5 %
c. by pencil and paper calculation	1.5 %	1.5 %
21. How would you calculate 35 + 147 ?		
a. by mental calculation	79.4 %	64.8 %
b. by a calculator	9.8 %	6.5 %
c. by pencil and paper calculation	11.3 %	29.4 %
22. Do you feel your ability in mental calculation has deteriorated since using calculator ?		
a. Deteriorated much	12.3 %	3.5 %
b. Deteriorated a little	38.2 %	26.8 %
c. No deterioration	46.6 %	60.9 %
d. No deterioration; certainly with improvement	2.0 %	5.4 %
23. Do you consider that constant use of calculators would weaken one's ability in mental calculation ?		
a. strongly disagree	5.4 %	5.3 %
b. disagree	12.3 %	16.4 %
c. not sure	27.9 %	38.5 %
d. agree	45.1 %	29.4 %
e. strongly agree	8.8 %	7.5 %

24. When you use a calculator to get an answer, do you ever consider that the answer is reasonable ?

a. usually not	15.7 %	16.6 %
b. sometimes	38.2 %	48.4 %
c. usually	33.8 %	23.8 %
d. definitely yes	11.8 %	7.7 %

小學數學教師對學生使用計算機的意見調查*

隨着電子工業的發達，計算機的應用日漸普及，它的運用已深入社會各階層。在學校教學方面，尤其是數學的教學，亦受到這項科技發展的影響。

對於學生應用計算機於數學學習及測驗，筆者與葉照坤先生於一九八〇年曾調查中學數學教師（大部份爲中五數學科教師）及中六學生的意見，調查結果發表於香港數理教育學會會刊（一九八一年六月版）。大致而言，中學數學教師及中六學生都歡迎在高中年級（中四及中五）的數學教學及測驗准許使用計算機。對於在較低年級，尤其是在小學階段的學生使用計算機，他們則表示不甚贊同。

香港課程發展委員會於一九八〇年編印一套新的小學課程綱要數學科初稿。在其緒言中的一段謂：「•••計算訓練仍舊是不可缺少的。但是，社會的需要已與前不同了。隨着計算機的使用日漸普遍，繁複計算技巧的訓練已無必要。因此，計算練習只應引用簡單數字。」。緒言又強調：「要讓學生先明白然後進行練習。知其然而不知其所以然的計算練習是沒有多大用處的。」

爲徵詢小學數學教師對學生使用計算機的意見，在一九八一年四、五月間，筆者進行了這次問卷調查。問卷題目與上次調查中學數學老師的題目基本上相同。問題內容請閱表一。

參予填寫這次問卷的老師計有２５１人。他們分別在參加教育署舉辦的多次研討新數學課程會議時，即席填寫及交還問卷。根據分析，這些小學老師，百分之八十五是任教小學五、六年級的數學教師。他們平均每人家中都有兩部計算機（在２５１人中，只有１０人回答沒有計算機），另家中有兩位成員在工作中使用計算機。問卷各題詳細分析見表一。

綜合而言，這次調查小學數學教師所得的結果，與前次調查中學數學教師意見相比較，大致相同。由表一數字分析看來，參予這次調查的小學老師大部份同意小學的算術課程最重要的目標是整數、分數及小數的計算能力。他們亦同意「算術的計算速度並不比明白如何及何時利用

*本調查得蒙馮源先生及鄭偉謙先生提供寶貴意見，並蒙馮源先生協助收發問卷，與及各老師樂意填答問題，謹此致謝。

不同算術運算更爲重要」。對於計算機的運用，使成爲數學課程的一部份，他們雖然同意可施行於中一至中三年級，但反對施行於小學，尤其是小一至小三年級。他們亦不贊成學生在做家課及參加測驗時利用計算機。第十一題問及在學校的情況下使用計算機的主要目的，老師們絕大部份認爲是在節省繁複計算時間，使能多作數學思考。有關筆算與計算機的教授程序，參予調查的小學數學老師大多數都同意計算機的運用，應該與筆算同時教授，使學生在兩方面都能勝任。他們亦同意在數學的教學過程中，學生如果對數學有正確的認識，則採用計算機實可代替筆算步驟。

在海外各地已有很多對學生使用計算機的研究。英國 Nottingham 大學附屬的 Shell Centre for Mathematical Education 最近發表了一份在小學的計算機實驗報告 (A Calculator Experiment in a Primary School)。這份報告詳述研究發現，並指出計算機在某些課程範圍可以應用的地方。同樣地有關計算機的教學、課程及遊戲的書籍及文章亦相繼出版。Charlotte L. Wheatley 在一九八一年期的 School Science and Mathematics （第620－624頁）發表了一篇文章名 Calculator Use in the Middle Grades 。文內介紹了四種利用計算機作爲教學工具的活動。其中一種解決疑難的活動述說如下：—

1 〝你的心臟每一次心跳可泵輸65毫升血液。如果你的心臟每分鐘跳68次，那麼在1分鐘內泵輸多少血液？在1小時內？在1日的24小時內？在1星期內？〞

2 〝當小明跑步時，他的心跳每分鐘達145次。問他的心臟每分鐘泵輸多少血液？在16分鐘內？〞

3 〝小英的心臟每分鐘泵輸4·7升血液。問她的心臟每分鐘跳動多少次？〞

4 讓學生找出他們的心跳次數。然後問〝你的心臟每分鐘泵輸多少血液？〞

在此項活動，計算機會十分方便去發展學生解決疑難的能力，並加強十進制單位的運用。又數學在健康教育的應用亦可充份顯示出來。

這次調查僅提供了小學數學科教師對學生使用計算機的某些意見。隨着計算機的普及化，計算機在教育上可能引致成爲數學教育的關注事項。這項現代科技發明具有重組我們學校數學課程的潛力。謹祈望對這發明在教育功能上能有更多的研究及廣泛的咨詢。

表 一

問　　題	選答百分率（N＝251人）				
	十分不同意	不同意	不能肯定	同　意	十分同意
1　小學的算術課程最重要的目標是整數、分數及小數的計算能力。	2·4	27·3	2·0	59·8	8·4
2　中一至中三的算術課程最重要的目標是整數、分數及小數的計算能力。	4·4	36·3	20·3	29·5	2·4
3　算術的計算速度並不比明白如何及何時利用不同的算術運用更爲重要。	3·6	21·9	10·8	43·4	12·4
4　計算機的運用應是小一至小三數學課程的一部份。	41·8	53·4	2·0	0·8	0·4
5　計算機的運用應是小四至小六數學課程的一部份。	13·5	47·8	13·1	21·9	1·2
6　計算機的運用應是中一至中三數學課程的一部份。	2·4	10·8	10·0	63·3	9·6
7　學生做家課時，應該可以利用計算機。	9·6	45·4	16·3	21·5	2·0
8　小四或以上的學生參加測驗時應該可以利用計算機。	14·3	69·3	6·4	7·2	0·8
9　小六或以上的學生參加測驗時應該可以利用計算機。	6·4	31·5	13·1	41·4	4·8
10　運用計算機的技巧對學生將來在學業上的成就是必須的。	2·4	19·5	19·9	47·8	6·0
在第11、12、13題的情況下，在學校使用計算機，你覺得怎樣？					
11　應用計算機的主要目的					
(甲)　在使數學的課程更爲充實，	5·6	26·3	15·9	33·5	0·8
(乙)　在加強學生的學習動機，	2·4	26·7	15·9	35·1	0·8
(丙)　在提供各類數學遊戲，	2·4	19·5	13·5	41·8	1·6
(丁)　在節省繁複計算時間，用以多作數學思考。	1·2	4·8	4·4	60·2	23·5
12　計算機的運用，應該與筆算同時教授，使學生在兩方面都能勝任。	0·8	16·7	6·4	60·6	12·4
13　在數學的教學過程中，學生如果對數學有正確的認識，則應用計算機實可代替筆算步驟。	0·8	7·2	4·8	62·9	19·9

A proof of $m_1 m_2 = -1$ for F.2 pupils

In the C.D.C. Syllabus of Mathematics for Junior Secondary forms the concept of "slope, m" of the line joining two points $P_1(x_1, Y_1)$, $P_2(x_2, Y_2)$ defined as $\frac{y_2 - y_1}{x_2 - x_1}$ is introduced in the stage of form two.

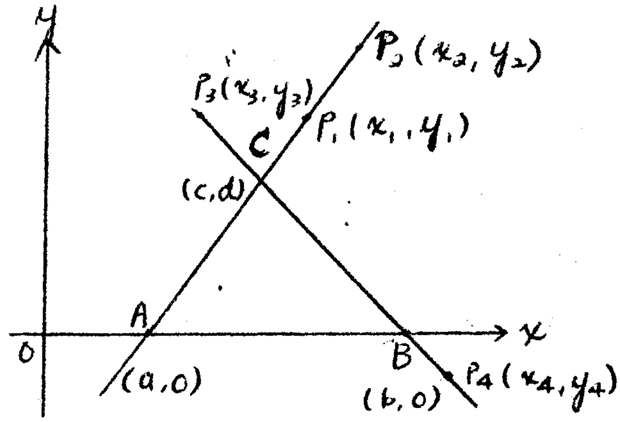
Before the introduction of slope, pupils are equipped with the following background knowledge of mathematics.

F.1 Unit 1 Number and countiny
" 2 Formulae, open sentences and equations
" 3 Use of protractor and compasses and basic properties of angles and simple shapes
" 4 Percentages
" 5 Approximation and measurements
" 6 Area and Volume
" 7 Negative numbers and the extended number line
" 8 Introduction to co-ordinates
" 9 Algobraic expressions
" 10 Angle and line segment bisection
" 11 Angles and parallel lines
" 12 Negative numbers
" 13 Statistical data
" 14 More about algelraic expressions

F.2 Unit 1 Rate, ratio and proportion
" 2 The angles in a triangle
" 3 Approximation and significant figures
" 4 Pythagoras
" 5 Polynomials
" 6 The sine, cosine and tangent ratios
" 7 Trigonometric relations
" 8 Use of formulae
" 9 More about co-ordinates
(i) Distance
(ii) Gradient (slope)

When the pupils come across the angle at which two lines intersect, they are not taught to calculate the size of the angle because they are not intended to have the knowledge of trigonometric compound angles. They are led to see the relation of slopes when two lines are perpendicular. They can verify the relation "$m_1 m_2 = -1$" if and only if the 2 lines cut at right angles. All the textbooks (more than 10) I have read either do not have a proof or have one which is not accessible by form two pupils.

Here I am suggesting a proof which will be accessible by form two pupils.



Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $P_4(x_4, y_4)$ be any four points.

P_1P_2 (or produced), P_3P_4 (or produced) at the x-axis at $A(a,0)$, $B(b,0)$ respectively, and the intersection of the lines P_1P_2, P_3P_4 is at $C(c, d)$.

The slope of P_1P_2, m_1 say, is $\frac{y_2-y_1}{x_2-x_1}$, and equal to $\frac{d}{c-a}$.
Similarly the slope of P_3P_4, m_2 say, is $\frac{y_3-y_4}{x_3-x_4}$, and equal to $\frac{d}{c-b}$.

$\angle C$ is a right angle if and only if

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$$

$$(b-a)^2 = (c-a)^2 + d^2 + (b-c)^2 + d^2$$

$$b^2 - 2ab + a^2 = c^2 - 2ac + a^2 + d^2 + b^2 - 2bc + c^2 + d^2$$

i.e. $$-2d^2 = 2(c^2 - bc - ac + ab)$$

$$\therefore m_1 m_2 = \frac{d}{c-a} \cdot \frac{d}{c-b}$$

$$\frac{d^2}{c^2-bc-ac+ab} = -1$$

And it is readily seen that if $m_1 m_2 = -1$

then $$\frac{d^2}{c^2-bc-ac+ab} = -1$$

$$\Rightarrow (b-a)^2 = (b-c)^2 + d^2 + (c-a)^2 + d^2$$

$$\Rightarrow \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$$

$$\Rightarrow \angle C = 90^\circ$$

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NEW & RENEW

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If you have a BASIC program stored in your microcomputer and you key in NEW accidentally, it is quite difficult to recover the program again. Although some microcomputers with disc facilities can recover the program easily by simply RENEW, yet with a micro without disc facilities, you can hardly do anything. In this article, I want to share my experience in handling this problem, viz. to recover a BASIC program after NEW has been keyed in, and to explain the underlying principle behind the series of operations needed. To make my presentation precise, I shall use TRS-8Ø Level II BASIC 16K RAM as my sample model.

First of all, let us examine the internal coding of a BASIC program such as

```
1Ø PRINT "HELLO"
2Ø PRINT "NEW & RENEW"
3Ø END
```

It is known that the coding starts at memory location 17129. We can key in the following line to print out the contents of memory locations 17129 through 17168 :

```
FOR I = 17129 TO 17168 : ?I; PEEK(I), : NEXT
```

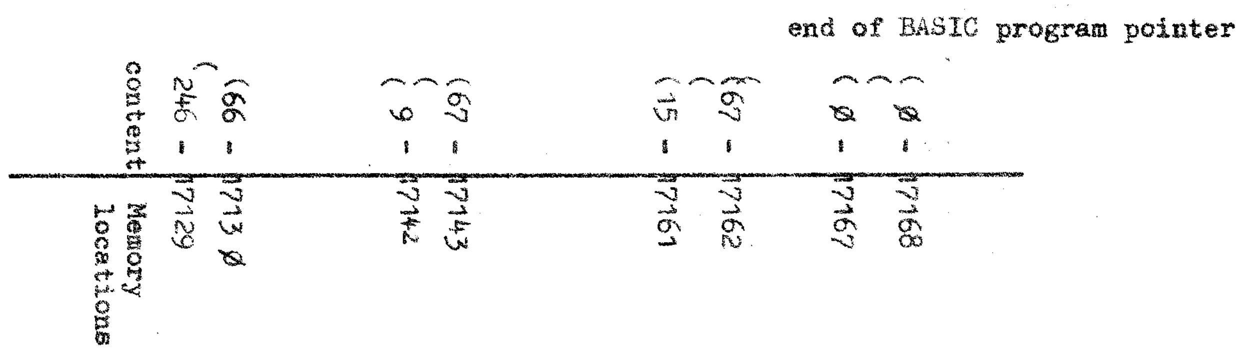
The display may look somewhat like this :-

17129	246	17130	66	17131	1Ø	17132	Ø
17133	178	17134	34	17135	72	17136	69
17137	76	17138	76	17139	79	1714Ø	34
17141	Ø	17142	9	17143	67	17144	2Ø
17145	Ø	17146	178	17147	34	17148	78
17149	69	1715Ø	87	17151	32	17152	38
17153	32	17154	82	17155	69	17156	78
17157	69	17158	87	17159	34	1716Ø	Ø
17161	15	17162	67	17163	3Ø	17164	Ø
17165	128	17166	Ø	17167	Ø	17168	Ø

The content of memory location pair 17129 & 1713Ø represents the starting address 17142 (i.e. 246 + 66 * 256) of line 2Ø. The following table shows the contents of certain memory location pairs and the corresponding addresses they represent.

Memory location pair	Content	Corresponding address	Explanation	Memory locations	Explanation
17129 1713Ø	246 66	17142	starting address of line 2Ø		
17131 17132	1Ø Ø		line no. 1Ø	17133 to 17141	PRINT "HELLO"
17142 17143	9 67	17161	starting address of line 3Ø		
17144 17145	2Ø Ø		line no. 2Ø	17146 to 1716Ø	PRINT "NEW & RENEW"
17161 17162	15 67	17167	start of end of BASIC program pointer		
17163 17164	3Ø Ø		line no. 3Ø	17165 - 17166	END
17167 17168	Ø Ø		end of BASIC program pointer(2bytes)		

The structure of the internal coding is as follows:-



/.....

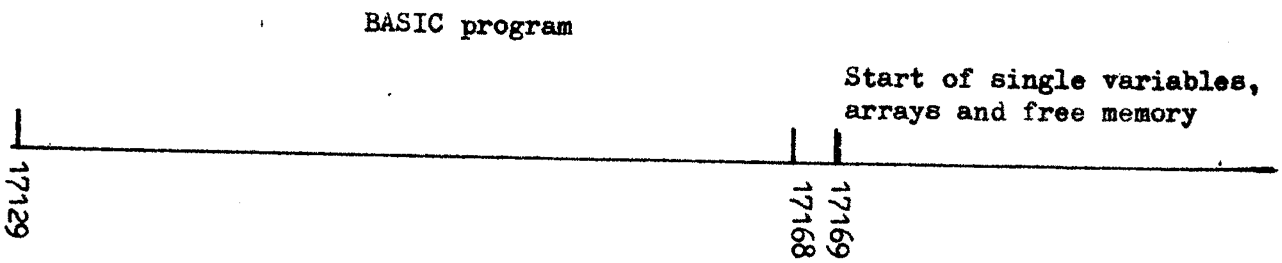
So 17168 is actually the end of this BASIC program. However, this is not the end of the story. What about memory locations 17169 onward? From 17169 onward, the memories are used

(i) to store simple variables (including variable types - integer, single precision, double precision or string, variable names and values),

(ii) to store arrays, and

(iii) as free memory.

The structure of the memories are as follows :-



The starting addressess of these dedicated memories are stored in memory location pairs 16633; 16634, 16635; 16636, and 16637; 16638 respectively. You can print out the contents of the above memory location pairs by keying

?PEEK(16633); PEEK(16634), PEEK(16635); PEEK(16636), PEEK(16637); PEEK(16638)

The display will look somewhat like this :-

17 67 17 67 17 67

Since 17 + 67 * 256 = 17169, we know that simple variables, arrays and free memory actually start at 17169.

Now if you key in NEW and examine the contents of memory location pair 17129 & 17130, you will get

Ø Ø

which is the end of BASIC program pointer. Also, if you examine the contents of memory locations 16633 through 16638, you get

235 66 235 66 235 66

/.....

This means that as soon as you key in NEW, the three pointers (i.e. start of simple variables pointer, start of arrays pointer and start of free memory pointer) reset to 17131 (i.e. 235 + 66 * 256).

To sum up, after keying in NEW there are four significant changes in certain memory locations, namely :-

Memory location pair	Content from	Content to	Remark
17129 17130	246) 17142 66)	Ø Ø	Reset end of BASIC program pointer. Linked list broken by pointer.
16633 16634	17) 17169 67)	235) 17131 66)	The 3 pointers, namely start of single variables pointer, start of arrays pointer and start of free memory pointer, reset to 17131.
16635 16636	17 67	235 66	
16637 16638	17 67	235 66	

In order to RENEW your BASIC program, you have to recover the linked list by POKEing appropriate values to memory location pair 17129 & 17130 and to reset the three pointers. The following procedures may help you to have your program back :-

Procedure	Remark
(1) POKE16634, 1ØØ : POKE16636, 1ØØ: POKE16638, 1ØØ	Set the three pointers to a rather high memory location. (For a longer BASIC program, you have to set the pointers to an even higher memory location)

Procedure	Remark
(2) FOR I = 17133 TO 17384 : IF PEEK(I) = Ø THEN X = INT ((I+1)/256) : Y = I+1-X*256 : POKE 17129, Y : POKE 1713Ø, X : ELSE NEXT	Calculate the link and POKE the values to memory locations 17129 & 1713Ø

Now, type LIST. With a bit of surprise, you will find your program back to the screen again.

減少數學練習中多餘的書寫

（節譯自教育署出版「數學通訊」第三期（1961））

在香港小學生的數學練習簿中，可看到不少多餘的書寫。數學本來已經是比較困難的科目，但有些教師還把不少多餘的工作加在學生身上。這是一種因循的做法，當年他們做學生時老師也是這樣做，可是現在自己卻缺乏意願和勇氣去改正。

常見的多餘的書寫有下列五類。

(一)抄寫題目

要學生把文字題抄在簿內，對數學學習而言，是一種妨礙，抄題目所費的時間往往比計算的時間為多。有些教師認為抄題目可免去溫習時翻書的麻煩，可是，翻一翻書會比抄數以百計的題目更辛苦嗎？

也有人說抄題目可以幫助語文學習。這實在誇大了一些。這只是抄書練習而已。在低年級裡，要是不計較所費的時間和精力，多做些抄書未嘗無益。但是，捨本逐末，把抄書作為數學練習，寧可多寫字而減少計算的練習，無論如何是不值得讚許的事。何況，究竟有多少教師改簿時有足夠的時間去批閱這些「抄書」？

「變相抄題」也是多餘的工作，請看下例：

〔題目〕　哥哥有3元，弟弟有2元。兩人共有多少元？
〔解答〕　哥哥有　3元
　　　　　弟弟有　2元
　　　　　兩人共有　3元＋2元＝5元

解答中的開頭兩行並非計算，只是變相抄題而已，何必要在這上面耗費時間？

減去多餘的抄寫，把省下的時間用在更多的數學練習上，學生實際得益更多。

(二)徒勞無功

不知是誰製訂出下列的格式：

5 x 10 = ?

5 x 10 = 50

答：50

$$\begin{array}{r} 5 \\ \times\ 10 \\ \hline 50 \end{array}$$

聰明的敎師會對上例作這樣的批評：

(a)無須抄題（ 5 x 10 = ? ）

(b)這是式題，不是應用題，故不必在最末處再寫答案。

(c)在答案下劃雙線，本屬多餘，劃了三次，更加倍多餘。

(d) 10 的乘法，用心算或唸口訣已能解決，無須再要算草。

有些敎師從不訓練學生心算，所以在高年級的練習簿中滿是下列之類的不必要的算草：

$$\begin{array}{r} 1 \\ +\ 1 \\ \hline 2 \end{array} \qquad \begin{array}{r} 1 \\ 3\overline{)\,3} \\ 3 \\ \hline 0 \end{array} \qquad \begin{array}{r} 5 \\ 10\overline{)\,50} \\ 50 \\ \hline 0 \end{array}$$

這是多麼累贅的計算：

$$\begin{array}{r} 10 \\ \times\ 9350 \\ \hline 90000 \\ 3000 \\ 500 \\ 00 \\ \hline 93500 \end{array}$$

但是，比起下例，它還不是最累贅的。

7 架汽車 + 8 架汽車 + 9 架汽車 = ? 架汽車

7 架汽車 + 8 架汽車 + 9 架汽車 = 24 架汽車

7 架汽車

8 架汽車

\+ 9 架汽車

24 架汽車

答： 24 架汽車

這是書法練習，耐性訓練，抑或是想扼殺學生對數學的興趣？

另一點值得提及的是應用乘法分配律很多時使計算快捷些，例如：

(a) $47 \times 99 = 47 \times (100 - 1)$
$= 4700 - 47$
$= 4653$

(b) $5 \times 4\frac{2}{13} = 5 \times (4 + \frac{2}{13})$
$= 20\frac{10}{13}$

(c) $23 \times \frac{17}{18} = 23 \times (1 - \frac{1}{18})$
$= 23 - \frac{23}{18}$
$= 21\frac{13}{18}$

若教師有興趣研究，當能發現更多簡化計算的方法。

(三)過於拘執

用間尺劃等號也是一項常見的費時失事的工作。小學生費盡心機，可是結果往往不是兩條線距離太闊，就是貼在一起。

$\frac{6}{2} = 3$　　　　　　$\frac{6}{2} = 3$

有些教師甚至規定這兩條線的長度，繪長一些短一些也不准許。這簡直是太過份了。

(四)不使用符號及縮寫

符號及縮寫是為節省時間設立的，例如平方厘米的符號是 cm^2，最大公因數的縮寫是 H.C.F.。可是有些教師不教學生用符號或縮寫代替文字書寫，以致無意義地浪費時間。

(五)過多的「詳細練習」

應否要求學生解答習題時把計算的各步驟詳細列出？在低年級裡，學生尚未掌握寫的技巧，不應要求太高，但在高年級裡，詳細列式是有需要的。一方面訓練學生逐步解答問題的技巧，另方面可讓教師檢查計算過程中有無錯誤。但是，命學生把每一條算題都詳細解答卻不是最有效的做法。若數量太多，學生負担過重，教師也無足夠時間批改。

爲了節省學生的時間，練習可分兩類：

(a)詳細練習　重要的、有代表性的題目要詳細地做出。教師細心批閱，用符號標出錯處。學生自己改正。

(b)簡略練習　學生在堂課簿或草稿簿上用簡捷的方法計算。教師不一定親自批改，但必須通過口頭核對答案或其他的方法，找出及解答學生疑難之處。

上面只是指出一些弊端，至於怎樣的做法才算合適，那是要靠教師自己去決定。當然，一向因循於舊習慣的教師會以這樣或那樣的理由去反對任何改革。我認爲，問題不在於這一個算式或者那一句文字是否多餘，最重要的是教師應適時注意發現缺點或不合理的地方，不斷改進工作。

消除多餘的書寫，愛惜學生的時間，不讓他們在無助於數學學習的事情上消耗時間。

（完）

〔註〕

這篇文字（原文是英文）是一位督學廿多年前寫的。令人嘆息的是，文內所提的各點弊端，除了最後一項「詳細練習」外，到今天仍存在。我們認爲這篇文字值得每一位小學數學教師閱讀，所以把它譯成中文刊登出來。至於「詳細練習」這一點，今天與廿多年前的情形比較，有了180度的轉變。近年來，由於公開考試流行使用「多項選擇題」及「短答案題」，不少教師在平日的數學練習中，只要求學生寫出答案就算，忽略了列算式或用其他方法寫出計算步驟的訓練，因而又出現了另一種弊端。我們認爲寫出計算過程的訓練，是數學教學的重要部份——訓練學生的分析、綜合、組織及表達能力，絕對不是多餘的書寫。在平日的練習中要注意，在校內考試中也不應忽略。因此，校內考試的試題中，應該有一部份的試題規定學生列式作答，顯示解答的過程，以考查學生在這方面的能力。

Test for Divisibility by 7

Even primary school students know that there are simple tests for divisibility by all the whole numbers up to 12 except 7.

It is theoretically and practically unsatisfying that a test for 7 remains unknown. The following analysis shows how such a test can be performed.

Now $10^n = (7 + 3)^n$

$$= 7^n + {}^nC_1\, 7^{n-1} . 3 + \ldots\ldots + {}^nC_{n-1}\, 7 . 3^{n-1} + 3^n \quad \text{(Binomial Theorem)}$$

As all terms except the last contain 7 as a factor, it goes without saying that 10^n leaves a remainder of 3^n when divided by 7. So any number of the form $a_o\, 10^n + a_1\, 10^{n-1} + \ldots\ldots + a_{n-1}\, 10 + a_n$ when divided by 7 leaves a remainder

$$R = a_o\, 3^n + a_1\, 3^{n-1} + \ldots\ldots + a_{n-1}\, 3 + a_n$$

$$= ((((\, a_o\,)\, 3 + \ldots\ldots + a_{n-3}\,)\, 3 + a_{n-2}\,)\, 3 + a_{n-1}\,)\, 3 + a_n$$

Suggesting a possible test for divisibility by 7.

Example I

What is the remainder when 895 is divided by 7 ?

$R = (8 \times 3 + 9)3 + 5 = 104 = 7(14) + 6$

$\therefore$ Remainder = 6

(The working may be shortened by removing the multiples of 7 at any stage)

```
  [8]              [9]              [5]
 X 3         -->  + 3         -->  + 1
 ----             ----             ----
  2 4              1 2               6  <-- Remainder
 -2 1             -  7
 ----             ----
    3               5
                  X 3
                  ----
                   1 5
                  -1 4
                  ----
                     1
```


/ Example II ....

Example II

Is 455 divisible by 7 ? (The working may be done mentally)

$\boxed{4}$	$\boxed{5}$	$\boxed{5}$
X 3	+ 5	+ 2
1 2	1 0	7
- 7	- 7	- 7
5	3	0 ← R = 0, therefore 455 is divisible by 7.
	X 3	
	9	
	- 7	
	2	

(In fact most of this may be done mentally, and with practice the test becomes quick and simple)

Generalising

When a number of the form $a_o 10^n + a_1 10^{n-1} + \dots + a_{n-1} 10 + a_n$ is divided by $10 - x$, the remainder is $a_o x^n + a_1 x^{n-1} + \dots + a_n$

or $$(((a_o) x + \dots + a_{n-2}) x + a_{n-1}) x + a_n$$

Let us consider some values of x :-

i) when x = 1, this gives a test for divisibility by 9. $R = a_o + a_1 + a_2 + \dots + a_n$; the sum of the digits must be divisible by 9.

ii) when x = 2, we have a test for divisibility by 8. As 8 divides 1000, the test need only be made on the last three digits. The test may be performed in the same manner as example II.

iii) when x = -3, this provides a test for divisibility by 13.

Example III

Is 793 divisible by 13 ?

7	9	3
X -3	- 8	- 3
- 21	1	0 ⟶ 793 is divisible by 13
13	X -3	
- 8	-3	

The test can be extended indefinitely for other values of x (even negative values) but the practical value becomes negligible.

Lucus

怎樣教授負數乘法

負數雖然已有長久的歷史，但負數的恰當定義和被列入學校課程內，只是近幾十年的事。

在負數的教學中，負數乘法，特別值得研究。數學家克萊茵（Klein）曾說：「"負乘負得正"這句話，往往成為學生在學習上的障礙。」而負數乘負數這一種運算，差不多可說是學生第一次遇到的非直觀性概念。

如果我們想學生徹底了解及熟習負數乘法的運算，怎樣去教方能收到最好的效果呢？翻尋有關負數乘法的教學書籍，方法雖然有很多種，但對不同教法的比較，則少之又少。那種教法最為有效？教師也許會感到難於選擇。

有鑒於此，以色列 Weizmann 學院的 Abraham Arcavi 和 Maxim Bruckheimer 兩位，搜集了多種教法，進行比較。他們將繁多的教法分成五大類，分述如下：

I) 呆記法

學生緊記

$(+)(+)=(+)$

$(+)(-)=(-)$

$(-)(+)=(-)$

$(-)(-)=(+)$

即「同號相乘得正，異號相乘得負」。不作任何解釋。

(II) 歸納法

此法的精義是學生從眾多例子中歸納出结果。現在舉出這種方法中的兩種不同方式：

(一) 數字運算法

假定學生已熟悉正數的乘法，先着他們計算下列各題

$(+4)(+2)=$

$(+3)(+2)=$

$(+2)(+2)=$

$(+1)(+2)=$

$0\ (+2)=$

再擴展下列各題

$$(-1)(+2)=$$

$$(-2)(+2)=$$

$$(-3)(+2)=$$

學生會由首五題的答案．推出 $(-1)(+2)=(-2)$，$(-2)(+2)=(-4)$ 等．

再計算下列各題

$$(+3)(+3)=$$

$$(+3)(+2)=$$

$$(+3)(+1)=$$

$$(+3)\ 0=$$

$$(+3)(-1)=$$

$$(+3)(-2)=$$

$$(+3)(-3)=$$

由此推出正數乘負數，或負數乘正數，其結果為負數．其積的絕對值是該兩數的絕對值相乘之積．簡稱「正負得負」。

最後再計算

$$(-3)(+3)=$$

$$(-3)(+2)=$$

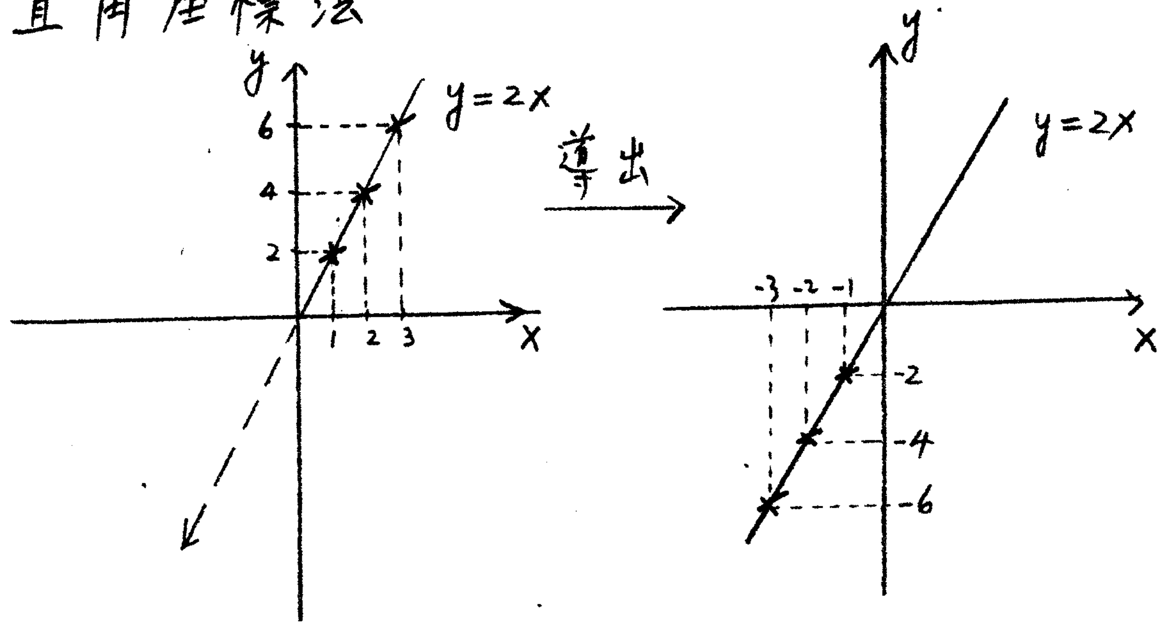
$$(-3)(+1)=$$

$$(-3)\ 0 =$$

$$\begin{cases} (-3)(-1) = \\ (-3)(-2) = \\ (-3)(-3) = \end{cases}$$

由最後三行引出「負負得正」。

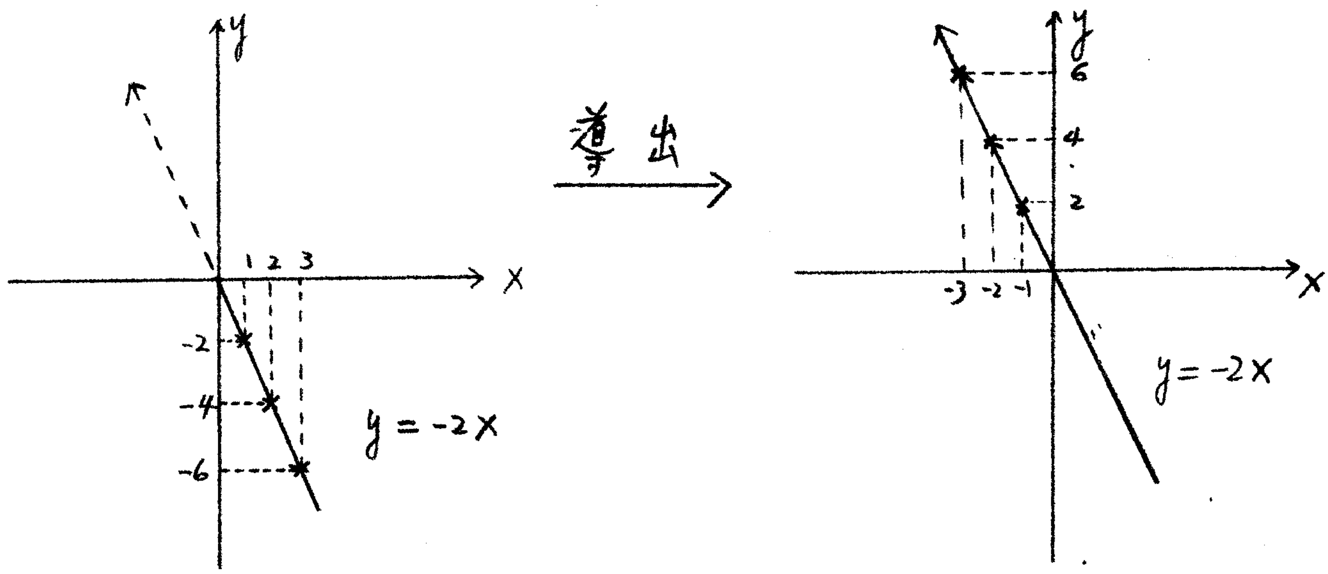
(二) 直角座標法



橫軸數(值)乘(+2),答案在縱軸上. 如(+2)(+1)=(+2), (+2)(+2)=(−4)等(左圖)。延長此直線至第四象限(右圖). 同理得(+2)(−1)=(−2), (+2)(−2)=(−4)等.

上圖是由正數乘(+2)推至負數乘(+2)。

同樣地,可利用下繪兩圖,由正數乘(−2)導出負數乘(−2)



上述兩種方式的歸納卻有賴學生的直觀推理，由已承認的事例而導出結果。其優點是易教和易明。同時學生只須從觀察數型及其中關係，發現新的結果，不必套用一些與數學無關的事例去推理。此外，學生更能夠藉此機會、親身體會到數學理論的創造過程。

不過，此方法的可行性，決定於學生能否悟到"正確"的引伸、因而達到所需的結果。

III 演繹法

這方法須假設在正數集中已成立的定律，如結合律、交換律、分配律、消去律等在負數集中亦能成立，從而推出所需的新法則。

㈠ 不嚴謹的演繹法

尤拉（Euler）曾以"欠債"為例，和用（偽）演繹法推出負數乘法。現略述如下：

正數相乘而得正數，沒有問題，所以（+a）乘（+b），應得（+ab），但（+a）乘（−b），和（−a）乘（−b），其結果如何，則要分別討論。先以（−a）乘（+3）（即3），如果把（−a）看成負債（+a）元（+3）（−a）便是負同樣的債三次，可看成負了相等三倍於（+a）的債項，因而成為（−3a）。所以（−a）乘（+b）得 −ba，亦即 −ab……。至於（−a）乘（−b），其積的絕對值應是ab，而ab前的符號，只有"+"和"−"兩種。但是"−"是不可能的。因為上文已闡明（−a）（+b）=（−ab），（−a）乘（−b）的答案不能和（−a）乘（+b）的答案相同，所以（−a）乘（−b），其積應該冠以"+"號，即（−a）（−b）=（+ab）。

㈡ 較嚴謹的演繹法

杜必奇（Dubisch）利用交換律，消去律，分配律等，用在數例導出結果。舉例如下：

$$(+2)\ 0 = 0$$
$$(+2)\left[(+3)+(-3)\right] = 0$$
$$(+2)(+3) + (+2)(-3) = 0$$
$$(+6) + ? = 0$$
$$\therefore\ (+2)(-3) = (-6)$$

又同樣地

$$(-2)\ 0 = 0$$
$$(-2)\left[(+3)+(-3)\right] = 0$$
$$(-2)(+3) + (-2)(-3) = 0$$
$$(-6) + ? = 0$$
$$\therefore\ (-2)(-3) = (+6)$$

IV 模型法

雖然許多課本都用這方法去講解負數的乘法，但仍未有一種模型能夠包含所有整數的性質。以下是三種常見的模型：

(一) 只着重正負號而忽略乘積的模型

設有一城，城內只有好人和壞人兩種，這些人經常出入這城。假若我們順理成章地用"+"號代表好人，則"−"號

代表壞人。另一方面，用"+"號代表入城，用"−"號代表出城，因此，好人入城，符號為(+)(+)，對城內一切，均有好處，結果用"+"表示

即 (+)(+)＝(+)

好人離城，符號為(+)(−)，對這城來說是一種損失，結果用"−"號表示：

即 (+)(−)＝(−)

同樣地利用

壞人入城說明 (−)(+)＝(−)

壞人出城說明 (−)(−)＝(+)

(二) 符號和乘積絕對值兼顧的模型

以電影攝影機，拍攝抽水入(或出)一玻璃水槽的過程為例。假如放映機的菲林向前轉為"+"，向後轉為"−"。泵水入水槽為"+"，抽水出槽為"−"。假設水泵每分鐘可泵水30升，攝影機和水泵同時開動兩分鐘，拍攝泵水入槽的情形。然後再開動兩分鐘，拍攝將水抽出的情形。所拍攝的菲林，沖

晒後，從放映機將拍攝得的影像，放映到銀幕上。利用銀幕上所見水槽內水位的高低，解釋負數乘法的結果。

若泵水入水槽(+)，而放映機向前轉(+)，銀幕上所見的水位應是60升。因水槽本來是空的，所以增加了60升，從而導出(+30)(+2)=(+60)。但是將泵水入槽(+)的一段菲林後轉(−)，放映到銀幕上，則可看到水位逐漸降低，從原來的60升降至0升，所以槽內的水減少了60升，從而導出(+30)(−2)=(−60)。如將水抽出水槽(−)那一段前轉(+)，放映到銀幕，會看到銀幕上的水位在降低，從而導出(−30)(+2)=(−60)。最後將抽水出槽一段後轉，放映出來又可以看見水位增高，從而導出(−30)(−2)=(+60)。

(三) 數線模型

數目可用向量位移表示。乘以正數 a，即是將向量的線段變為原有長度的 a 倍，而保持原有方向。乘以負數 b，線段變為原有長度的 |b| 倍，而方向和原有方向相反。

V 公理法

上述四法，在數學觀點上，未見妥善。嚴謹的解法乃是用兩個正整數得出的序偶 (a,b) 來表示負數。此法可在大學用書中看到。

Abraham Arcavi 和 Maxim Bruckheimer 在 1979/80 年度曾將三十二班第七級的學生，分成四組，分別用上述前四種方法，每組用其中一法講解負數乘法，然後用試題及問卷考查這四種方法的效果。統計的結果，頗出人意表，除去班與班之間學生原有質數的差異外，接受這四種不同教法的學生，無論在成績或在學習態度上，都沒有（顯著的）分別。

如果因為得到這樣的結果，便立刻認為這四種教法的效果沒有分別，所以教師可以任其好惡，隨便選擇其中一種教法，就可能太武斷。原因是所用的測驗題目，可能未有顯出各種教法的强弱，又可能是在一短時間內（統計在教授學生後四個月進行），教法的優劣未

有顯示出來。另外的一個結論，是當我們採取某一種教法時，不要主觀地認為此種教法會比其他教法為優越。

本文取材自"Mathematics in School Nov 81 Vol 10 No 5"

——汪原漢

原漢按：亦不要憑若干個例子就武斷地認為教學法沒有優劣的分別。

FIFTH INTERNATIONAL CONGRESS ON MATHEMATICS EDUCATION

Adelaide, Australia - Friday 24 August to Thursday 30 August 1984

The International Commission on Mathematical Instruction (ICMI) has accepted an invitation from the Australian Academy of Science to hold the Fifth International Congress on Mathematical Education at the University of Adelaide.

The Congress Program

Determination of the major emphasis in the Congress Program is the responsibility of the International Program Committee (IPC), appointed by ICMI. Dr M.F. Newman of the Australian National University is the chairman of IPC.

It is expected that the program will span all levels of education and discuss problems of general interest while recognising different cultural perspectives.

A principal objective of the Congress will be to facilitate both professional and personal contact amongst its participants. In particular, the organisers seek to encourage existing working groups in mathematics education to meet at the Congress and to encourage overseas participants to visit Australian colleagues in their home educational institutions.

Languages

The official language of the Congress is English. There will be translations of important sessions into several languages. There will also be provision for translating abstracts or summaries of presented papers into several languages. The selection of languages will be dependent upon the needs of the participants and also on the availability of assistance in providing translations prior to the Congress. Languages considered currently for selection include Japanese, Chinese, Indonesian, French, German, Russian and Spanish.

Future Announcements

The ICME 5 Organising Committee in Australia expects to issue a first announcement by May 1982. This will contain general information of relevance to prospective participants. A second announcement is expected to be available by May 1983. It will contain details of the scientific program and ancillary activities and include a registration form. The second announcement will automatically be sent to all respondents to the first announcement.

Request for Comment

The Organising Committee requests comment from prospective participants which might assist it in planning Congress activities. Remarks on the weaknesses and strengths of previous Congresses and other relevant international meetings will be greatly appreciated. Responses received before July 1982 will be especially helpful.

Please write to

ICME 5,
Wattle Park Teachers' Centre,
424 Kensington Road,
WATTLE PARK, SOUTH AUSTRALIA 5066,
AUSTRALIA.

University lecturers, College of Education lecturers and mathematics teachers who wish to contribute articles for publication are more than welcome. Contributions need not be typed. For further information, please contact the Editor, School Mathematics Newsletter ay 5- 774001 ext. 36.