SCHOOL MATHEMATICS NEWLETTER

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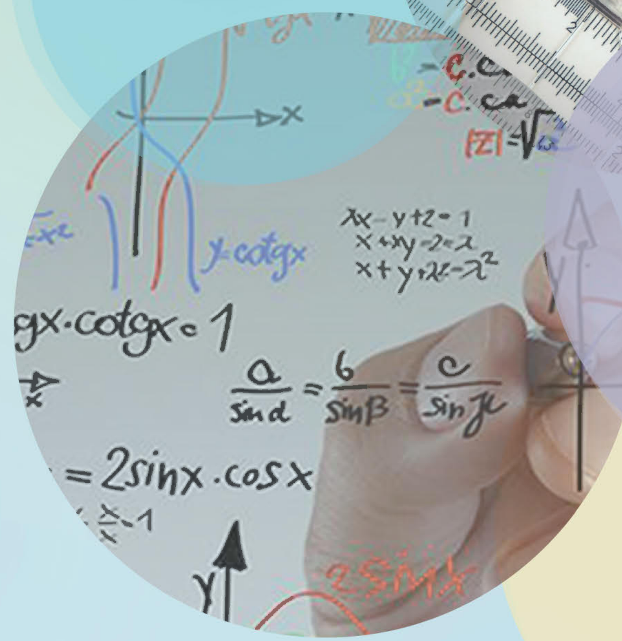
SMN



SCHOOL MATHEMATICS NEWSLETTER

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School Mathematics Newsletter（SMN）

Foreword

The School Mathematics Newsletter（SMN）is for teachers. SMN aims at serving as a channel of communication for mathematics education in Hong Kong. This issue includes articles written by academics, principals, teachers and curriculum officers. Articles related to hot topics such as STEM education and e-learning are included.

We welcome contributions in the form of articles on all aspects of mathematics education as the SMN is meant for an open forum for mathematics teachers. Please send all correspondence to:

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We extend our thanks to all who have contributed to this issue.

Contents

1. Using Tablet Devices in the Mathematics Classroom

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It is commonly believed that good use of digital technologies can improve mathematics teaching and learning. In recent years, while more tablet devices are available in the classrooms, some attention is shifted to resources and pedagogies that are tablet compatible. In this article, I share some examples of teaching materials and discuss possible use of tablet devices in the secondary mathematics classroom.

My suggestion is based on the following consideration.

1. I focus on materials specific to mathematics learning although we may also consider many other materials that are useful in tablet supported environment regardless of subjects, such as general purpose presentation or collaboration tools.

2. Some (but not many yet) new apps make use of specific features in tablets to support mathematics learning. Still many other materials are basically existing online interactive resources but converted to tablet friendly formats (mainly Java- and Flash-free versions).

3. I do not consider textbook related materials, such as mathematical content organised for self-study or assessment. My interest is different kinds of dynamic mathematical content that allow students to access abstract mathematical objects through manipulation of useful representations.

I have gathered a list of items in my website [1] for sharing with teachers. Although there is quite a long list of apps related to mathematics learning in some way, my major suggestion is web based resources that can be accessed in tablets. These include established resources from well-known educational sites that are gradually converted to tablet friendly formats, with good quality maintained. Here are a few major examples:

1. GeoGebra [2] is not just a powerful multipurpose tool, but also a means to develop resources, communicate and collaborate in a huge international network of educators. As part of this network, the GeoGebra Institute of Hong Kong [3] is a local community of teachers, educators and researchers promoting the use of good dynamic materials in the classroom. The tool itself, together with its sharing and management platform GeoGebraTube [4], allows use of all applets in tablets and some mobile devices. A recent QEF project [5] of CCC Tam Lee Lai Fun Secondary School, led by Alex Chik, successfully provides iPad ready teaching

and assessment materials, covering many topics in our mathematics curriculum.

2. The Illuminations website [6] of the National Council of Teachers of Mathematics (NCTM) has been a portal of various useful applets for K-12 mathematics teaching. Recently, a number of its existing applets are converted for use in mobile devices. Figures 1 and 2 shows two such useful applets in the website, which allows students to manipulate and explore 3D models in order to understand important geometric properties of those solids.

3. Sketchpad Explorer [7] is a free app for opening files made with Geometer's Sketchpad, which has been a leading dynamic geometry tool for many years. You can search for iPad compatible examples in its resource centre [8]. The Dynamic Number Project [9] is also a good example of continuing research-based development of dynamic content for learning of specific concepts, namely number and algebra. For example, there are tasks designed for visualising, exploring and practising factorization of trinomials through manipulation of area models (figure 3).

4. Wild Maths [10] is part of a family of projects including the famous NRICH [11] portal, providing well designed problems, explorations, games and puzzles. Newer projects of the family, like Wild Maths, integrate interactive elements that can be run in new platforms.

Apart from well-established web based resources, entirely new online tools are also being developed, compatible with new devices in the classroom. Desmos Calculator [12] is a noteworthy example. It is basically a graphing calculator with a friendly interface, but can be used in many different kinds of learning tasks. You can find many good examples of teaching materials and students' products shared in the project website. In the latest development [13] of the project, teachers can create interactive graphing tasks for collaborative learning in the classroom. The platform allows students' work on the same graph, while using individual mobile devices, to be displayed simultaneously on the same screen for sharing and discussion. This is a new and powerful means of facilitating collaborative learning while capitalising on mobile devices in the classroom. Figure 4 shows the result of a group of students plotting points on the same graph, entering through individual devices. Note that the students' responses can be examined separately, as shown in figure 5.

Comparing with web resources, new apps may not be so easily integrated with our secondary mathematics curriculum. Nevertheless, some examples are opening up new possibilities. They may also be products of long term research projects that explore and promote new means of developing particular mathematical concepts. Center for Algebraic Thinking [14] develop apps that illustrate various approaches of algebra learning. DragonBox Algebra [15] and iPuzzle [16] are also examples of new approaches to develop key algebraic concepts through games and puzzles in the mobile supported environment. Motion Math Zoom [17] utilises manipulation on number line for improving number sense.

Whether we are using tablets or other digital technologies, understanding the nature and possibilities of dynamic content for developing meaningful mathematical concepts is always a challenge for us.

[1] http://arthurlee.weebly.com/tablets.html
[2] https://www.geogebra.org/cms/
[3] http://www.geogebra.org.hk
[4] https://www.geogebra.org/materials/
[5] http://www.geogebra.hk
[6] http://illuminations.nctm.org/content.aspx?id=3855
[7]http://www.dynamicgeometry.com/General_Resources/Sketchpad_Explorer_for_iPad.html
[8] http://sketchexchange.keypress.com/search
[9] http://www.dynamicnumber.org
[10] http://wild.maths.org

[11] http://nrich.maths.org/
[12] https://www.desmos.com
[13] https://teacher.desmos.com
[14] http://www.algebraicthinking.org/tech
[15] http://dragonbox.com
[16] http://ttalgebra.edc.org/iPuzzle
[17] http://motionmathgames.com/motion-math-zoom/

Figure 1

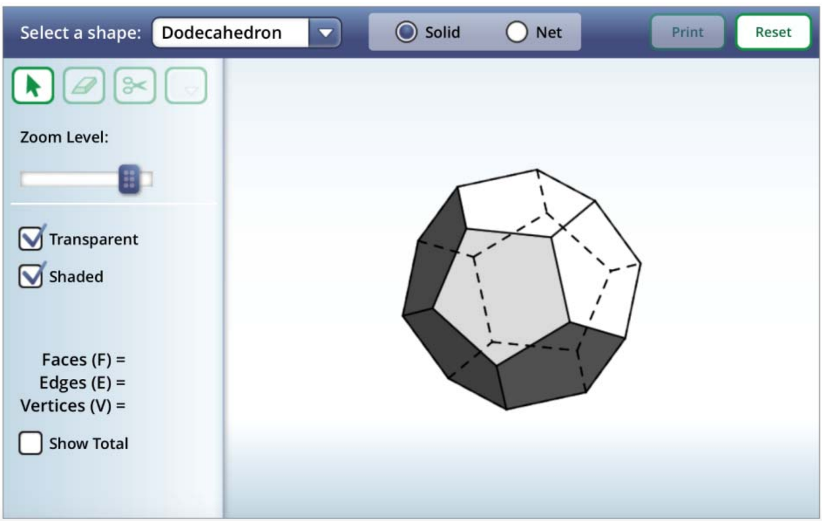


Figure 2

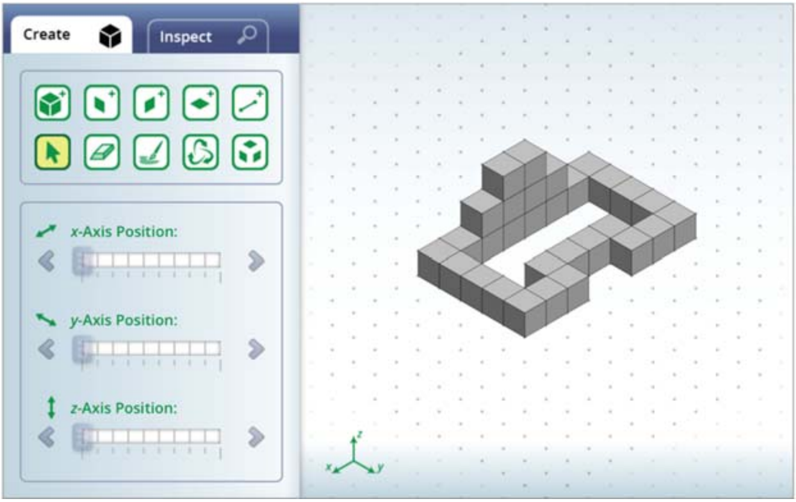

Create
Inspect
x-Axis Position:
y-Axis Position:
z-Axis Position:

Figure3

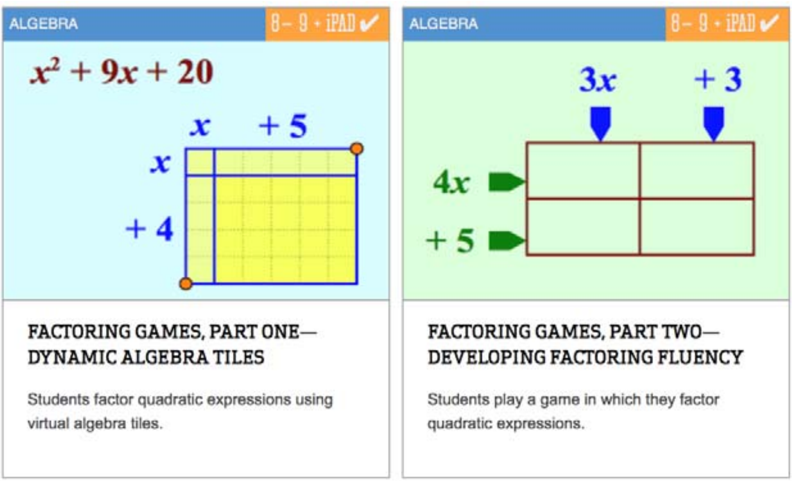

ALGEBRA
8– 9 • iPAD
$x^2 + 9x + 20$
x
$+ 5$
x
$+ 4$
FACTORING GAMES, PART ONE—
DYNAMIC ALGEBRA TILES
Students factor quadratic expressions using virtual algebra tiles.
ALGEBRA
8– 9 • iPAD
$3x$
$+ 3$
$4x$
$+ 5$
FACTORING GAMES, PART TWO—
DEVELOPING FACTORING FLUENCY
Students play a game in which they factor quadratic expressions.

Figure 4

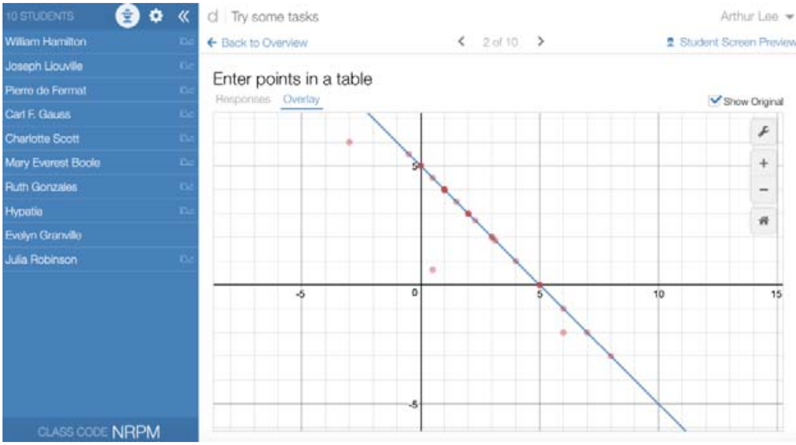
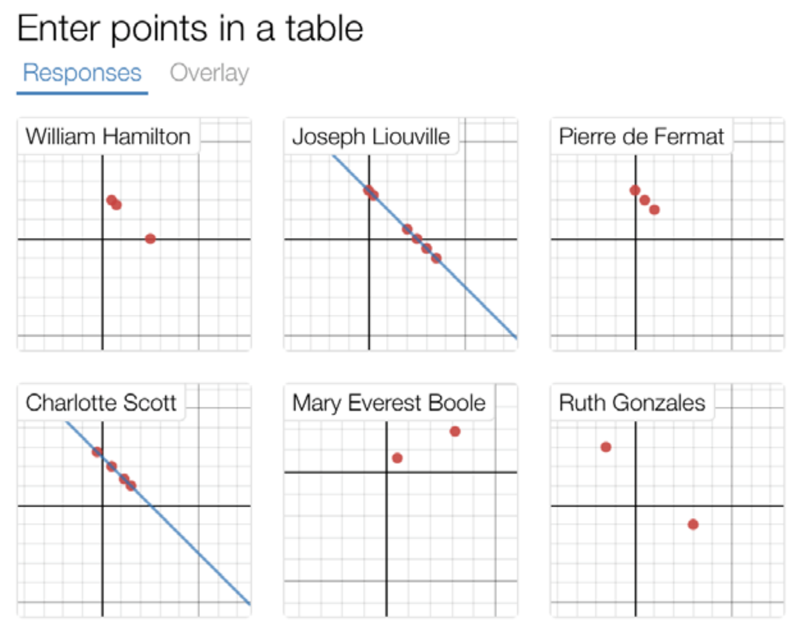


Figure 5



2. WeBWorK – an E-Homework-Assistant for Teachers

Cheung Leung Fu, Lam Ka Wing

Introduction

In today's world filled with E-teaching technological gadgets, a teacher may be overwhelmed by the very many software in the market. Questions such as: "which tool is most appropriate for my class?" or "do the students really benefit from such e-tools?" are abundant.

In the midst of such a confusing situation, we would like to swim against the current and introduce to the readers "yet" another such software – WeBWorK.

Why we venture to this? Because WeBWorK is different! It is a non-profit making open-source software. Originally, it was developed by two mathematicians at the University of Rochester, viz. Michael Gage and Arnold Pizer, about two decades ago. And because it is open-sourced, everyone can join and develop it further.

What makes WeBWorK interesting – a personal view

For us, as teachers, it is the "symbolic answer-checking" ability of WeBWorK which makes it so interesting.

Now comes the question – how does it check symbolic answers? At this point the authors can only guess, because we still haven't had time to understand this part of WeBWorK's source code. But before giving the readers our guess, let me ask the readers to think about one thing – how does a teacher typically check a symbolic answer.

Perhaps an example would be more telling:
Consider the question:
"Compute the derivative of $f(x) = (1 + e^x)^2$."
There are different ways to answer this question. Just to name two of them,

(i) one can expand the function and get $1 + 2e^x + e^{2x}$ and then differentiate term-by-term to obtain the answer $2e^x + 2e^{2x}$.

(ii) one can make use of the Chain Rule and obtain $2(1 + e^x)e^x$.

Both of these answers, though in different forms, are about the same function.

Now we are back to a typical difficulty which confronts a teacher often times – how to verify the fact that they are one and the same function?

One way is to follow the arguments of the student line-by-line. But this is time consuming, especially if the class size is large. Are there other ways?

One "approximate" way is to use some means (say, a computer software) to plot the graphs of the two functions and check that the two graphs overlap.

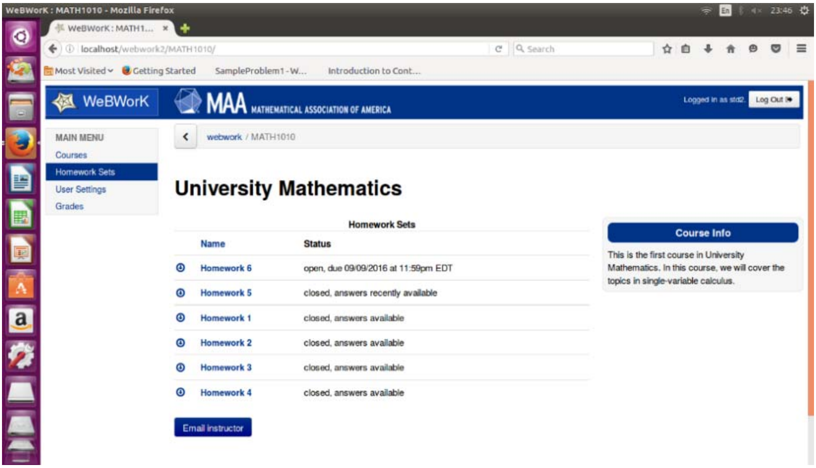
And maybe there are other more clever ways.
In short, the inventor(s) of WeBWorK managed to work out methods which enable the software WeBWorK to check the “sameness” or “equivalence” of solutions in situations like the one described above.

As such, it alleviates immensely the everyday workload of an already overloaded teacher. On top of this, WeBWorK has many more functionalities, some of which will be outlined below.

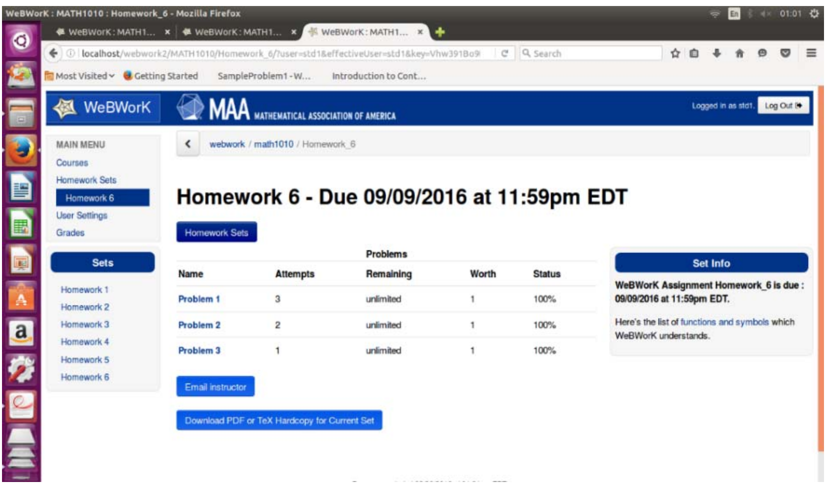
Basic Arrangement in WeBWorK

In this section, we will guide you through part of a student’s session of WeBWorK.

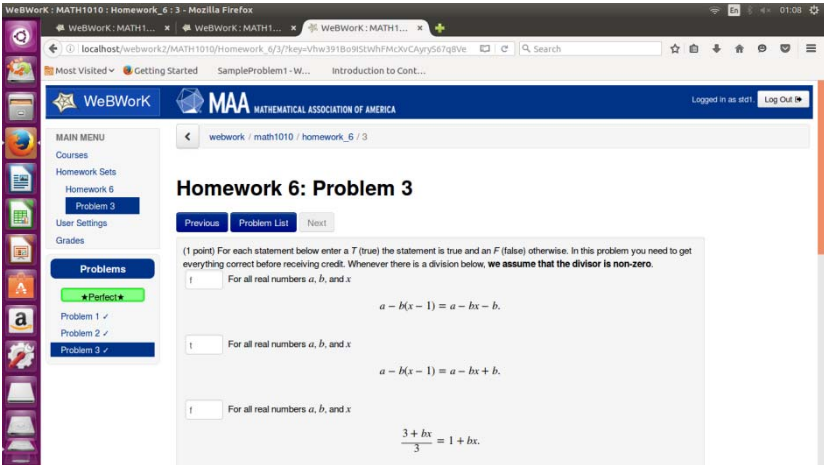
The situation is the following – imagine that you were a student, asked by the teacher to work out several questions in the course “University Mathematics” in WeBWorK. Now you type the necessary internet address, enter the relevant userid and password and will then see the following screenshot.
Now the student can choose the problem set to work on (in the example below, we assume that the student chooses Homework 6).



Having chosen Homework 6 (by clicking there), the student can then choose which problem in Homework 6 to work on (for example, Problem 3).



Then the student clicks on problem 3 and the following appears on screen:



This is a typical True/False question.

This brings us to types of questions WeBWorK can handle:

Types of questions

WeBWorK has options for several types of questions:

(i) True/False questions (such as the one shown in the previous screenshot).

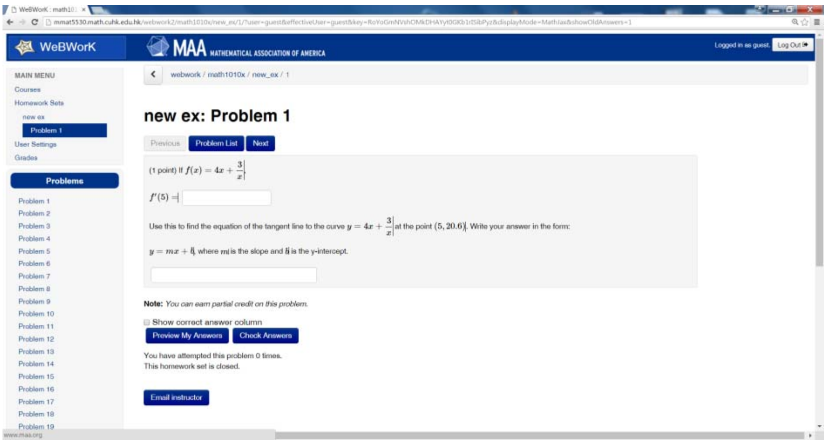
(ii) Multiple choice (in short, MC) questions.

(iii) Questions requiring "symbolic answers" (i.e. the answer involves some mathematical expressions which contains both English words, numerals and mathematical symbols).

WeBWorK's strength – checking questions involving "symbolic" answers

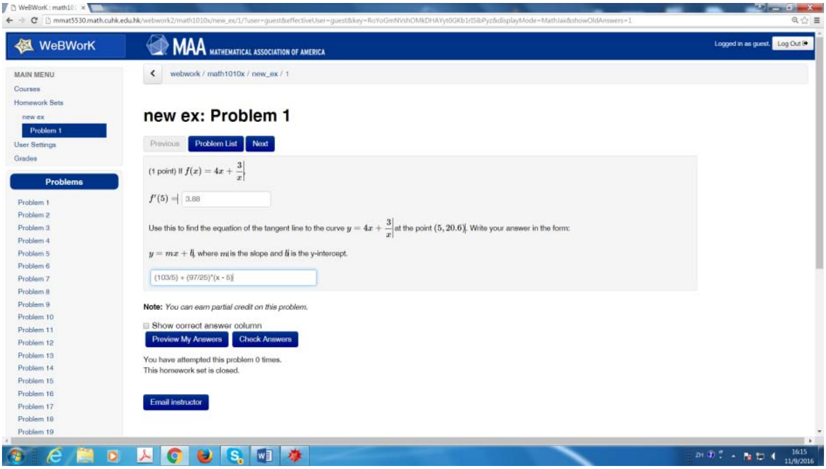
In the past, numerous software programs have been developed to handle MC questions as well as True/False questions. It is the capability to check "symbolic answers" which makes WeBWorK unique.

We will see what the above means by way of an example. Consider the following question:

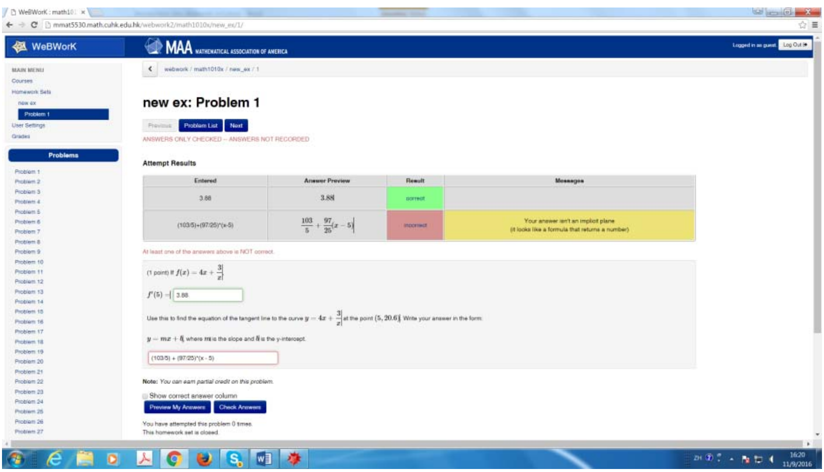


For this question, there are two fields to fill in the answer, the first of which asks for a numerical solution, while the second one asks for a "symbolic" solution.

Let us concentrate on the second "symbolic" solution. Suppose the student types the following



Upon clicking the button "Check Answers" (or "Submit" answers), the following screenshot appears:



Here the software program sends an error message to the student, i.e. the student should have typed $y = \left(\frac{103}{5}\right) + \left(\frac{97}{25}\right) *$ $(x - 5)$ instead of only the right-hand side (of course, any other equivalent forms for the right-hand side such as

- $20.6 + 3.88 * (x - 5)$

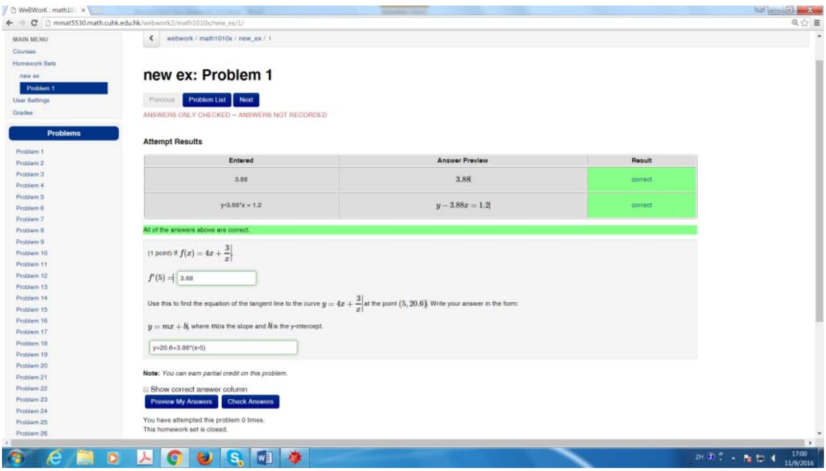
- $1.2 + 3.88 * x$

would be equally acceptable.

If we type as answer

$$y = 20.6 + 3.88 * (x - 5)$$

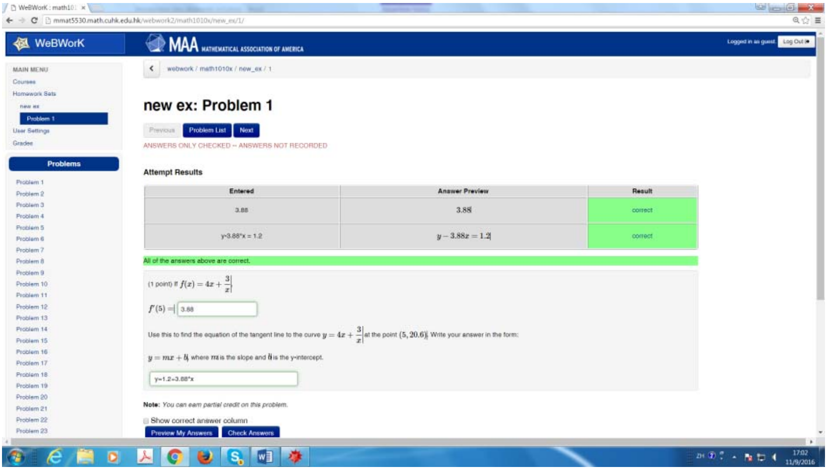
and "check/submit" answer, we see the following screenshot:



- and if type

$$y = 1.2 + 3.88 * x$$

we get the following one:



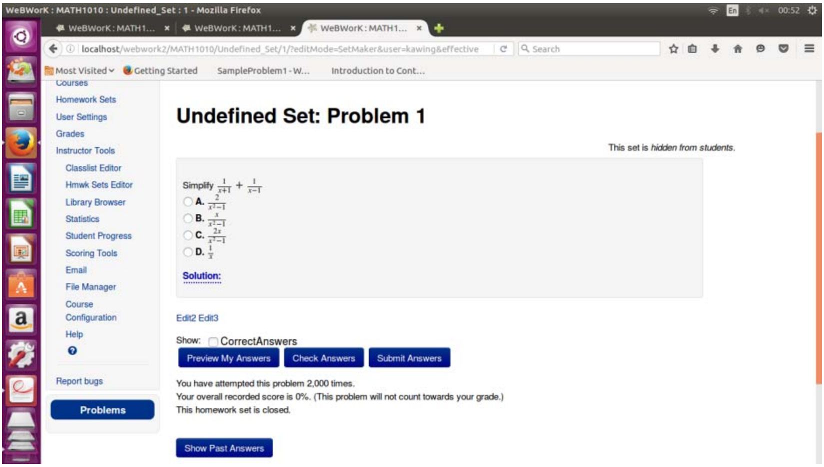
Other traditional types of questions

So much for "symbolic" answer type questions. Indeed, one thing which makes WeBWorK stands out in comparison to other related software lies in its "symbolic-checking" functionality.

To make the description slightly more complete, we now show the screenshots of some traditional types of question, such as MC questions and True/False questions.

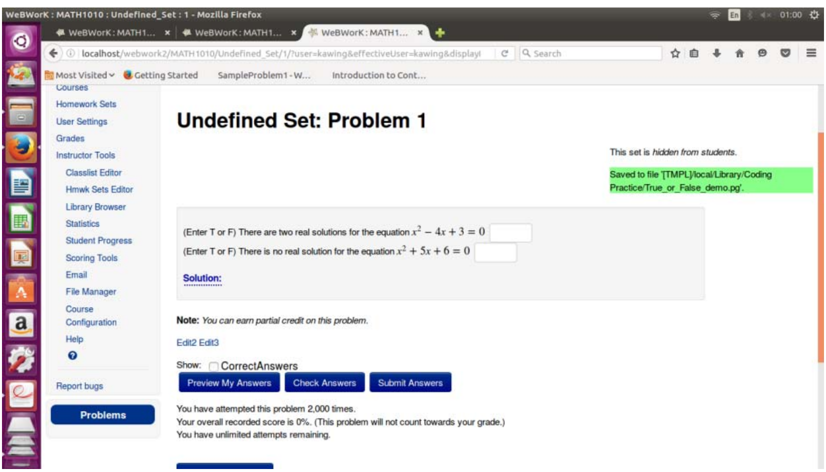
Example of an MC question

This is an example of a MC problem:



Finally, we have the following screenshot of a true/false question.

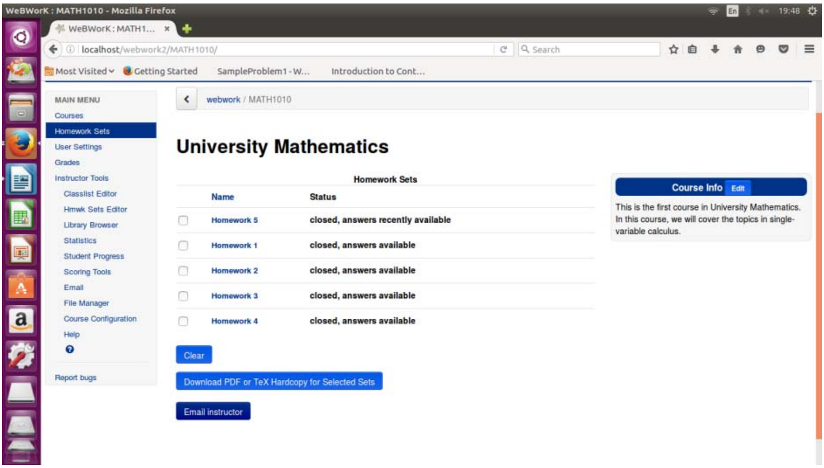
Example of a true/false question



Instructor’s Point of View

In this section, we briefly describe the various layouts on the monitor screen from the instructor’s perspective.

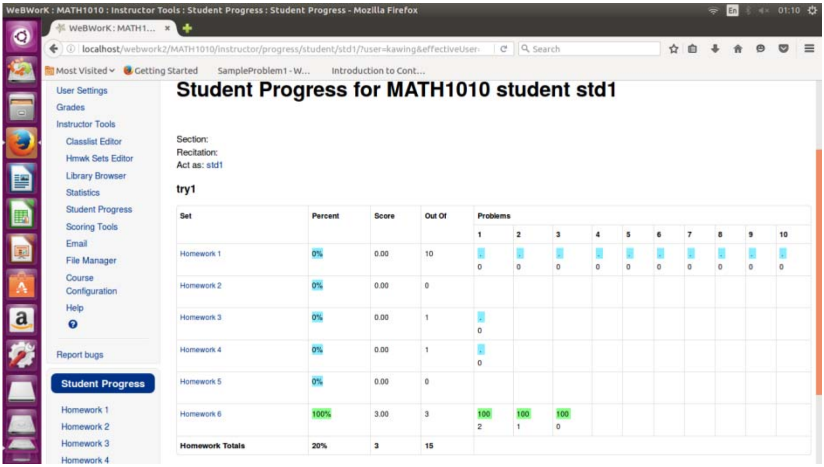
This is a typical layout which an instructor sees after logging the course he/she is teaching.



Now he/she can edit and do lots of things for his/her course. Among these various things, we mention just one of them.

Tools for an instructor – student statistics

For one thing, the instructor can view “performance statistics” of any student in his/her class on the screen. This is shown below:



Summary

In the above, we gave a very quick selected tour through the software WeBWorK. In particular, we highlighted its symbolic answer checking capability.

What we have not described are things like

- How the instructor sets the questions/homework deadline for each individual student.
- How the instructor sets the various access rights for each individual student.
- More detail about the source code of the program.

Remarks

A careful reader should have noticed the differences in some of the screenshots shown above. The reason behind is that these screenshots have originated in two different version of WeBWorK, one of them is based on a single DVD and the other based on a server. Despite this small difference, the functionality of the program is basically the same. Any reader who is interested in the program can freely download either of these two version from the Mathematical Association of America website.

3. A Constructional Approach to Learning Mathematics

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Department of Mathematics and Information Technology
The Education University of Hong Kong

Introduction

Mathematical concepts are abstract. They only become concrete and meaningful when one masters all the essential components of the concepts. Mastering the concepts requires forming a coherent concept image. More specifically, concept image is about examples or graphical representations of a concept intertwined with logical relationships that evolve over time as the learner gains better insight of the concept (Tall, D. and Vinner, S. 1981).

The formation process of a concept image can be thought of as constructing a simple house. First, there needs to be a good foundation. Then up go the pillars or walls to support the integrity of the structure. And finally the roof goes on top of the pillars or walls and connects all the supporting pieces into an integral structure. Studying Mathematics is similar to building a house. It has three key stages that are analogous to the three house construction stages. First, one needs to have an adequate amount of prior mathematical knowledge as foundation. Then one makes observation of certain properties and investigates the attributes of these properties. From the investigation one arrives at a set of conditions that abstract the properties and then devises a theorem that connects the conditions with the original observations. These three stages of studying Mathematics are essentially how one conducts research in Mathematics.

To facilitate such a process, there can be learning tasks organised is such a way that allows students to construct the concept (Freudenthal, 1991). It is worthwhile to point out that there are always conceptual difficulties or obstacles pertaining to a given mathematical concepts. Upon a deeper analysis of these difficulties, one arrives at a set of core difficulties and may then devise a set of solutions specifically for the difficulties. With the solutions in hand, learning tasks can be designed to tackle the difficulties. In principle, these learning tasks may run without any teaching aids other than a pen and paper. But as Mathematics is abstract, in many instances, it is best supported by some visualisation instruments, such as IT tools.

In what follows, we present three examples of how to implement such a process in a classroom. These three examples cover topics on areas of closed figures, fraction arithmetic and the rigorous definition of limits. The first two are elementary Mathematics and the last one is an advanced one. These three examples shall collectively demonstrate that our approach can work with different levels of Mathematics.

Three examples

I) Areas of Closed Figures

The topic of areas of closed figures is taught at the primary school. Students often first start the area of a square and/or the area of a rectangle along with their area formulae. Then they move on to areas of other geometric figures, such as parallelograms, triangles and trapeziums and their area formulae. There are three known types of conceptual difficulties (Kospentairs, Spyrou and Lappas 201; Naidoo and Naidoo. 2007; Yu and Tawfeeq, 2011) in going from rectangular figures to other closed figures.

1) *the lack of the concept of area conservation, with a misunderstanding that the area of a figure is not the same before and after dissection;*

2) *the failure to identify a base and its corresponding height for area calculation;*

3) *the misconception that only regular closed figures like squares and rectangles have measurable area and corresponding mathematical formulae for area calculation; while other irregular closed figures have none.*

To overcome the above three difficulties, we first think of the meaning of area. It is about measurement units of closed figures. The conventional area unit is square: given a closed figure, one tiles it with identical squares and the number of squares that completely cover the figure represents a measurement of the "area". Note that it is perfectly permissible to use other geometric shapes to tile closed figures and arrive at a measurement, but we settle for the most versatile choice - squares. The evaluation of an area is nothing more than counting the number of squares that completely cover the closed figure.

Learning tasks below are designed to be exploratory with a focus on unit squares. They require the students to be competent in integer arithmetic. Also, they are logically linked from one to another so to facilitate a cognitive development of the concept.

Task 1: Assuming students are competent with basic arithmetic, such as counting, adding and multiplying, we can start with a given square as the measurement unit and proceed to finding the area of a bigger square or a rectangle. The emphasis here is to lay down the idea of measuring area in terms of unit squares. We are to tile the bigger square or the rectangle with unit squares and the maximum number of unit squares used is the area. This shall naturally lead to the establishment of the area formulae for squares and rectangles in unit squares.

Task 2: For the area of parallelogram, exploratory tasks are set up for the students to discover the relationship of the parallelogram area and the rectangle area via cut and paste operations. This can be done with paper but it is more versatile and precise with a computer. A GeoGebra-based applet may be developed to offer a graphical interface with a dynamic coordinate plane that accurately represents geometric objects. The dynamic functions of the GeoGebra applet allow users to manipulate, duplicate and rotate the geometric objects for a clear visualisation of the cognitive processes behind the actions on the geometric objects, which facilitate the realisation of the height and the base of parallelogram are actually the width and length of rectangle and thus helps set up the area formula for parallelogram. Also, the cut and paste operations enhance students' grasp of the concept of area conservation.

<u>Task 3:</u> Students are to diagonally cut a parallelogram into two identical triangles. This part will introduces the idea that the area of parallelogram is twice the area of triangle. To establish this, an arbitrary triangle is given and with the duplication and rotation functions of the applet, students can assemble a parallelogram from the two triangles and find that the area of triangle is indeed half of the area of parallelogram. Furthermore, it follows that the formula for triangle is simply half of the product of height and base.

The three difficulties listed above are overcome by the three learning tasks with the help of the applet.

Tasks	Key objectives
1	a) Units of measurement b) Areas of Square and Rectangle and their formulae
2	Area of Parallelogram and its formula
3	Area of Triangle and its formula

Table 1: Key Objectives in the learning tasks of areas

II) <u>Fractions Addition and Subtraction</u>

Fraction arithmetic is an important and difficult topic in primary education. It is particularly hard to students when it comes to arithmetic with different denominators. In the case of fraction addition and subtraction with different denominators, there are three key core properties (Carraher, 1996; Skemp, 1986) that a student needs to grasp in order to master the arithmetic.

1) *the concept of a common denominator*

2) *the knowledge of fraction equivalence*

3) *the application of prior knowledge of whole number addition / subtraction to adding / subtracting of the fractions*

The above three difficulties stems from the problem handling denominators. Recall that for any positive integer *a*, it could be interpreted as walking *a* steps from 0 rightward along the number line with each step accounting for 1 unit. Now, addition of two integers a and b is simply walking a unit steps from 0 rightward along the number line and then follows by walking another b unit steps rightward along the number line. The total number of steps taken here is what is referred to as the sum of a and b. Thus, it is clear that addition of integers arises from counting. Now, for fraction $\frac{a}{b}$, it can read it as $a \times \frac{1}{b}$ and thus can be interpreted as walking a steps from 0 rightward along the number line but the steps here are not the unit steps. Rather, each step is $\frac{1}{b}$, a unit fraction. Thus, if we were to add two fractions with different denominators, say $\frac{a}{b} + \frac{c}{d}$, and display the sum as a fraction itself, we have a problem. The key to fraction arithmetic lies in the understanding and utilisation of unit fractions.

In what follows, we present learning tasks for fraction addition and subtraction with respect to unit fractions. There are four exploratory tasks that require students to have competence in integer arithmetic. Also, each task builds from the previous work and prepares students for the next task.

Task 1: Unit Fractions

The objective here is to introduce the meaning of unit fractions and the motivation of learning such a concept. A unit fraction is the length of an equally partitioned portion of the unit length and is denoted by $\frac{1}{b}$, where b is the number of portions.

Task 2: Fraction Addition and Subtraction with Same Denominators

(a) Take unit fraction $\frac{1}{b}$, students are to learn that adding a of them together is akin to integer multiplication, that is $a \times \frac{1}{b}$, and this sum is written as $\frac{a}{b}$. Mathematically, the work done here is $\underbrace{\frac{1}{b}+\frac{1}{b}+\cdots+\frac{1}{b}} = a \times \frac{1}{b} = \frac{a}{b}$,

where a is called the numerator and b is called the denominator.

(b) Students are to explore fractional equivalence and thereby establish the relationship, for any positive integer k,

$\frac{1}{b} = k \times \frac{1}{kb} = \frac{k}{kb}$, where b is a non-zero integer.

(c) Students can now proceed to adding or subtracting two ordinary fractions of the same denominator $\frac{a}{b} \pm \frac{c}{b}$. By (a), the fraction $\frac{a}{b}$ is the sum of a unit fractions of $\frac{1}{b}$ and the fraction $\frac{c}{b}$ is the sum of c unit fractions of $\frac{1}{b}$. It is clear that there are $a \pm c$ pieces of $\frac{1}{b}$ in $\frac{a}{b} \pm \frac{c}{b}$, and by (a) it can be written as $\frac{a \pm c}{b}$. In other words, we have

$\frac{a}{b} \pm \frac{c}{b} = \left(a \times \frac{1}{b}\right) \pm \left(c \times \frac{1}{b}\right) = (a \pm c) \times \frac{1}{b} = \frac{a \pm c}{b}$, where b is a non-zero integer.

Task 3: Fraction Addition and Subtraction with Different Denominators

(i) Students are to apply techniques learned in Task 2 to work with unit fraction addition and subtraction with different denominators here. As it is done in Task 2, there needs a common denominator and students will find a common denominator and in light of Task 2 (b), they can determine the corresponding numerators. Finally, they make use of Task 2 (c) to perform the arithmetic operations. In other words, we have

$$\frac{1}{b} \pm \frac{1}{d} = \frac{d}{bd} \pm \frac{b}{bd} = (d \pm b) \times \frac{1}{bd} = \frac{d \pm b}{bd},$$

where non-zero integers b and d are different.

(ii) Students are to extend their work in (i) to ordinary fractions addition and subtraction of different denominators. As students have dealt with two unit fractions with different denominators, they now apply the same procedure to first determine the common denominator. Thus, we have

$$\frac{a}{b} \pm \frac{c}{d} = \left(a \times \frac{1}{b}\right) \pm \left(c \times \frac{1}{d}\right) = \left(a \times \frac{d}{bd}\right) \pm \left(c \times \frac{b}{bd}\right)$$

$$= \frac{ad}{bd} \pm \frac{cb}{bd}$$

$$= (ad \pm cb) \times \frac{1}{bd} = \frac{ad \pm cb}{bd},$$

where non-zero integers *b* and *d* are different.

Tasks	Key objectives
1	Concept of unit fraction and its geometric representation
2	Fraction addition and subtraction with same denominator (a) $\underbrace{\frac{1}{b} + \frac{1}{b} + \cdots + \frac{1}{b}} = a \times \frac{1}{b} = \frac{a}{b}$ (b) $\frac{1}{b} = k \times \frac{1}{kb} = \frac{k}{kb}$ (c) $\frac{a}{b} \pm \frac{c}{b} = \left(a \times \frac{1}{b}\right) \pm \left(c \times \frac{1}{b}\right) = (a \pm c) \times \frac{1}{b} = \frac{a \pm c}{b}$

3	Fraction addition and subtraction with different denominators (i) $\frac{1}{b} \pm \frac{1}{d} = \frac{d}{bd} \pm \frac{b}{bd} = (d \pm b) \times \frac{1}{bd} = \frac{d \pm b}{bd}$ (ii) $\frac{a}{b} \pm \frac{c}{d} = \left(a \times \frac{1}{b}\right) \pm \left(c \times \frac{1}{d}\right) = \left(a \times \frac{d}{bd}\right) \pm \left(c \times \frac{b}{bd}\right) = \frac{ad}{bd} \pm \frac{cb}{bd} = (ad \pm cb) \times \frac{1}{bd} = \frac{ad \pm cb}{bd}$

Table 2: Key Objectives in the learning tasks of fraction addition and subtraction

III) The Rigorous Definition of the Limit of a Sequence

The limit concept has always been a difficult topic for senior secondary and university students. The concept of the limit of a sequence has significant impact on other related concepts in advanced mathematics (Tall and Vinner, 1981 and Tall, 1992).

Formally stated, L is the limit of a sequence $\{a_n\}$ if for any positive real number ε, there exists a natural number N such that for every natural number n greater than N, the absolute distance between a_n and L is less than ε. Symbolically, this is usually written as

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that for $n > N$, $|a_n - L| < \varepsilon$.

As in Cheng and Leung, 2014, one can break the $\varepsilon - N$

definition down into three core components:

1) *The geometric meaning of the inequality* $|a_n - L| < \varepsilon$

This inequality is an essential part of the definition that captures the concept of convergence. The other components in the definition are conditions under which this inequality holds. A visualisation tool that can lead to the intended definition should graphically represent this inequality in a way such that students can manipulate it as a dynamic *measuring* tool which can be used to discern the idea of convergence of a sequence.

2) *The condition for n to be larger than N* ($n > N$)

Students are often confused with what *n* and *N* represent since *N* does not explicitly enter the inequality and *n* relates to the part of the sequence that comes after *N* where *N* is implicit in the inequality "<" once an ε is decided. Students should be able to freely manipulate the dynamic measuring tool to find *N* so that ascertaining the existence condition $\exists N$ becomes a critical visual cognitive activity for students during the exploration process.

3) *The dependency between* ε *and N* ($\forall \varepsilon > 0, \exists N$)

The dependency of *N* on the choice of ε is the most difficult, yet critical, feature for students to grasp. The dynamic measuring tool should allow students the flexibility to explore and explain possible dependency relationship between ε and N in the process of reconstructing (or re-inventing) the limit definition. It may be pedagogically more challenging *not to* embed rigidly the intended dependency relationship into the dynamic measuring tool. This will make the exploration more open for critical discernment.

A GeoGebra-based dynamic Applet (Cheng and Leung, 2014) is specifically designed to help students learn the three components of the rigorous concept of the limit of a sequence.

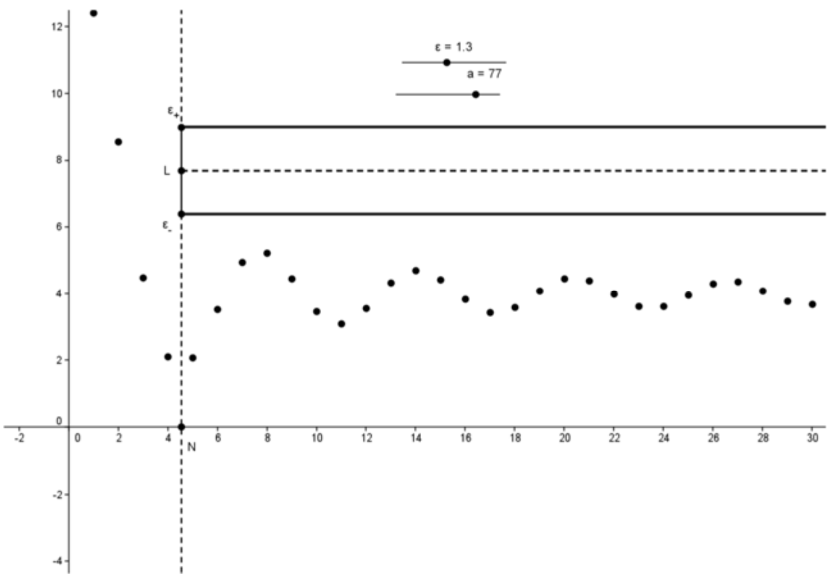

Figure: A draggable open-ended rectangular strip - the $\varepsilon -$

Prior knowledge required here are general senior secondary school mathematics and an intuitive understanding of the concept of the limit of a sequence from an introductory Calculus course. There are four learning tasks for students here.

Task 1: Students are given five sequences:

$\left\{\frac{5}{n}\right\}$, $\{n^2\}$, $\{\sin^2 n\}$, $\left\{\tan\left(2n\sin\left(\frac{1}{n}\right)\right)\right\}$ and $\left\{\frac{10\cos n}{n}\right\}$.

They are to use the applet to plot the sequences and to determine if the sequence tail can be contained in the draggable rectangular strip of given width ε. The objective here is to relate what they know of the convergence of sequences to the containment of the sequence tail in the rectangular strip. This rectangular strip is a graphical representation of the inequality of the rigorous definition.

Task 2: With the same set of sequences as in Task 1, students to learn of the role of N, the anchoring point (left edge x-value) of the rectangular strip in relation the value of the running index n of the sequence. The goal here is to establish that $n > N$ and the implicit role infinity in the definition.

Task 3: Again, with the same set of sequences as in Task 1, students are to explore the dependency of ε and N by exploring the existence of one when the other is given. In convergence cases, the existence of N is achieved given any positive ε. However, this phenomenon does not occur in the divergence cases. Students are to record all their findings and use them in the next task.

Task 4: Students summarise and combine results from the previous tasks to form a rigorous definition of sequence convergence. Note that students are not expected form an exact version of $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that for $n > N, \ |a_n - L| < \varepsilon$.

Nevertheless this task will allow students to associate all they have found and present their findings in one package.

Tasks	Key objectives
1	Geometric meaning of the inequality $\|a_n - L\| < \varepsilon$
2	Condition for n to be larger than N ($n > N$)
3	Dependency between ε and N
4	Construction of the definition

Table 3: Key objectives in the learning tasks of the rigorous definition of the limit of a sequence

Discussion on Key Attributes of the Three Examples

The overall design philosophy is to facilitate a logical build-up of a concept. A concept can be broken down into several pieces of core attributes that are logically linked to each other. It is through this logical chain of relationships that students build up their understanding of the concept.

I. Areas of Closed Figures: With the introduction of square units for measurement, students construct the area formulae for squares and rectangles, and then proceed to parallelograms and finally arrive at triangles. As they apply the cut and paste procedure with the applet in their exploration, they may continue their work to other closed figures, such as trapezia.

II. Fraction Addition and Subtraction: The work with fraction addition and subtraction starts with unit fractions and its geometric interpretation. The geometric interpretation is critical to all subsequent tasks as it offers the students a visual representation of fractions that allows them to find common denominators and more importantly, it facilitates students' grasp of the fraction equivalence. The ensuing tasks deal with fraction addition and subtraction of same denominator with unit fractions learned in the first task. The final task on fraction addition and subtraction of different denominators, which is often the biggest challenge for students, is now straight forward by applying knowledge learned from the previous two tasks.

III. Rigorous Definition of the Limit of a Sequence: As this definition is advanced and complex, the design principle of the tasks is the break to definition into core parts and explore their properties or characteristics in turns. Also, as students are assumed to have an intuitive knowledge of the convergence of a sequence, which is the prior knowledge here, they work through the tasks with this prior as a beacon of reasoning. The first three tasks are the core attributes of the definition and with knowledge gained from the first three tasks, students are to assemble a version of the rigorous definition in the final task.

Principles observed from the examples.

1. Strong emphasis on the use of prior knowledge
2. Student explorations are designed to use prior knowledge as a reference during investigation. Also, when possible, the explorations should run along a path where one section logically leads to the next section.

The keys to success are that the teacher needs to isolate the many properties a concept may have and focus only on the critical ones. As soon as these critical ones are identified, the teacher can start designing the tasks where students explore each key property in turn. This is in line with the axiomatic approach in modern Mathematics.

Conclusion

Assuming that students are mathematically prepared for a mathematical concept, that is, they have an adequately grasp of prior knowledge, a traditional lesson starts with motivating examples to draw students' attention and then presents the concept and then perform further illustrations of or investigation on the concept. In essence, in view of the house building metaphor, the roof is shown first and then the whole house is displayed to the students. The pillars are subsequently constructed at the end. Quite often, the whole process is done by the teacher while students watch and listen. This is an effective way to tell students what a certain mathematical concept is, but it is less effective in helping students form a clear concept image. This is primarily due to the fact that the order of presentation is not natural and also, more to the point, the teacher builds the house while the students stand aside. The end result is simply that the students remember what the house looks like from the outside but only have a vague impression of what is inside. It is, however, the inside of the house that counts when attempting to form a good concept image.

If we take a flash back at how Mathematics is done through its history, we see that when mathematicians see a mathematical phenomenon or patterns or encounter a problem. They investigate the nature of the problem through a series of mathematical experiments or calculations. As they gain a deeper understanding of the problem, they theorise what they find and then construct a proof for it. Of course, there are numerous occasions where a proof is never found or is still being studied. Nevertheless, this is the approach of how Mathematics is created. We advocate to take learning Mathematics along this path. In the past, even though H. Freudenthal proposed it to the mathematical community, few had followed in practice. But with the advancement of information technology and its user-friendliness, we can develop simple yet powerful software for Mathematics classes and set the students on a journey of mathematical discovery.

References

[1] Bonwell, C. and Eison, J. (1991). *Active Learning: Creating Excitement in the Classroom AEHE-ERIC Higher Education Report No. 1*. Washington, D.C.: Jossey-Bass.

[2] Cheng, K and Leung A. (2014). A Dynamic Applet for the Exploration of the Concept of the Limit of a Sequence, to appear.

[3] Carraher, D.W. (1996). Learning about fractions. In P.P. Steffe, P. Nesher, P. Cobb, G.A. Goldin and B. Greer (Eds), Theories of mathematical learning, 241-266. NJ.: Lawrence Erlbaum.

[4] Cory, B. and Garofalo, J. (2011). Using dynamic sketches to enhance preservice secondary mathematics teachers' understanding of limits of sequence. *Journal for Research in Mathematics Education*, 42(1), 65-96.

[5] Freudenthal, H. (1991). *Revisiting Mathematics Education*. China Lectures. Dordrecht: Kluwer Academic Publisher.

[6] Kong, S. C. (2008) The development of a cognitive tool for teaching and learning fractions in the mathematics classroom: A design-based study. *Computers & Education*

51, 886–899

[7] Kospentaris, G, Spyrou, P. and Lappas, D (2011). Exploring student's strategies in area conservation geometrical talks. *Educational Studies in Mathematics*, 77 (1), 105-127.

[8] Mamona-Downs, J. (2001). Letting the intuitive bear on the formal: A didactical approach for the understanding of the limit of a sequence. *Educational Studies in Mathematics*, *48*, 259-288.

[9] Naidoo, N. and Naidoo, R. (2007). Using Blended learning to facilitate the mathematical thought process of primary school learners in a computer laboratory: A case study in calculating simple areas. *Journal of College Teaching and Learning*, 4(7), 79-94.

[10] Roh, K.H. (2008). Students' images and their understanding to definitions of the limit of a sequence. *Educational Studies in Mathematics*, 69(3), 217-233.

[11] Skemp, R.R. (1986). The psychology of learning mathematics (2nd ed.). Middlesex: Penguin.

[12] Tall, D. (1992). The transition to advanced mathematical thinking: Functions, Limits, Infinity and Proof. *In Grouws*

D.A (ed.) Handbook of Research on Mathematics Teaching and Learning, Macmillan, New York, 495-511.

[13] Tall, D. and Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169.

[14] Yu, P.W.D. and Tawfeeq, D.A. (2011). Can a kite be a triangle? Bidirectional discourse and student inquiry in a middle school interactive geometric lesson. *New England Mathematics Journal*, 43, 7-20.

4. 設計跨學科 STEM 活動

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我設計的 STEM 活動

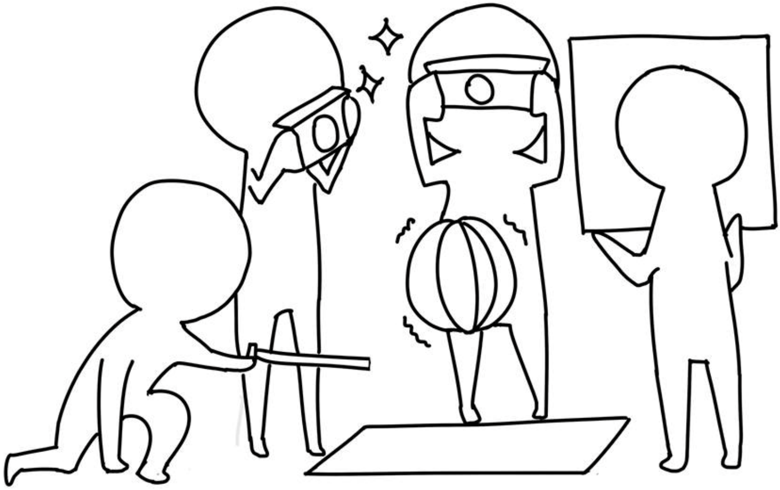
我在設計跨學科 STEM 活動時，有四個出發點:

1. 學生覺得有趣、好玩
2. 能關聯學科知識
3. 能提升學生的手動及解難能力
4. 所需材料簡單、價錢平

按以上四點，本校進行了不少跨學科 STEM 活動，現介紹一個跨學科 STEM 活動作參考實例。

「球」同存異

活動是利用靜電棒將保溫鋁箔 PET 膜（鋁箔條）升起並拍照，再用電腦軟件 ImageJ 量度角度和長度。利用中二數學課程內的誤差概念，比較百分誤差、最大誤差值共加上誤差的標準差（Standard Deviation）去計算實驗的準確度。同學需裁剪鋁箔條及拍照，然後收集數據並計算當中誤差，再在研習報告寫出總結並設計創意鋁箔條樣式。



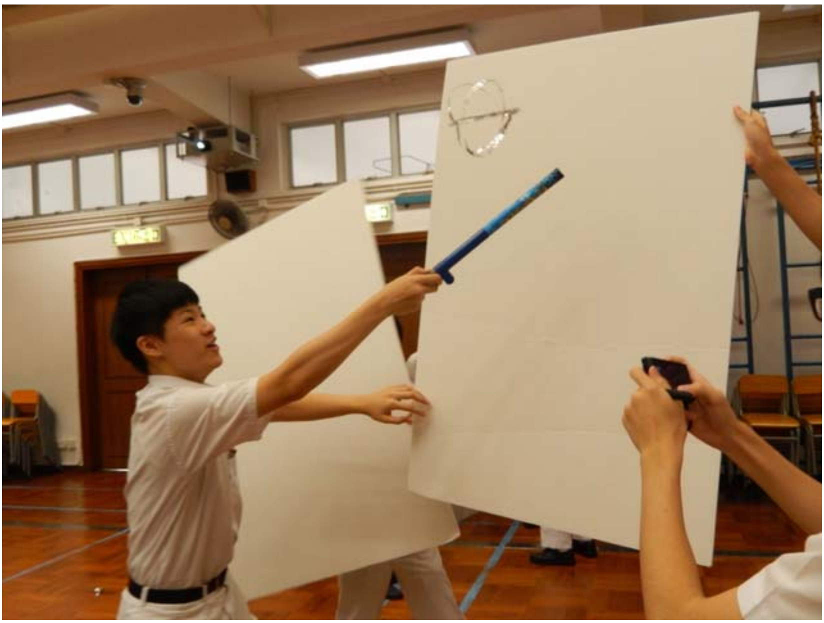
S

認識靜電棒的結構及科學原理

靜電棒的內部為「范德格拉夫發電機」，當中透過膠帶高速轉動，把棒子頂端的電子給帶到棒子末端，產生靜電。而鋁箔條中的電子也會被帶到棒子末端，因為電磁同極相距的關係，鋁箔條會與棒子相斥，就好像把兩塊同極的磁石放在一起的情況一樣，從而使鋁箔條飄浮在空中。

原理與「范德格拉夫發電機」相同，當啟動靜電棒時，馬達會推動膠帶高速轉動，把銅片的電子從棒子頂端帶到棒子末，從而產生靜電。

當鋁箔條放到棒子頂端上，電子便傳到鋁箔條。因為同極相距的關係，鋁箔條與鋁箔條之間便會撐開，成一球狀。與棒子相距而形成飄浮狀態。

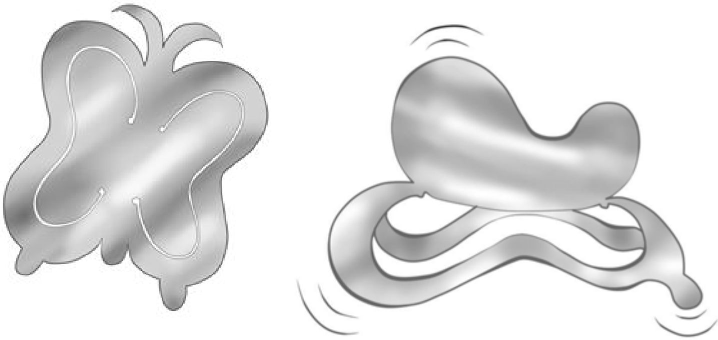


E

培養手動及創造能力 感受流體力學

在科學素質中，手動能力是非常重要的一環。活動中要同學用美工刀裁剪鋁箔紙成實驗所需的鋁箔條來形成球體。看似容易的步驟卻讓同學們怨聲載道，因要準確的裁剪成指定大小，一不小心又會把鋁箔條弄斷，要重新開始!

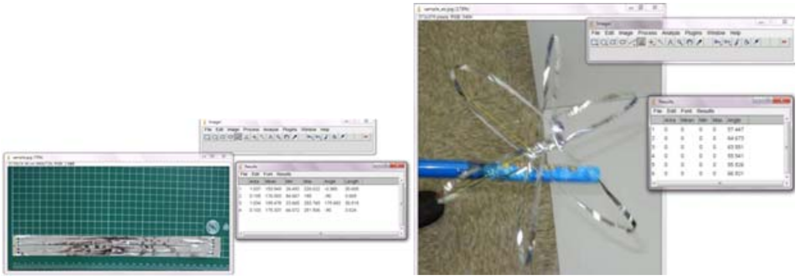
在進行數據分析的同時，同學亦要設計一張可在空中飄浮的鋁箔紙。同學要嘗試裁剪出不同形狀，測試那種設計可使用靜電棒使其在空中飄浮，從中領略工程學中的流體力學。



利用電腦輔助作精確量度

當同學成功拍照後，便需要量度角度作分析。最簡單當然是用量角器，可是要精確量度，便要借助電腦軟件了。

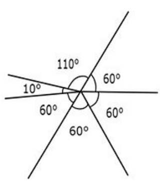
在本活動中，是使用了一個公共的圖像處理軟體，ImageJ。它能夠進行圖片的區域統計，量度間距、角度甚至面積。希望同學學會使用軟件的技巧後，可應用在其他科目身上，例如在生物科量度樹葉的面積或是在地理科量度山坡的斜度等。



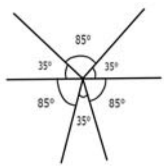
那個統計方法最合用?

在假設鋁箔條平均地張開成一個球體時，透過∠s at a pt. 的定理，每個角度應該為 360°/6＝60°。在實驗的過程中，同學需要製造鋁箔條，進一步計算每個角度是否等於六十度，從而判斷其假設的準確性。在初步測試中，同學已確定球體不會張開成標準的 60°。可是，怎樣去判斷那一個球體的誤差最小呢? 因此，同學需利用數據作出分析，包括比較平均百分角度誤差、最大角度誤差值及角度誤差的標準差（Standard Deviation）等。

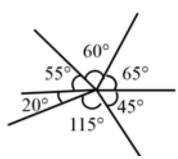
如以下例子:



例子一



例子二



例子三

	平均百分角度誤差	最大角度誤差值	角度誤差的標準差
例子一	27.78%	50°	23.57°
例子二	41.67%	25°	0°
例子三	33.33%	55°	20.41°

例子一的平均百分角度誤差是最小的，但其角度誤差的標準差卻是最大的。由此可見，採用不同計算角度誤差的方法所得出的結果並不相同。因此，同學要知道沒有那個方法是最準確，在查看統計報告時，要小心檢視其分析方法是否恰當免被誤導。亦希望同學能明白到分析數據時不能只偏信於一種計算方法，而要多用各種不同計算的方法去得到更客觀的分析。(註)

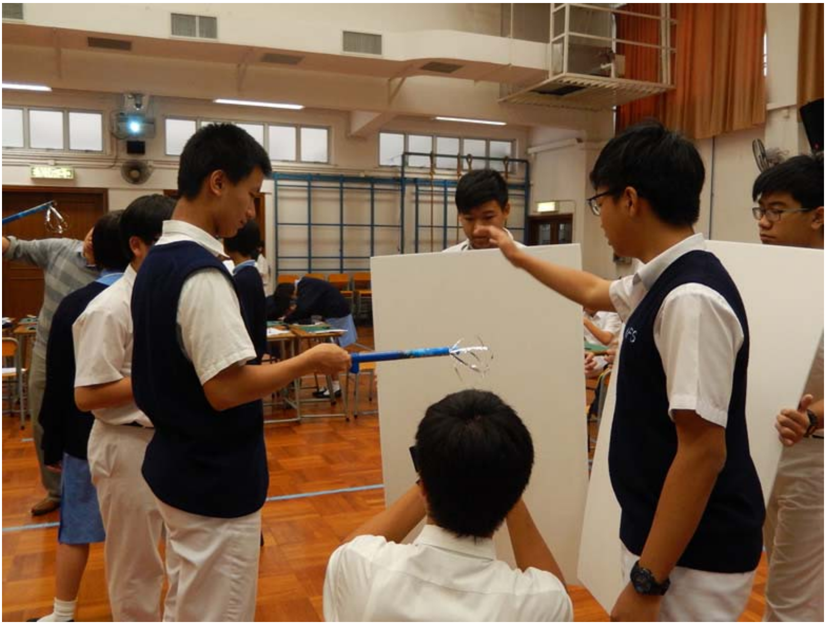
實驗數據和理論的差異

進行這次活動，除希望同學快樂地玩一下科學玩具、應用一下電腦及數學外，亦希望同學可透過實驗讓同學發現數據不一定與科學理論一致。如本活動建基於 angles at a point 的理論上，可是要做到每一角度等於六十度，在現實中根本是不可能! 當中包含很多因素，所以同學要在報告中指出實驗的誤差(source of error)何在，從而希望同學注意實驗設計的細節及理論應用的條件，不要只憑空想像，閉門造車!

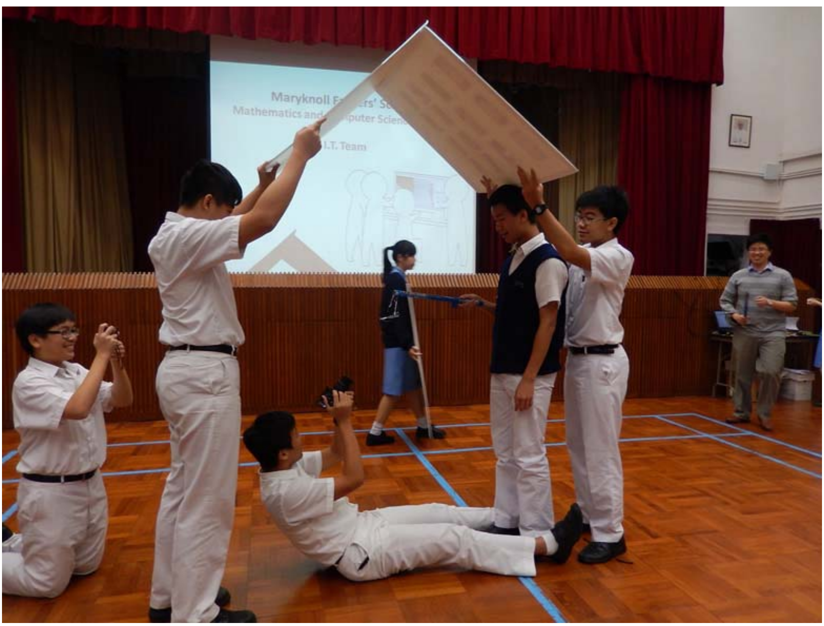
何為 STEM 活動?

最近鋪天蓋地的在說 STEM，大家都說推動 STEM 對學生有好處，可是科學、科技、工程及數學可說是有關係，可是要把四個方向揉合在一起，又很是讓人卻步!「球」同存異是少數包括全部四個元素的 STEM 活動。不過，在設計 STEM 活動時，是沒有要求一次滿足四個願望。其實，只要有 STEM 四項中的最少兩項也符合了 STEM 活動的定義了。如此一來，設計 STEM 活動便容易多了。可以是數學加科學的，也可是科技加上數學的。

我校還曾進行地理跨數學、中文跨科學，甚至是體育跨科學的活動。要說能推出一個又一個活動的要訣是甚麼? 便是要有一班志同道合、一心為學生的教師團隊。一位常識豐富的教師，也不可能知道其他學科需要甚麼。但有各學科參與其中便可以把各學科對同一活動的想法及學術要求說出來並加以改良，讓學生做活動時得到最佳效益，這也便是成功設計的條件了!



同學嘗試以不同角度以拍出最佳效果。



註：

1. 平均角度誤差(μ_{Ae}):

$$\mu_{Ae} = \frac{\sum_{i=1}^{6}|\angle A_i - 60°|}{6}$$

$$= \frac{|\angle A_1 - 60°| + |\angle A_2 - 60°| + |\angle A_3 - 60°| + |\angle A_4 - 60°| + |\angle A_5 - 60°| + |\angle A_6 - 60°|}{6}$$

2. 平均百分角度誤差:

$$\frac{\mu_{Ae}}{60°} \times 100\%$$

3. 最大角度誤差值:

$$\text{Max}\,(|\angle A_1 - 60°|, |\angle A_2 - 60°|, |\angle A_3 - 60°|, |\angle A_4 - 60°|, |\angle A_5 - 60°|, |\angle A_6 - 60°|)$$

4. 角度誤差的標準差:

$$\sqrt{\frac{\sum_{i=1}^{6}(|\angle A_i - 60°| - \mu_{Ae})^2}{6}}$$

$$= \sqrt{\frac{(|\angle A_1 - 60°| - \mu_{Ae})^2 + (|\angle A_2 - 60°| - \mu_{Ae})^2 + (|\angle A_3 - 60°| - \mu_{Ae})^2 + (|\angle A_4 - 60°| - \mu_{Ae})^2 + (|\angle A_5 - 60°| - \mu_{Ae})^2 + (|\angle A_6 - 60°| - \mu_{Ae})^2}{6}}$$

5. STEM 跨學科學習計劃－磁浮列車創作活動

張永泰

伯裘書院

科學、科技、工程與數學（Science, Technology, Engineering, Mathematics, 簡稱 STEM）的科際整合教育議題近年來受到各界的關切與重視，香港教育局明確表示學校推廣「STEM」教育的意向，指出首要目標是培養學生對相關學科的興趣，提升他們的跨學科學習和應用能力，從而發揮潛能和創新能力，加強「動手實踐」、「培育創意」及「連繫不同學習領域」等科技教育理念的元素[1]，將理論導向的設計、探究等策略納入動手實作過程中，並將學會的知識推廣至科技發展及應用項目上，藉此培養能夠整合理論與實務的科技人才。

本校本年度推展 STEM 跨學科學習計劃，以全校參與的形式推廣 STEM 項目，並以三大方面作推展，包括建構 STEM 學科學習活動、課後培訓計劃以及推行全校性 STEM 活動。學教師於課堂活動上加強學生跨學科的解難能力及數學應用，讓學生學習上取得成功感。其中專題研習是用來緊扣科學教育、科技教育和數學教育相關的學習元素。學生從一個真實問題展開研習。在研習的過程中，學生探究解決問題的各項相關知識，並自行從不同的學習領域引入相關學習元素，綜合所學的知識與技能，把它們應用在真實生活的情境中。

磁浮列車是全球鐵路運輸系統其中一個重要的發展項目。

磁浮列車是一種靠磁浮力:磁吸力和排斥力，來推動的列車。由於軌道的磁力使之懸浮在空中，列車行走時不需要接觸地面，因此其阻力只來自空氣的阻力。截至二零一六年四月，全球共有三條商業磁浮列車線路，包括中國上海磁浮列車系統。國內的鐵路系統發展成熟，磁浮列車系統亦積極發展，亦透過「一帶一路」的經濟項目將國內的鐵路技術引進其他國家。為了提升學生對磁浮列車項目的認知，本校舉辦了簡化版磁浮列車創作活動,作為專題學習主題,強化學生綜合和應用知識與技能的能力、培養學生的創造力、協作和解決問題的能力。

是項活動是以第三學習階段課程作框架的延伸性活動。透過動手製作磁浮列車，當中連貫了不同學習領域的學習元素。例如在數學學習領域中，學生能探究常見的公式，並代入數值於公式，以理解生活中速率、距離及時間之間的關係及處理量度時的誤差。活動中亦配合科學學習領域中的能源轉換知識及電學基本原理。另外，在科技學習領域中，學生能製作短片或簡報以展示學習成果，從不同的學習活動中豐富學生的學習經歷。

製作工具與材料

學生製作磁浮列車的外形

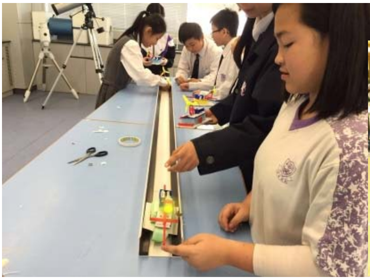
下表列出有關磁浮列車創作項目於不同學習領域的相關學習課題。

學習領域	數學	科學	科技
學習課題	代入法與解方程 單位換算	能源轉換，電磁學–電路設計	短片製作 電路組裝
學習目標	• 運用代入法及方程解決問題 • 處理速率、距離及時間之間的關係 • 轉換日常生活度量的單位，包括:長度、重量及時間	• 掌握磁浮列車能源轉換 • 掌握電學基本原理。	• 運用短片或簡報發表活動成果 • 應用基本電學原理(電路開關及閉合電路)和掌握相關知識。

活動項目先由教師與數理科技學會中四級學生聯合草擬設計，包括路軌製作、磁浮列車測試及短片指引製作。為了簡化磁浮列車的製作，磁浮列車模型以永久磁鐵作為磁浮力，並利用摩打推動扇葉以作推動力，而路軌亦有一排相同極性的磁鐵條放於兩則作磁浮之用。學生可運用相同極性的磁石粒，透過排斥力令列車模型浮於路軌之上。

活動開始時，教師先給予學生製作指示，並且將指示拍成短片以方便學生觀看。學生需要根據磁浮列車的製作要求組合列車，包括磁石的極性、列車路軌的闊度及電路基本裝置。活動時，學生會獲得一些製作材料，例如卡紙、海

棉、磁石等，學生自由借用製作工具，發揮自己的創意，以小組形式設計磁浮列車工具，當中亦包括設計和擬定具體的實踐方案。過程中，如學生在設計上遇到困難，教師可利用問題引導，以加強師生互動的關係。



磁浮列車測試

學生與列車設計作品合照

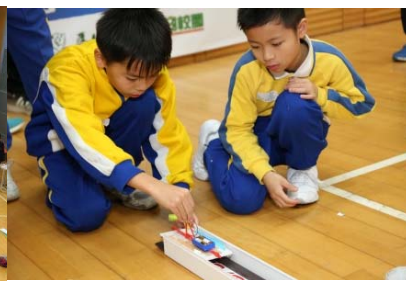
學生由動手做列車底板，到把整架磁浮列車造出來，需時約十多分鐘。學生完成磁浮列車後，都懷著興奮的心情進行測試，可是在初次測試時，部分學生未必即時成功，例如出現電路裝置錯誤、列車的闊度偏差、列車的重量不平均等等問題。學生透過吸取不同失敗的經驗，把列車的設計作出改進，當中可與同學互相學習及交流，以達至成功的列車設計。學校在推展磁浮列車活動上發展了不同的項目，除了讓學生可在課堂進行活動及測試外，也在初中全級舉行磁浮列車創作比賽，學生在限時內設計自己隊伍的磁浮列車並進行速度比賽。

學生完成列車設計後，需要量度列車行走指定路程所需的時間，並利用代入法計算磁浮列車的平均速度。由於學生在小學時已對速率的概念有一定的認識，所以他們在計算速率、距離及時間之關係時均表現理想，亦能在進一步處

理相關率與比的問題。部分能力高的學生會透過不同的行走距離以測試磁浮列車的平均速度，及比較各組同學磁浮列車平均速度不同的因素。學校透過工作紙了解學生在學習上的成效，從而評估相關教學設計是否達到預期目的以及如何改善活動流程。

本校 STEM 教育計劃的推行，除了加強校內的數理科技文化外，亦希望藉著活動項目，與其他學校分享 STEM 教育計劃的經驗及活動體驗。因此於 2016 年 3 月 19 日，本校聯同多個機構及組織，包括政府資訊科技總監辦公室、香港大學 Technology-Enriched Learning Initiative (TELI)，聯合舉辦數理科技學習匯 2016 (STEM Learning Fair 2016)。當日舉行全港學界首屆 STEM 磁浮列車創作比賽，吸引了超過三十多隊共二百名小學生參與。學生利用大會供應的物資與工具，即場製作屬於自己隊伍的磁浮列車，並與其他隊伍作出比拼，於最短時間完成磁浮軌道。透過活動後問卷調查得悉大部分參加者認為是次比賽有助他們提升數理科技的興趣，以及運用數學知識於磁浮列車創作上。

磁浮列車創作比賽大合照



學生進行比賽

即場製作磁浮列車　　　　　本校學生指導參賽者

推行 STEM 跨學科學習計劃在學校各層面都能有所得益。於學校層面，有助獲得推動學校數理科技的活動的經驗，提升學校的數理科技學習文化和氣氛。而於教師層面，此計劃能促進學校數學科組與其他科組的協作交流，例如與科學、科技學科組，加強教學活動的設計及教學資源的運用，藉此提升學生運用數學於生活問題上的解難能力。而於學生層面，學生可透過動手製作活動，擴闊他們的視野及科技知識，提升創意思維及解難能力，這正是 STEM 教育一個學習的重要元素。

參考資料：

1. 教育局(2015)- 推動 STEM 教育-發揮創意潛能(概覽)。

6.　小學生運算能力的培養

張美儀

順德聯誼總會何日東小學

良好的運算能力是學好數學的先要條件，是發展邏輯思維及培養解決數學難題能力的基礎。要培養學生的運算能力，需要讓學生掌握感知數學特徵的能力、對數的分解組合能力、靈活運用計算法則與運算定律的能力等。運算能力強的學生能靈活地運用合理的運算方法，計得準、速度快。從實際教學經驗所見，在小學時期，尤其是在初小階段，就得開始培養學生的運算能力。

學生運算能力欠佳的原因

現今很多小學生處理運算題常會出錯。以解應用題為例，算式列對了，但計算時經常失誤，以致未能求得正確答案；也有些學生未能靈活運用數學知識，只會機械式地運算，導致運算過程繁複，速度也變得很慢，更甚者是運算過程欠合理，算不出正確的答案來。學生未能培養良好的運算能力，總括而言，主要原因有三，其一是還未能透徹理解及掌握運算的法則和定律，其二是對數學特徵的觀察和推理能力較差，其三是缺少良好的運算習慣。

培養學生運算能力的方法

一. 讓學生透徹理解運算的法則和定律

要提高運算技巧，反覆的練習確實可讓學生熟練運算法則，但在進行運算練習之前，必須讓學生透徹明白各項運算方法的原理，還要了解使用某種運算方法的原因。當學生明

白為什麼要這樣計算時，不但不易遺忘學過的法則，而且能將法則靈活地運用在不同的題型上，具有正面的遷移作用。

以計算小數除法為例，如果教師只教導學生移動小數點的口訣，而未有讓學生理解移動小數點的意義和目的，他們就不能明白這運算技巧的數學原理，容易出現移位錯誤，例如計算 12.76÷2.2，容易算成 1276÷22。學生常在小數點移位的方向或在轉移的位數上出錯，教師必須讓學生明白被除數和除數移動小數點的方向及位數必須一致，是因為這兩個數必須按同一比例擴大或縮小，才不會影響計算的答案，它們移動的步數主要取決於除數的小數位，因為移動之目的是要把除數擴大成整數，方便運算。

二. 熟練運算基本技巧及熟記特別數據

學生除了要明白運算法則所依據的數理，對一些運算的基本技巧必須熟練掌握，例如整數、小數、分數的四則運算，以及小數、分數、百分數之間的互化運算，在明白互化的概念和技巧後，若能熟記一些互化的數值，例如：

$$\frac{1}{2}=0.5=50\% \qquad \frac{3}{8}=0.375=37.5\%$$

$$\frac{1}{4}=0.25=25\% \qquad \frac{5}{8}=0.625=62.5\%$$

$$\frac{3}{4}=0.75=75\% \qquad \frac{7}{8}=0.875=87.5\%$$

$$\frac{1}{8}=0.125=12.5\% \qquad \frac{1}{16}=0.0625=6.25\%$$

學生便能在計算一些小數、分數四則混算題時，自然熟練地把特別的小數和分數換算，提升計算效率。

此外，訓練學生恰當地採用簡捷算法進行運算，也是十分重要的，因此，教師不但要讓學生掌握數學概念，還要培養他們的觀察力，使他們靈活運用加法結合和交換律、乘法結合和交換律及乘法分配律等於計算上。假若學生能熟記一些乘法結果，例如：5×2＝10，25×4＝100，125×8＝1000等，也有助學生進行運算。只要學生能細心觀察，加上敏銳的思考，便能簡化計算的程序，快而準地算得答案。

三. 加強心算的訓練

心算是指不借助紙筆或任何計算工具，只憑思維直接進行計算。心算是培養運算能力的基礎，更是發展筆算和估算的關鍵。在小學數學教學中，可以分階段持續培養學生建立準確熟練的心算能力，例如在一、二年級安排 20 以內的加和減法、乘法表內的乘除法，100 以內的加減法等基本心算訓練；三、四年級安排 100 以內一位數乘、除兩位數等心算訓練，五、六年級則安排數的分解與組合及一些利用運算定律進行的心算題，以訓練學生靈活運用數學知識的能力。

心算練習須恆常性的在課堂上落實進行，教師可於每節課預留 3 至 5 分鐘的時間，進行不多於 10 題的心算題，選題必須配合學生在該階段的學習內容，適當地安排複習內容，既能溫故知新，亦能逐漸加強學生的運算技能。進行心算練習時，應要求學生算得既準且快。學生答對了題目，應

即時給予稱讚和鼓勵，加強學生學習的動機。

四. 加強「一題多解」的訓練

鼓勵學生探討不同的解題方法，不但有助發展其思維能力及運算能力，更能增強學生學習的積極性和培養其創造力。在課堂教學中，教師宜多舉不同的解難方法供學生參考，透過師生或生生之間的共同討論，讓學生多分析和比較，以引導他們找出最簡捷的計算方法，增強學生解題的技巧和培養他們獨立思維的能力。遇到學生提出有創意的解法，教師應給予肯定和鼓勵；遇到學生提出錯誤的解法，教師也應對其積極思考的態度，給予肯定和讚賞，然後引導學生明白錯誤所在，讓他們自我修正。

五. 培養估算的能力

培養學生估算的能力，讓他們判斷計算結果是否合理及對其判斷加以解釋，可提高他們運算的準繩度。估算教學可滲透在一些適合的課題上，若學生已懂得利用四捨五入法求近似值，教師可引導學生把這技巧應用在多位數四則運算或小數乘、除法中，進行簡單的估算或對運算的結果進行粗略的檢驗。以之前所舉的小數除法 12.76÷2.2 為例，學生可能錯誤運算成 1276÷22，計算結果是 58，是原來答案的 10 倍，如果先估算，原式答案應是接近 13÷2 的值，即大約是 6，學生便可很容易發現錯誤並迅速地作出修正。

六. 培養良好的運算習慣

有效益的運算練習能幫助學生鞏固數學概念，強化和熟練運算技巧。教師宜有規劃地安排由易到難、多元化和質量

佳的練習，以激發學生的興趣，提高他們運算的效率。進行運算練習前要確保學生已清楚理解有關公式和定理，教師還要強調學生須認真審題、仔細運算和按步檢查，並同時思考所用的方法是否正確。完成後要讓學生檢驗計算結果是否合理。從小養成嚴謹的態度和自我檢驗的習慣，不但能培養和發展學生的思維能力和判斷能力，更能減低錯誤。

結語：

良好的運算能力是數學學習的重要一環，在小學數學教學上必須加強培訓，教師應透過不同的方法，在教學內容上安排適當的訓練，培養學生良好的運算習慣和認真嚴謹的學習態度。在教學過程中，關鍵在於培養學生在運算法則的理解和運算技能的掌握，不是機械地使用法則，而是通過掌握其意義然後加以應用。此外，加強培養學生的邏輯思維能力及對數學特徵的觀察推理能力，使其能靈活運用不同方法解決問題，並能對結果進行合理性的判斷，這樣，學生便能循序漸進地獲得良好的運算能力，可以合理、簡捷、和正確處理數學問題了。

7. 數學題的陷阱

馮德華

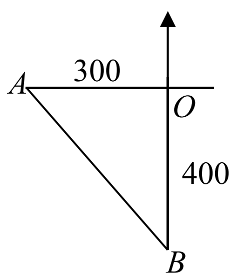
香港數理教育學會

有一位教授曾說:「很多數學教師只是在課堂中教授課本，而不是教授課程。」意指很多數學教師只是將課本的課題內容照本宣科講述，並沒有關心課程的設計、處理該課題的深淺度以及在整個課程的定位等，於是課本的課題內容過深、過淺、甚或出錯，教師未必察覺。課本的作者可各師各法地選取及演繹書內課題，其中有些題目似是而非，令教師產生疑惑！教師應時刻思考課本所陳述的內容，否則容易掉進陷阱而不自覺。教師不應因為課本列在適用書目表上，便對其內容深信不疑，而應透過專業知識作出正確判斷！

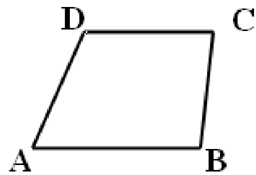
請你先完成以下四題課本例子，以便討論。

例 1. 若 $A = 1.8$ (準確至一位小數)，求 A 的最大值。

例 2. 在圖中，A 點在 O 點的西方 300 m，B 點在 O 點的南方 400 m。求由 A 點測得 B 點的真方位角，答案準確至一位小數。



例 3. 已知 $ABCD$ 是一個梯形，其中 $AB//DC$。若 $\angle A = 50°$ and $\angle B = 130°$，證明 $ABCD$ 是一個平行四邊形。



例 4. 若 $\Delta ABC \cong \Delta PQR$，且 $AB = 10$，求 PQ 的值。

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以下是筆者的見解：不同的定義，產生不同的效果。若不同意例子的說明，希望大家能在下一期的中學數學通訊賜教、討論。

例 1.最大值不存在。

[註：最少上界為 1.85]

例 2. $\tan \angle OAB = \dfrac{400}{300}$

$\angle OAB = 53.1300235°$

由 A 點測得 B 點的真方位角 $= 90° + 53° = 143°$

[註：筆者認為真方位角是一個三位有效數字的整數。]

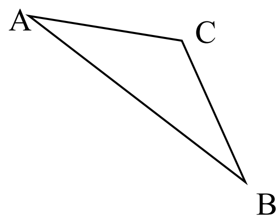
例 3.證明成立與否，要視乎「梯形」的定義。

若「梯形」的定義是「一個四邊形，其中**<u>只有</u>**一對平行線」，則證明必不成立。

例 4.如圖，因為同一三角形必然全等，

即 $\Delta ABC \cong \Delta ACB \cong \Delta BAC \cong \Delta BCA \cong \Delta CAB \cong \Delta CBA$

[註：筆者認為表示全等三角形時不須理會字母位置。]



由於不須理會 $\Delta ABC \cong \Delta PQR$ 中的字母位置，所以不能求得 PQ 的值。

8. The 57th International Mathematical Olympiad held in the Hong Kong University of Science and Technology, Hong Kong from 9 to 16 July 2016

CHENG Sze-man

Introduction

The International Mathematical Olympiad (IMO) is an annual competition for secondary school students around the world. It has been held every year since 1959 (except in 1980, because the host country, Mongolia, went into financial difficulty) and has been organised 56 times until July 2016.

Hong Kong was very lucky to have the second chance to hold the IMO. The first IMO held in Hong Kong was in 1994. There were 69 countries, a total of 385 contestants (including male and female) participated in the event. This year, the International Mathematical Olympiad Hong Kong Committee Limited (IMOHKCL) and the Hong Kong University of Science and Technology (HKUST) are respectively the Host Organisation and the Host University of this event, and the Education Bureau (EDB) of the Government of the Hong Kong Special Administrative Region is the Supporting Organisation of the event.

Every year, each country/region may send a maximum of six students to participate in the contest. Contestants must be no more than 20 years old and must not have any post-secondary school education. There is no limit to how many times a person may participate in the IMO, provided that the individual meets the age and schooling requirements. With an exception of Mr LEUNG Yui-hin, Arvin, all members of the Hong Kong team had participated in IMO 2015. Particularly, Mr YU Hoi-wai and Mr CHEUNG Wai-lam participated in IMO 2014 as well.

Contestants have to sit for two days of contests, each lasting for 4.5 hours. On each day of the contest, there are 3 problems, each carrying 7 marks. The problems are mainly of essay type, and require deep thoughts and careful observations to solve. Prizes are awarded on an individual basis. Each participant must work on their own and submit solutions in the language of his/her choice. About half of the contestants will be awarded medals, gold, silver or bronze. Also an honourable mention (HM) is awarded to contestants who do not obtain a medal but score full mark in at least one problem. In addition, a special prize is awarded to contestants who present elegant solutions to the problems posed.

The official period for IMO ran from 5 to 16 July 2016. The leaders of the participating countries/regions had to come to Hong Kong on 5 July 2016 or 6 July 2016 the latest to select the most appropriate problems for the contest before the arrival of the deputy leaders and the contestants on 9 July 2016.

The Hong Kong Team

The Hong Kong Team consisted of 6 contestants, a team leader and a deputy leader. The names of the team members were as follows:

Leader: Dr LI Kin-yin
Deputy Leader: Mr CHING Tak-wing
Contestants:
Mr CHEUNG Wai-lam (Queen Elizabeth School)
Ms KWOK Man-yi (Baptist Lui Ming Choi Secondary School)
Mr LEE Shun-ming (CNEC Christian College)
Mr LEUNG Yui-hin (Diocesan Boys' School)
Mr John Michael WU (Diocesan Boys' School)
Mr YU Hoi-wai (La Salle College)

Programme Overview

Date	Leaders	Deputies	Contestants
5 July (Tue)	Early arrival		
6 July (Wed)	Arrivals - IMO Advisory Board Meeting		
7 July (Thu)	Jury meeting		
8 July (Fri)	Jury meeting		
9 July (Sat)	Jury meeting	Arrival	
10 July	Jury meeting	Free time	

<table>
<tr><th>Date</th><th>Leaders</th><th>Deputies</th><th>Contestants</th></tr>
<tr><td rowspan="2">(Sun)</td><td colspan="3">Opening ceremony</td></tr>
<tr><td>Welcome Dinner</td><td colspan="2">Free time</td></tr>
<tr><td rowspan="2">11 July (Mon)</td><td>Jury Q & A</td><td rowspan="2">Excursion (9 a.m. – 1 p.m.)</td><td>Contest paper 1</td></tr>
<tr><td>Excursion (11 a.m. – 4 p.m.)
Poly U Visit (4 p.m. – 9 p.m.)</td><td>Free time</td></tr>
<tr><td rowspan="3">12 July (Tue)</td><td>Jury Q & A</td><td rowspan="3">Free time</td><td>Contest paper 2</td></tr>
<tr><td rowspan="2">Transfer to HKUST after lunch</td><td>Free time</td></tr>
<tr><td>Cultural Night</td></tr>
<tr><td rowspan="2">13 July (Wed)</td><td colspan="2" rowspan="2">Coordination Meeting</td><td>Excursion to Hong Kong Disneyland</td></tr>
<tr><td>Movie Night</td></tr>
<tr><td rowspan="2">14 July (Thu)</td><td colspan="2" rowspan="2">Coordination Meeting</td><td>Excursion to the Peak, St Stephen’s College and Stanley</td></tr>
<tr><td>Game Night</td></tr>
<tr><td rowspan="2">15 July (Fri)</td><td colspan="2">Excursion (8:30 a.m. – 12:00 noon)</td><td>Talk</td></tr>
<tr><td colspan="3">Closing Ceremony & IMO Dinner</td></tr>
<tr><td>16 July (Sat)</td><td colspan="3">Departure</td></tr>
</table>

Major Activities

During the official period of the 57th IMO, coordination meetings, jury meetings and site committee meetings were convened, and formal functions like excursions, lecture programme and visits were conducted.

At the coordination meetings, the team leader of each participating country/region would discuss with the coordinators of the IMO Coordination Committee on the performance of contestants of his team and agree on the scores of the competitors. If no agreement was reached, the cases (the scores for competitors) would be brought up again at the jury meetings.

Jury meetings would be held before and after the contests. At the jury meetings (before the contests), questions for the Olympiad were discussed and six were selected for the contest. The questions were then translated into different languages of the participating countries under the strict scrutiny of all team leaders. At the jury meetings (after the contests), contestants' queries at the beginning part of the competition were considered and answered, scores were confirmed and medallists decided. The team leaders would attend all the said meetings, while the deputy leaders would only attend the coordination meetings after the contests.

Formal Functions – Opening Ceremony, Closing Ceremony and IMO Dinner

The Opening Ceremony for the 57th IMO was held in the Queen Elizabeth Stadium on 10 July 2016 (Sunday). It began with the performance of the theme song "2016 IMO Overture", composed by Dr MUI Kwong-chiu and performed by Dr LUNG Heung-wing, Mr LEUNG Ching-kit, Mr CHOY Lap-tak, Mr CHAN Wai-hong, Greeners' Sound, Hong Kong Children's Choir, Die Jungen Konzertisten, Hong Kong Parents Choir and Hong Kong Youth Symphony Orchestra. These musicians also performed two other pieces of music in the ceremony, namely, "Mathematical Installation Percussion" and "In Love We Are One".

<table>
<tr>
<td>
Professor SHUM Kar-ping
Chairman of the International Mathematical Olympiad Hong Kong Committee</td>
<td>
Professor Tony CHAN
President of The Hong Kong University of Science and Technology</td>
</tr>
</table>

Welcome speeches were given by Professor SHUM Kar-ping (Chairman of the International Mathematical Olympiad Hong Kong Committee), Professor Tony CHAN (President of The Hong Kong University of Science and Technology), Mr NG Hak-kim (Secretary for Education, HKSARG) and Mr Geoff SMITH (Chairman of International Mathematical Olympiad Advisory Board).

<table>
<tr>
<td>
Mr NG Hak-kim
Secretary for Education, HKSARG</td>
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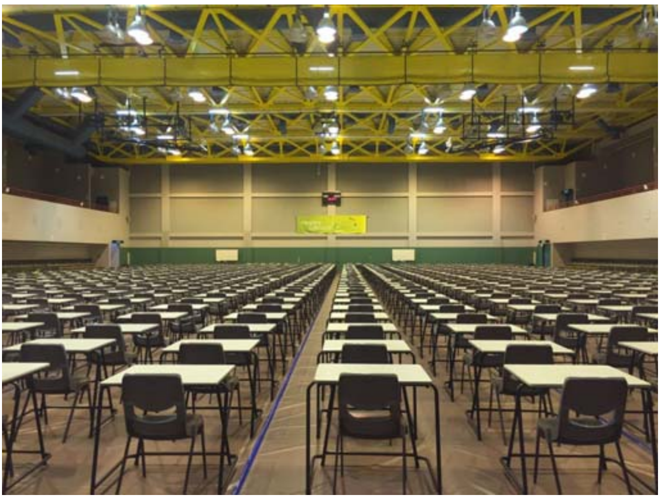
Mr Geoff SMITH
Chairman of International Mathematical Olympiad Advisory Board</td>
</tr>
</table>

The Closing Ceremony was held in the afternoon of 15 July 2016 (Friday) at the Hong Kong Convention and Exhibition Centre. 44 gold medals, 101 silver and 135 bronze medals were awarded at the event. Two performances, including the march in performance by the marching band of HKSYCIA Wong Tai Shan Memorial College and two songs, "鼓金齊鳴" composed by Dr MUI, and "龍騰虎躍" composed by Mr LI Min-xiong, performed by 一鳴鼓隊 were provided.

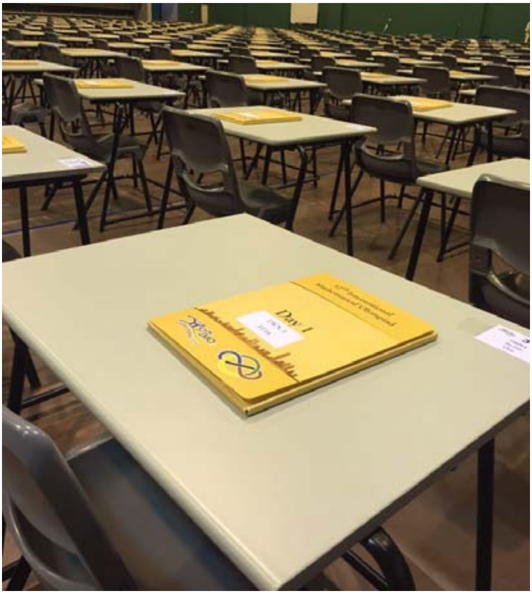
After the speeches given by the Chairman of the International Mathematical Olympiad Hong Kong Committee and the Secretary for Education, HKSARG, the contestants received their medals. Their names were called in groups according to the type of medals to be received, and they strode across the stage in all forms of attire, from suits to casual wear and national dress. A few draped themselves in their national flag. Each received a round of applause and had a photograph taken. The prize presenters were prominent mathematicians/professors, representatives from donors, organisers and supporters of the IMO2016. The 57th IMO ended in the IMO Dinner held at the Grand Hall, the Hong Kong Convention and Exhibition Centre.

Contests

The problems of the competition came from various areas of mathematics, such as topics included in mathematics curricula at secondary school. The solutions of these problems required exceptional mathematical ability and excellent mathematical knowledge in the part of the contestants. During the competition, contestants worked individually on six challenging problems, presenting their solutions akin to those produced by research mathematicians.



The students' contests took place in the mornings of 11 and 12 July 2016 at Sport Centre of the Hong Kong University of Science and Technology. 602 contestants from 109 countries/regions were required to answer three problems on algebra, geometry, number theory and combinatorics in 4.5 hours each morning.



Excursions and other activities

After contestants handed in their answer scripts in different coloured folders, people started scanning them into the computer and got them ready for marking. It was lucky that we had some student helpers (recruited from Diocesan Girls' School and Helen Liang Memorial Secondary School) to help us out in scanning these answer scripts.

Imagine that if each contestant wrote about 10 pages per question, then we needed to scan more than 40 000 sheets of paper in the afternoon of 11 and 12 July. Meanwhile the contestants took part in social and cultural events organised by the IMOHKCL and HKUST, including the film show, cultural nights and game night. The contestants also joined the tour to Hong Kong Disneyland on 13 July 2016. Excursions to the Peak, Stanley Market and St Stephen College were also arranged for the contestants in the morning and afternoon of 14 July 2016.

Achievements of Hong Kong Team

There were 109 countries/ regions sent teams to participate in the 57th IMO and the total number of contestants was 602. 44 gold, 101 silver, 135 bronze medals and 162 honourable mentions were awarded in the Olympiad.

Hong Kong Team won 3 gold medals, 2 silver medals and 1 bronze medal. Details of the awards and their scores obtained in individual questions were as follows:

Contestant	P1	P2	P3	P4	P5	P6	Total	Medal obtained
LEE Shun-ming	7	7	0	7	7	3	31	Gold
John Michael WU	7	7	0	7	7	3	31	Gold
YU Hoi-wai	7	7	0	7	7	3	31	Gold
CHEUNG Wai-lam	7	5	0	7	7	0	26	Silver
LEUNG Yui-hin	7	6	0	7	4	0	24	Silver
KWOK Man-yi	7	2	0	7	2	0	18	Bronze

In terms of the total scores of individual countries / regions, ***Hong Kong got a total of 161 and was ranked at the 9th place among the 109 countries / regions***, which was the ***best performance*** ever since 1988, the first year that Hong Kong Team participating IMO. The top 10 countries / regions on the list were:

Country	Score	Medals obtained
United States of America	214	6 Gold

Country	Score	Medals obtained
Republic of Korea	207	4 Gold and 2 Silver
People Republic of China	204	4 Gold and 2 Silver
Singapore	196	4 Gold and 2 Silver
Taiwan	175	3 Gold and 3 Silver
Democratic People's Republic of Korea	168	2 Gold and 4 Silver
Russian Federation	165	4 Gold, 1 Silver and 1 Bronze
United Kingdom	165	2 Gold and 4 Silver
Hong Kong	161	3 Gold, 2 Silver and 1 Bronze
Japan	156	1 Gold, 4 Silver and 1 Bronze

Some Observations and Recommendations

Overall speaking, the 57th IMO was conducted smoothly. The temperature was around 34°C in Hong Kong, which was quite hot, nevertheless, all participants enjoyed their lives here at HKUST during the IMO period. HKUST provided some well-equipped rooms for all participants, some of them even have lovely scenic views.

A guide (undergraduate student, postgraduate student and secondary school teachers), who accompanied the contestants throughout the entire period, was assigned to each team, though a few teams had to share the same guide and some teams had to change their guides during their period of stay, the Organising Committee tried very hard to minimise these kind of disturbance. All participants (as far as I knew) acknowledged the hospitality of their guides who tried their best to provide the participants the best service. The 57th IMO attracted a high number of participating countries and contestants, which was a big success.

All Hong Kong contestants ***were able to get medals in IMO 2016***. Hong Kong contestants mixed quite comfortably with competitors from other countries / regions. They made use of the opportunities to make friends with young mathematicians from other places and let them know more about Hong Kong and Hong Kong students.

9. Reflection on "Seed" Projects

CHAN Chi-man Benjamin
Helen Liang Memorial Secondary School (Shatin)

Introduction

It was a precious opportunity for me to be attached to the Mathematics Education Section, Curriculum Development Institute in the Attachment Programme for Teaching Staff in Government Schools to Education Bureau Sections from 11 January 2016 to 8 July 2016. During this period, I had a chance to assist two "Seed" projects. The first one was 'Exploration and Development of Self-directed Learning Strategies in Mathematics' (MA0215) and the other one was 'Exploration and Development of Strategies on the Learning and Teaching of Loci, Equations of Straight Lines and Transformations of Functions' (MA0315). It was beneficial for me to explore new pedagogy and develop effective strategies in teaching Mathematics. Now, I would focus on my experience in participating the "Seed" project in MA0315. Those strategies were also seldom mentioned in textbooks.

Strategies for the Learning and Teaching of Loci, Equations of Straight Lines and Transformations of Functions

Accompanied by the Curriculum Development Officers of the Mathematics Education Section, we visited the seed schools and conducted meetings with teachers to discuss matters about the project. Moreover, a series of well-organized activities were designed to address the learning difficulties of students. Although I had been teaching for more than 20 years, I learnt a lot from the teaching strategies used in the "Seed" projects. As the teaching strategies of the topic "Locus" impressed me most, I would like to share what I have learnt from this.

Reflection on teaching strategies of the topic "Locus"

The teaching strategies of the topic "Locus" basically follows the suggestion of Explanatory Notes to Senior Secondary Mathematics Curriculum.

In the past, I seldom used the following teaching strategies to introduce the concept of loci. After the "Seed" projects, I knew how to use them in order to enhance the effectiveness of teaching in the topic "Locus".

Many teachers reflected that students generally had learning difficulties in understanding the concept of loci, describing and sketching the locus of points satisfying given conditions.

Use of real life examples

Real life examples are always effective and attractive for students to learn some abstract concepts. The real life examples used as an introduction of Loci convinced me much. Students can also experience the applications of mathematics through different tasks on the worksheets or assessment items. To arouse students' interests in learning loci, videos from YouTube concerning about why the wheels of a car should be in the shape of a circle are included in the Example 1. Two examples are illustrated and shared as follows.

Example 1

Photographs in the textbook and videos in YouTube can be used to give daily life examples to illustrate the concept of locus.

Video 1: Square wheels

www.youtube.com/watch?v=QF7odK55gkI

Video 2: Square wheels tricycle

www.youtube.com/watch?v=LgbWu8zJubo

I was impressed by the worksheets provided by Mathematics Education Section for their students. Some of the problems in the worksheets were real life examples that students were more interested. In Example 2, an incident of searching for a missing plane may help students understand the wide use of Loci. Students would be interested in this example as they may recall the news of the missing aeroplane Malaysia Airlines flight MH370.

Example 2

Disappearance of an aircraft

(a) It is reported that an aircraft with 432 passengers was missing in its flight from City A to City B. The distance between City A and City B is 100 km. The location of the aircraft is within 70 km from City A and within 50 km from City B. Mark two city A and B and shade the appropriate region in the box below. (1 cm stands for 10 km in the figure.)

(b) In addition, it is reported that the location of the aircraft is not within 40 km from City C. Making use of this information, shade the possible region of the aircraft in the box provided.

Most of the students got the correct answer in Figure 1 showed that the application of mathematics in real life could really help students to understand such abstract matters.

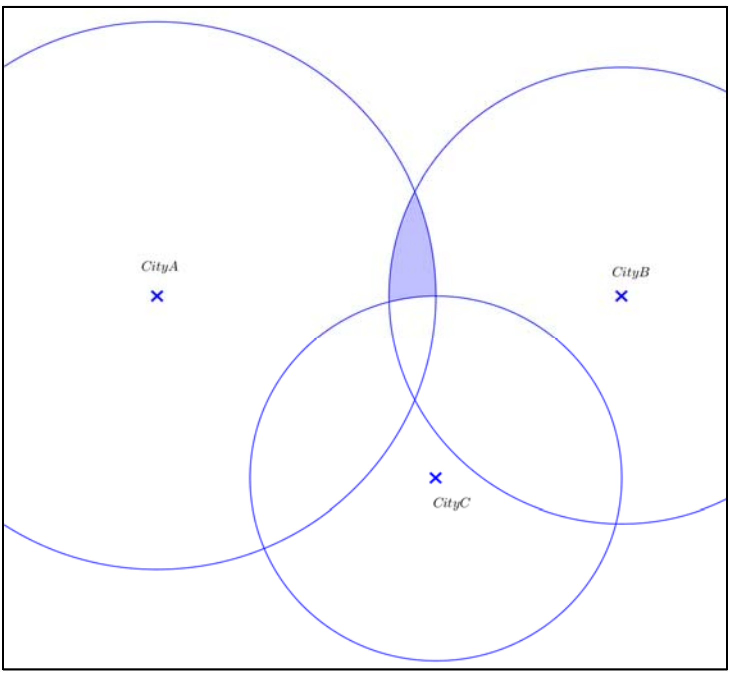


Figure 1

Focusing on the Definition

After the participation in the "Seed" program, I further confirmed that teachers always play a key role in leading students to have effective learning in classroom.

Teachers followed the suggestions of Mathematics Education Section that students should be made clear on the Mathematical terms such as "straight line", "line segment", "distance" and "locus", etc. in order to explore the concepts of locus.

I appreciated that the definition of the distance between two straight lines in the plane was clearly explained by the teacher. The teacher explained that the distance between two straight lines in the plane was the minimum distance between any two points lying on the lines. In case of two parallel lines, it was the perpendicular distance from a point on one line to the other line. For the intersecting lines, the distance between them was zero, because the minimum distance between them was zero (at the point of intersection).

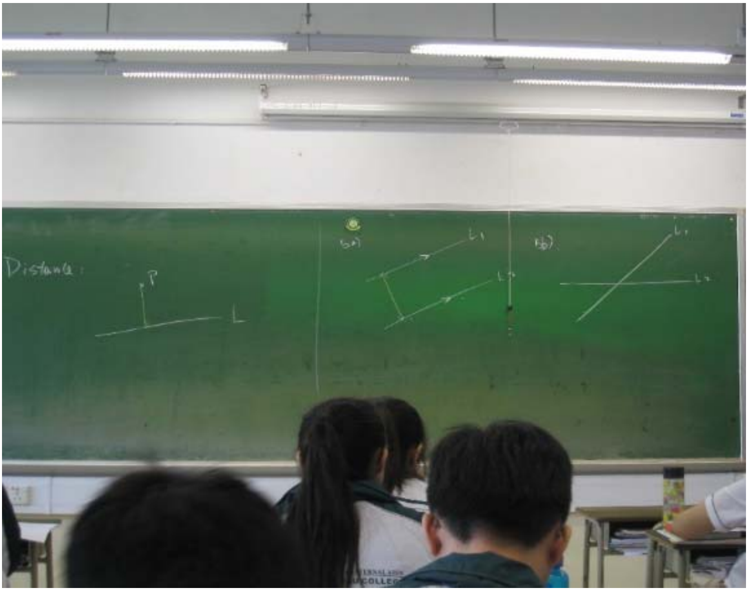


Figure 2
Teachers' explanations on the meanings of mathematical terms in Figure 2.

Use of IT

Before attachment, I hardly knew how to use GeoGebra. Through designing the GeoGebra files, my IT skills had improved a lot because I needed to prepare them in the worksheets. A lot of GeoGebra files had been developed and given to the teachers before the tryouts. By now, I can succeed to draw the different figures on Locus in the use of this software, which could certainly help to facilitate my teaching in the near future.

Use of models

After the "Seed" project, I knew how to use models, if possible, as a tool to help students sketch the locus. During the lesson, most of the students found difficulties in sketching the locus of *P* by rolling the square along the horizontal ground to the right in Exercise 1. Although the content was out of the syllabus, it would enrich the student's experience and exploration to find the locus.

Exercise 1

In the figure 3, *P* is one of the vertex in the square. The square rolls along the horizontal ground to the right. Sketch the locus of *P*.

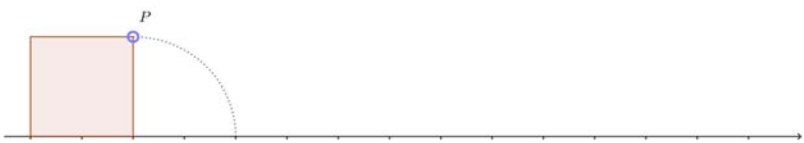


Figure 3

Some students wrongly described the locus of *P* which was a cycloid. The teacher provided a teaching aid which was a solid square, who allowed the students to roll it along the horizontal ground to the right in the worksheet. So they could traced the locus of *P* by the method of hand-on activity. Finally, all students in the class could understand how to obtain the locus of *P*.

In the Mathematics Education Section, there were internal professional development sharing sessions (TAMSS) among colleagues in the first and third Tuesday mornings of every month. I had an opportunity to share my experience of the above lesson observation to the colleagues. I was impressed by Dr NG YK's question that "Apart from the method of hand-on activity, what was the alternative method to find the locus of *P?*" He suggested that when more squares were drawn and packed them closely as shown in the figure. The locus of *P* can be obtained in figure 4.

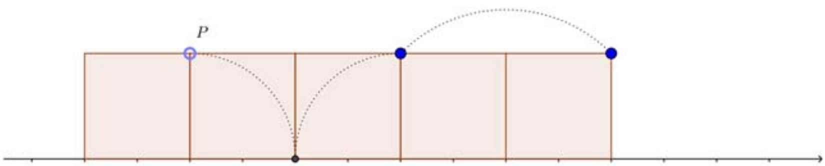


Figure 4

Use of a pair of compasses

On the whole, I was impressed by the proper use of the compasses to get the answer correctly.

With the help of compasses, students could locate the exact positions of the points on the locus with the guidance of the questions. It would be easier for students to sketch the locus and understand the reasons behind.

The teacher used the figure in one activity to let students sketch the locus of *P* which was equidistant from a line *L* and a fixed point A. Suppose the fixed point A was 2 units from L, the teacher explained on how to mark the points which were 1 unit and 2 units from A and L. Students would then use a pair of compasses to draw a circle with centre A and radius in 3, 4 and 5 units in Figure 5.

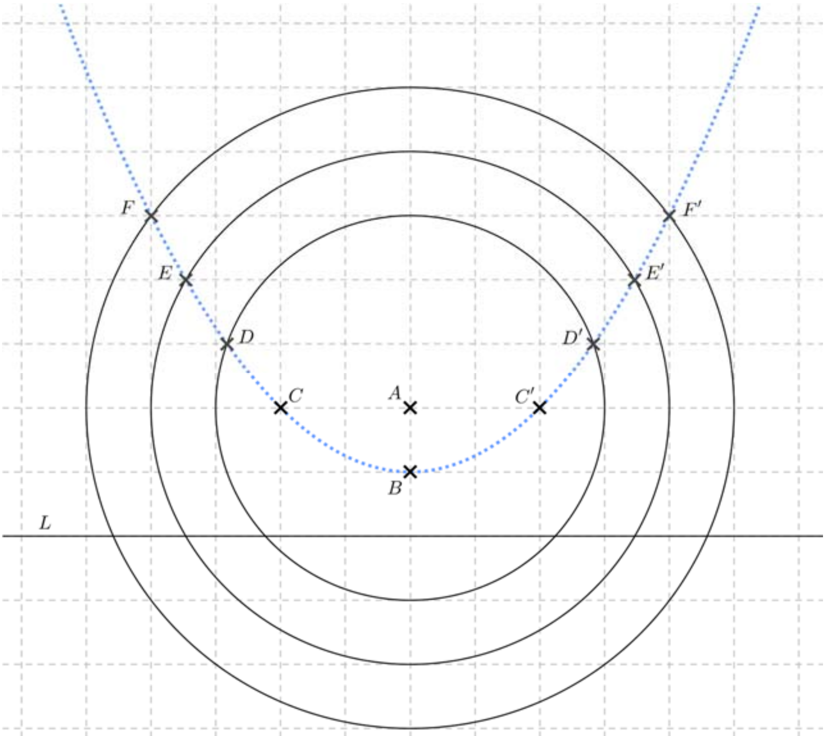


Figure 5

Use of Grid lines

Some activities in the worksheets, grid lines are provided to help students determine the distance between lattice points easier. From students' works in a lesson, many students made use of this advantage to sketch the correct loci. In part (a) of one activity as shown below, students could be able to locate 5 points equidistant from two parallel lines L_1 and L_2 instead of using the figure without grid lines printed on the textbook in Figure 6.

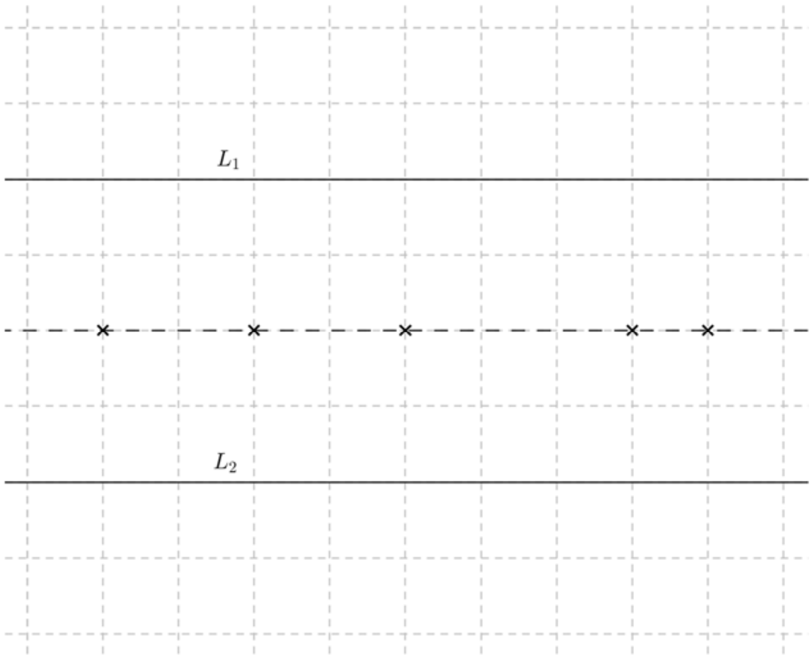


Figure 6

Most of the students could sketch the locus of a point P correctly. However, some students forgot to label the locus of P.

Use of paper strip or straight edge to construct parallel lines

Appropriate use of worksheets are always an important tool to help students to consolidate the knowledge.

Referring to the worksheet, the students were advised to cut paper strips in rectangular shape with width 1 cm to do one activity in Figure 7.

Figure 7

(a) In Figure 8, sketch the locus of the point Q which is at a fixed distance 1 cm from a straight line L_1.

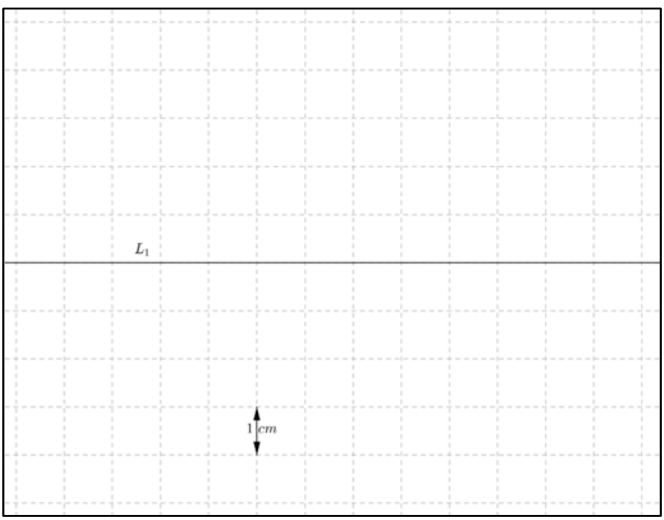


Figure 8

(b) Fig. 9 shows another straight line L_2 intersects L_1. Copy the locus obtained in (a) in Fig. 9.

(i) Cut the paper strip in Appendix. Use it to sketch the locus of the point R which is at a fixed distance 1 cm from a straight line L_2 in Fig 9.

(ii) How many point(s) on the plane which is at a fixed distance 1 cm from L_1 and L_2?

(c) Mark the points which are at a fixed distance 2 cm from L_1 and L_2. (Use S_1, S_2 etc. for the points required.)

(d) In Fig. 9 sketch the locus of the point P such that it is always equidistant from L_1 and L_2.

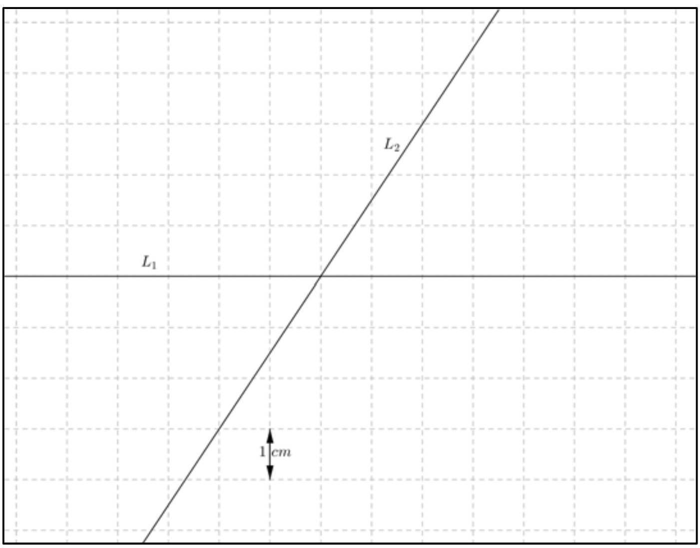


Figure 9

In part (a), almost all of the students could label correctly the locus of the point P which was at a fixed distance from a straight line L_1. However in part (b), it was observed that some of them forgot to copy the answer in (a) and could not find how many points on the plane which was at a fixed distance from the two straight lines L_1 and L_2. Most of the students got the answers correctly liked in Figure 10.

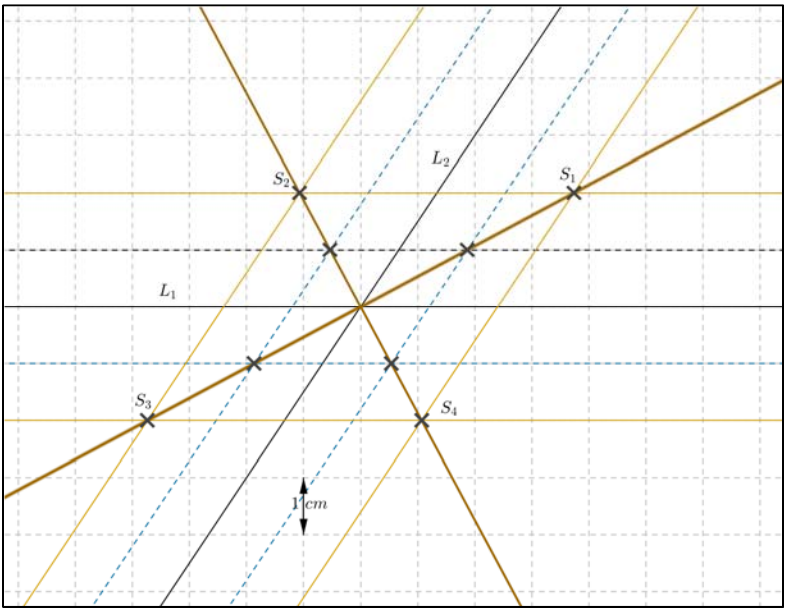


Figure 10

Discussion on "Wrong" solutions

I appreciated that the teacher did not directly explain the correct answer and made use of a "wrong" solution to arouse further discussion among the students. This convinced me that induced teaching is always great for students to correct the wrong thing by themselves, especially for those abstract concepts.

In the above activity, the students knew how to mark 4 points which were 1 cm from the two straight lines L1 and L2. By using similar technique, they knew how to mark another 4 points which were 2 cm from the two straight lines L1 and L2. However, they connected those 4 points by a curve and wrongly concluded that the locus was the parabola, was pointed out the reflection symmetry of the locus of P, L1 and L2 in Figure 11.

Fortunately, one student from another group found the mistakes and corrected that the locus was a pairs of straight lines because all the points on the locus must keep equidistant between L_1 and L_2. He mentioned that if the locus was a parabola, the vertex would not be equidistant between L_1 and L_2 in Figure 12.

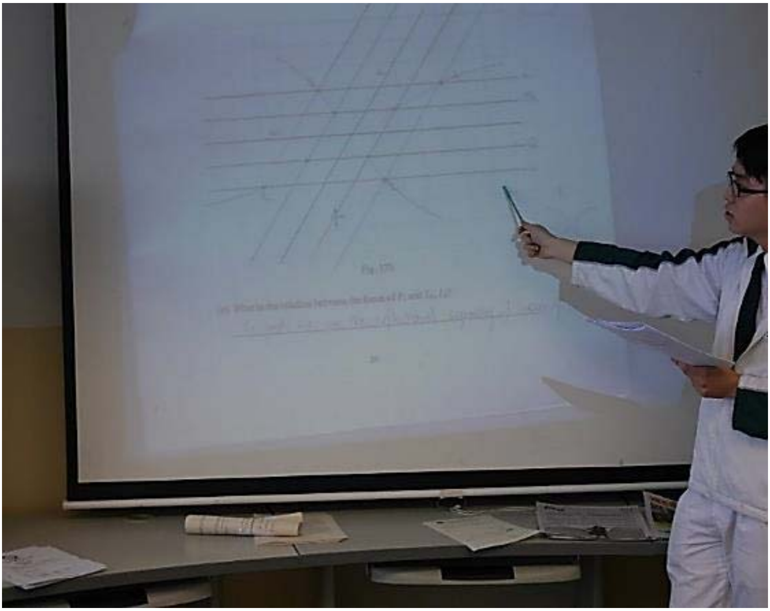


Figure 11

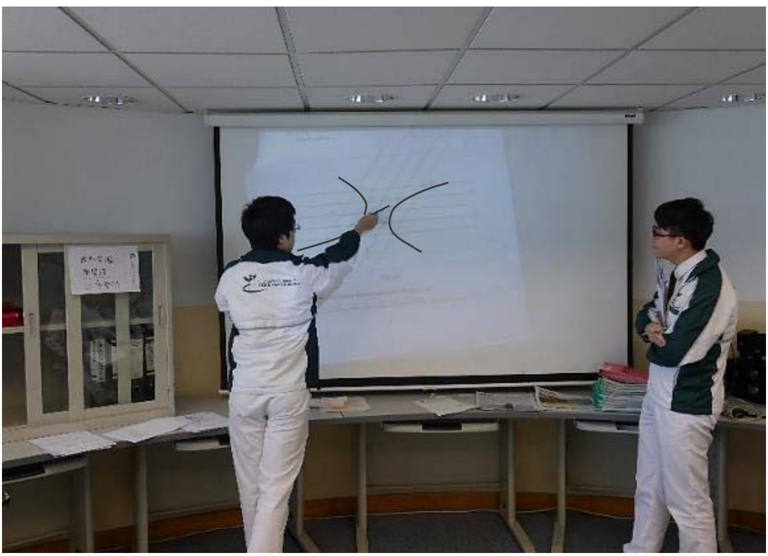


Figure 12

Another student drew the figure looked like a “Rhombus” to explain the loci of *P* which were the angle bisectors of the angles formed by L_1 and L_2 in Figure 13.

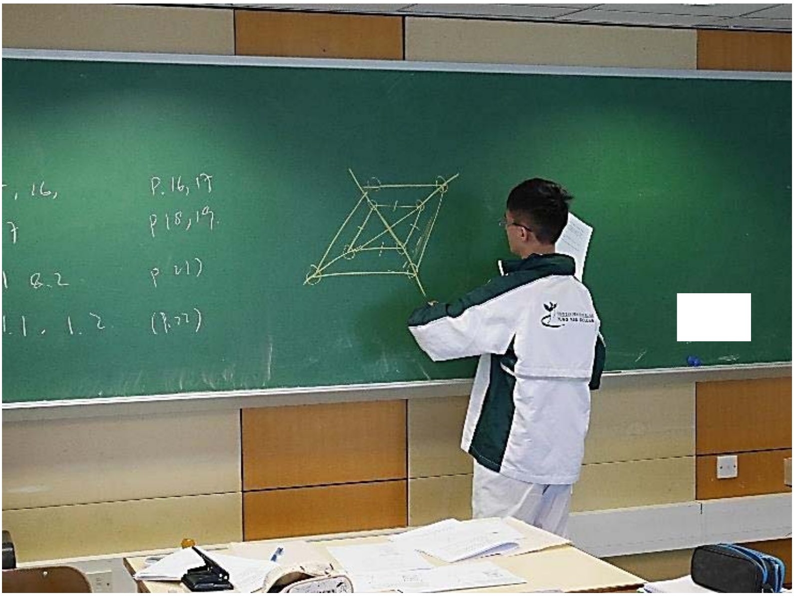


Figure 13

Conclusion

To sum up, the teaching strategies and experience I have gained during these six months in the Mathematics Education Section attached was very fruitful and useful for my professional development and future teaching life. This also gave me many precious opportunities to exchange teaching strategies and ideas with other teachers from different schools. I could further enrich my practical ideas and activities in the learning and teaching of Mathematics to improve lesson design for students' learning of the topics.

Vote of thanks

I expressed my sincere thanks to the Curriculum Development Officer, Mr LEE Kin-sum who taught me a lot the useful techniques of teaching in these "Seed" topics. I also deeply expressed the most special thanks to my supervisor Mr LEE Chun-yue for his guidance and care during the six months. Moreover, I was glad to work with two seconded teachers, Ms LAM LC and Ms TONG ML together in the 'SEED" projects. We all have a common goal in mind to enhance the effectiveness and motivation of learning in Mathematics. When I go back to school, I would make good use of the techniques and strategies learnt and will implement such strategies in the coming academic year.

References

CDC (2007). *Mathematics Curriculum and Assessment Guide (Secondary 4 – 6)*. Hong Kong: The Government Logistics Department.

Education Bureau (2009). *Explanatory Notes to Senior Secondary Mathematics Curriculum – Compulsory Part*. Hong Kong: The Government Logistics Department.

10. Reflection on 'SEED' project during my Attachment

TONG Man-ling
Sha Tin Government Secondary School

Attachment

It was really a precious time to be a teacher attached in the Mathematics Education Section for half a year. I felt so worried at the beginning since I haven't left the school-setting for more than twenty years. Fortunately, I met a colleague also came from Government Secondary school who was a good partner to investigate what we should do.

'SEED' project

One of the main duties of us was responsible for the 'SEED' projects for secondary schools. There were two projects in 2015-16, one for junior and one for senior forms. I was assigned to assist the projects of 6 schools out of 12 in the junior forms. Owing to the time clashed of meetings of other team members, I have the chance to visit 8 schools at last. It was a valuable experience to meet and exchange the ideas with teachers.

We have meetings with schools before implementing the try-out lessons. In the meetings, we have to discuss which part of the selected topic would be chosen. Our team would suggested some ideas and provide the materials that could be used according to the topic, may be pre-lesson videos or activities could be done in the lessons. We have to spend lots of time for searching resources and organize them to be user-friendly (Figure 1). Of course, frequent emails and phone calls with teachers were inevitable. We visited the schools for the try-out lessons (Figure 2) and conducted interviews with teachers and students after the lessons. Written reports in details of sessions should be handed in afterwards.

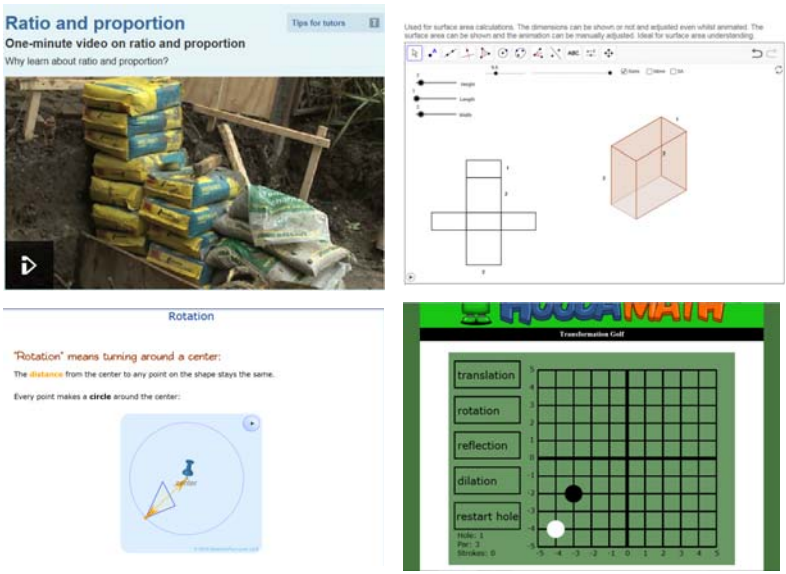


Figure 1 Materials provided to teachers

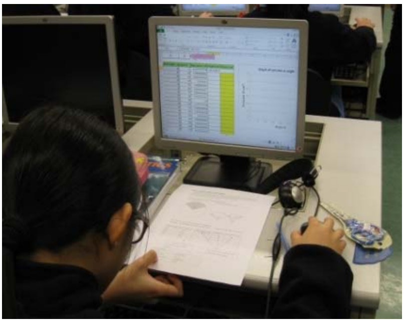


Figure 2 Lessons observation

The main theme of the 'SEED' project is to arouse the students to do the self-directed learning (SDL). The rationale of it was introduced by some seconded teachers before so I will not introduce here. In this passage, I would like to write down my own views in implementing SDL and try to design the teaching flow of a section in Coordinate Geometry.

My views and design on SDL

Topic: Section formula (S3)

Learning Objectives: ☞ To derive the section formula

☞To realize that mid-point formula is a case of section formula

Pre-requisites: Ratio and Similar triangles

1. Pre-lesson work

The students are asked to do some pre-lesson work that is closely related to what they are going to learn next lesson. We suggested watching a short video clip through internet and do an exercise. In brief, there are three objectives in the pre-lesson work. First, to revise some prerequisites that bridge with the new knowledge. Second, to introduce the new terms that are going to use so as to save the teaching time. The last one, I think it is the most important, to arouse the students' curiosity and interest in learning. So choosing good and relevant materials as pre-lesson work is a start to success.

The students should hand in their pre-lesson work in the morning of the day that the task implemented. The teacher just has a glance of the sheets to observe the common mistakes made by students without marking. The work will be distributed back to students for amendment after the lesson.

In my planning, I will ask the students to watch a short video clip (2min 43sec) which retrieve their memories of finding the lengths in similar triangles. In fact the last minute of the clip shows some questions, I have modified two of them as the first part in my pre-lesson worksheet. Besides that, I will ask the students to find the meaning of the terms ‘bisect’ and ‘trisect’ through searching internet. In addition, to find the mid-points of a horizontal line and a vertical line on the rectangular coordinate system is also mentioned. Finally, the students are asked to find the ratios of line segments when it was being trisected. The time used to watch the video and finish the worksheet is about 25 minutes.

Here is the design of the pre-lesson worksheet.

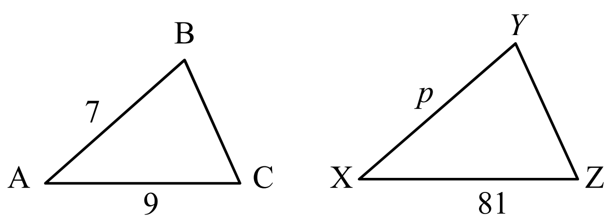
A. Ratio of sides of similar triangles

Please watch the following video clip and fill in the answers of questions 1&2.

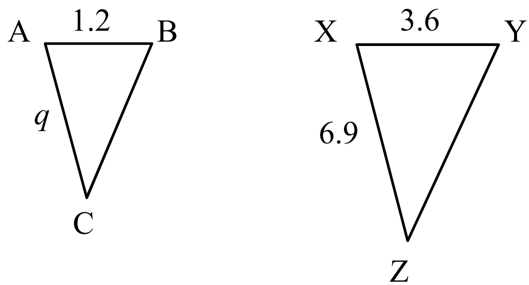
https://www.youtube.com/watch?v=DNNUYp

(00:00-02:43)

1.



Given $\Delta ABC \sim \Delta XYZ$, $p =$

2. 

Given $\Delta ABC \sim \Delta XYZ$, $q =$

Vocabulary

Please write down the meanings of the following words.

Word	Meaning
Bisect	
Trisect	

B. Bisect and Trisect a line segment

Figure 1 shows points A and B lie on a horizontal number line.

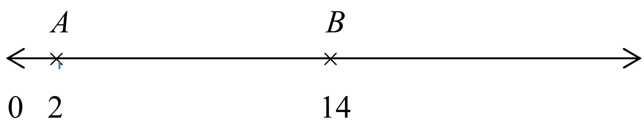


Figure 1

(a)(i) Given that P is the mid-point of line segment AB, that means to ______ line segment AB. In ratio, $AP : PB =$ ___ : ___. P locates on number

__________. You may find it by finding the distances of *OA* and *OB* first since *OP* = ________.

(ii) If the coordinates of points *A* and *B* are (2,6) and (8,6) on the rectangular coordinate plane respectively, the coordinates of *P* are ________.

(iii) How about if Q is the mid-point of points *B*(8,6) and *C*(8,18), what are the coordinates of Q? (Show your working)

(iv) Find the mid-point *R* of *A*(2,6) and *C*(8,18). (Show your working).

*Hint: Try to sketch a diagram.

*Point *R* is also called the *point of division* of the line segment *AC*. Since *R* is a point *on the line segment AC* such that *AR* : *RC* = 1 : 1, we say '*R divides AC internally* in the ratio 1 : 1.'

(v) Derive the *Mid-point formula* to find the mid-point *S* of $D(x_1,y_1)$ and E (x_2,y_2).

(b) In figure 2, we *trisect* the line segment *FG* with points *T* and *U*. Fill in the blanks of the following to show the ratio of lengths.

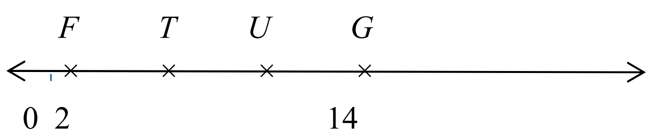


Figure 2

FT : *TU*: *UG* = ____ : ____ : ____   *FT* : *TG* = ____ : ____   *FU* : *UG* = ____ : ____

GU : *UF* = ____ : ____   *GT* : *TF* = ____ : ____

The ratios *FT* : *TG* and *GT* : *TF* are *same* / *different*. (Please circle)

C. Thinking Corner

How to find the *internal point of division* of 2 points with ratio *m* : *n*?

2. Classroom Activities

Since the students were asked to think about how to find the internal point(s) of division of 2 points with the ratio *m* : *n*, the activities conducted in the double lessons will lead the students to draw the conclusion.

Activity 1 (A or B)

The students are divided into groups of 4. Half of the groups do Activity 1A while another half do Activity 1B (i) and (ii). All of them are asked to trisect a line segment and find the 2 points of division of by using different methods.

Activity 1A:

Materials needed:

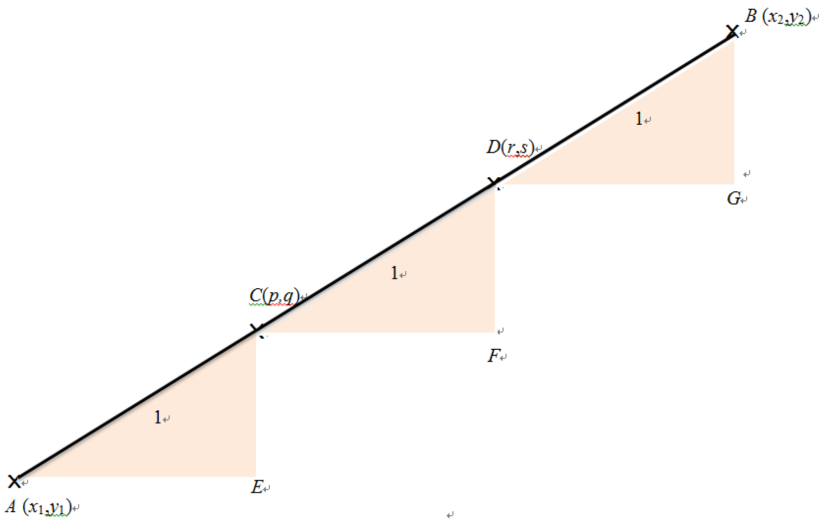
☞ an A4 sheet with a line segment *AB* where *A* is (x_1,y_1) and *B* is (x_2,y_2)

☞ 3 **congruent right-angled triangles**, all of them with hypotenuse = 1, the sum of hypotenuses of 3 triangles is the length of line segment *AB*.

☞ Blue-tac

Question: Find the internal points of division *C*(*p,q*) and D (*r,s*) of the line segment *AB* such that *AC* : *CD* : *DB* = 1 : 1 : 1. (Show your working)

Solution:



$\because \Delta ACE \cong \Delta CDF \cong \Delta DBG$

$\therefore BG = DF = CE$

$y_2 - s = s - q = q - y_1$

$y_2 - s = s - q$ --- (1)

$s - q = q - y_1$ --- (2)

From(1),

$s = \dfrac{y_2 + q}{2}$ --- (3)

sub. (3) into (2),

$q = \dfrac{y_2 + 2y_1}{3}$ and $p = \dfrac{x_2 + 2x_1}{3}$

Then use the same method to find *r* and *s*.

$r = \dfrac{2x_2 + x_1}{3}$ and $s = \dfrac{2y_2 + y_1}{3}$

Activity 1B:

Materials needed:

☞ Two A4 sheet with a line segment *AB* where *A* is (x_1,y_1) and *B* is (x_2,y_2)

☞ 4 **similar right-angled triangles**, two of them with hypotenuse = 1 and the other two with hypotenuse = 2. The sum of hypotenuses of 1 large and 1 small triangles is the length of line segment *AB*.

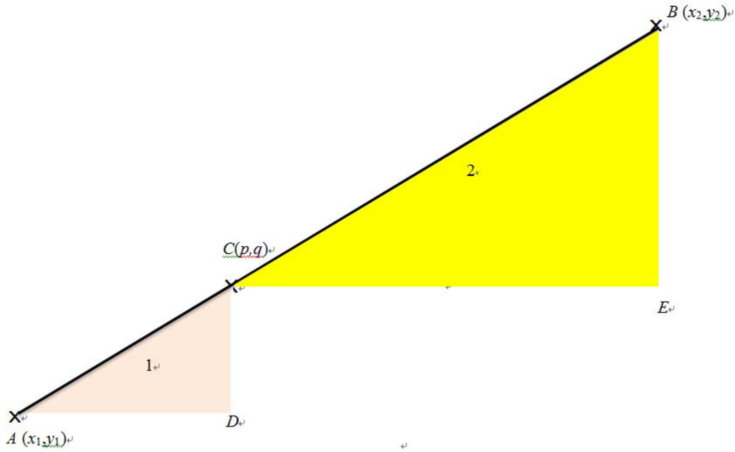
☞ Blue-tac

Question (i) : Find the internal point of division $C(p,q)$ of the line segment *AB* such that $AC : CB = 1 : 2$. (Show your working)

Question (ii) : Find the internal point of division $D(\mathrm{r},s)$ of the line segment *AB* such that $AD : DB = 2 : 1$. (Show your working)

Solution:

Question (i)



$\because \Delta ACD \sim \Delta CBE$

$\therefore \dfrac{BE}{CD} = \dfrac{CE}{AD} = \dfrac{CB}{AC}$

$\dfrac{BE}{CD} = \dfrac{2}{1}$

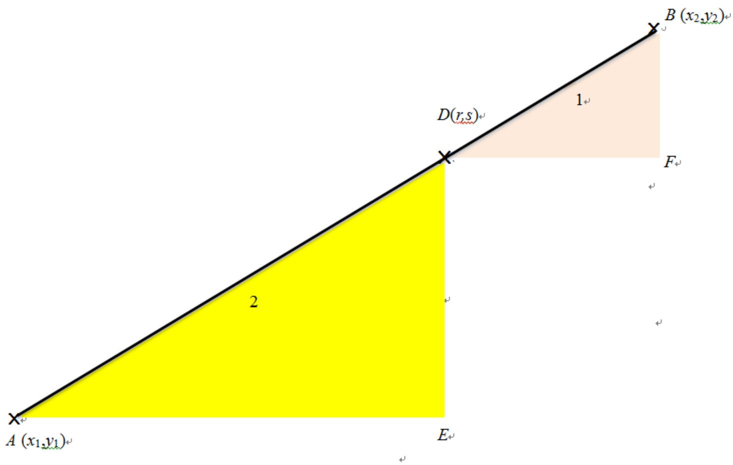
$y_2 - q = 2(q - y_1)$

$q = \dfrac{y_2 + 2y_1}{3}$ --- (3)

Similarly,

$p = \dfrac{x_2 + 2x_1}{3}$

Question (ii)



$$\because \Delta ADE \sim \Delta DBF$$

$$\therefore \frac{BF}{DE} = \frac{DF}{AE} = \frac{DB}{AD}$$

$$\frac{BF}{DE} = \frac{1}{2}$$

$$2y_2 - 2s = s - y_1$$

$$s = \frac{2y_2 + y_1}{3} \quad \text{--- (3)}$$

Similarly,

$$r = \frac{2x_2 + x_1}{3}$$

The students are asked to stick the given triangles on the line segment *AB* and find the 2 points of division. Select some groups to explain how to find the answer. The teacher may conclude that both congruent and similar triangles can be used to find points of division. In wordings, we can say ‘Point *C* divides *AB* internally in the ratio 1 : 2.’ Or ‘Point *C* divides *BA* internally in the ratio 2 : 1.’ Then ask the students to speak out the ratios of dividing the line segment for the point *D*.

Activity 2 (A or B)

Again, half of the groups do Activity 2A while another half do Activity 2B.

> Activity 2A:
>
> Materials needed:
>
> ☞ an A4 sheet with a line segment *AB* where *A* is (x_1,y_1) and *B* is (x_2,y_2)
>
> ☞ 2 **similar right-angled triangles**, one of them with hypotenuse = *m* and the other with hypotenuse = *n*. The sum of hypotenuses of two triangles is the length of line segment *AB*.
>
> ☞ Blue-tac
>
> Question: Point *E* (*a,b*) divides *AB* internally in the ratio *m* : *n*. Find the coordinates of *E*. (Show your working)

For Activity 2B, all the materials needed are same as 2A, while the Question becomes

> ‘Point *F* (c,*d*) divides *AB* internally in the ratio *n* : *m*. Find the coordinates of *F*.’ It is advised that the lengths of *m* and *n* are arbitrary.

The students may use the method learnt in Activity 1 to find the point of division. The teacher introduces the *section formula* and asks the students whether point E and F are same. He/ she must point out that to divide the line segment AB internally in the ratio m: n is different from dividing it internally in the ratio $n : m$. Finally, the teacher should help the students to investigate that the mid-point formula is a case of using the section formula when $m = n$, that means the ratio is 1 : 1.

3. Post-lesson work
   The teacher may give the pre-lesson worksheet back to the students and ask them to do the amendment. Besides that, a few numbers of questions will be selected from textbook for consolidation. Ask the students to watch the video clip if they find problem about the section formula.
   http://math.tutorvista.com/geometry/section-formula.html

The above plan was create by my imaginary after I have met colleagues from different schools. I didn't know whether it works or not. I will try to conduct it when I go back to school. If you are also interested in this, try it and let me know any comment and amendment needed through the follow email: mltong@edb.gov.hk. Thanks a lot!

11. 一個數學的有趣應用

程國基

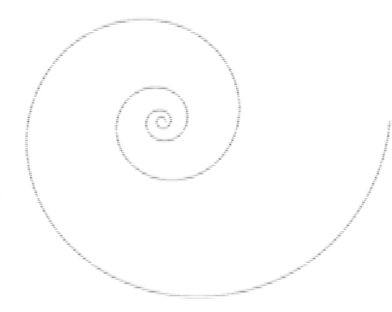
在一次由教育局數學教育組舉辦的數學歷史研習小組(註 1)的聚會中，有一位教師在簡報中，分享他的歐洲數學之旅。內容十分豐富和他的講解充滿趣味，使自己也要反省一下學習數學是否只限於紙上的運算？沒有數學家的生平歷史，是否失去認識數學根源的機會，使學習不夠完整？

從簡報中，他講解自己途經瑞士、法國、意大利等多個著名國家，沿途尋找數學家的足跡。先有伯努利家族(Bernoulli family)的三位數學家：雅各第一‧伯努利 (Jacob Bernoulli)，約翰第一‧伯努利 (Johann Bernoulli) 及丹尼爾第一‧伯努利 (Daniel Bernoulli)；其後有畢達哥拉斯(Pythagoras)、阿基米德 (Archimedes) 等。這些都是大名鼎鼎的數學家，他們對數學的貢獻很大。其中伯努利家族這三位數學家是比較近代的，他們都在微積分和概率方面有著很多不同的貢獻。在數學單元一和單元二中所提及的數字 e，它是由雅各第一‧伯努利 (Jacob Bernoulli) 所發現的(註 2)。從那次教師的簡報圖片中，我得知雅各第一‧伯努利過身後葬在瑞士巴塞爾大教堂內，碑上刻有一條螺線，如下圖：

圖一

為何會刻有這個圖案？相信是紀念這位數學家其中一項深

入的研究："等角螺線" (equiangular spiral)。值得留意的是，當日講者及一些同工很快指出這圖案不是"等角螺線"而只是"等速螺線" (Archimedean spiral) 吧！那麼"等角螺線"是怎麼樣呢？可看看下圖：

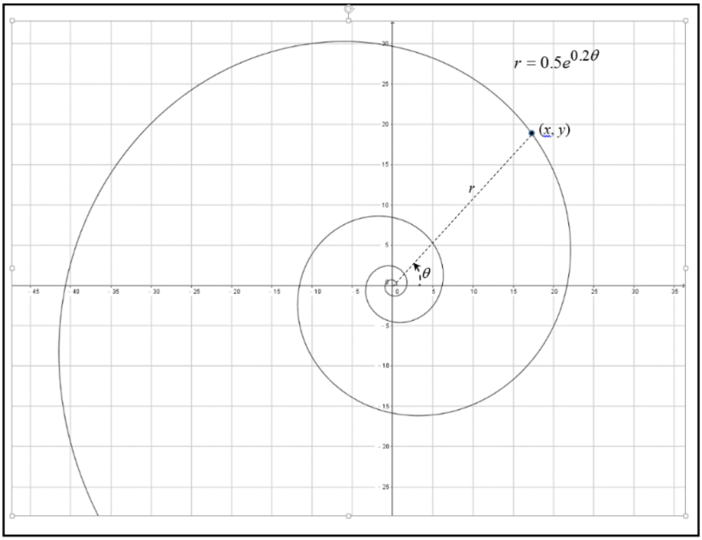


圖二："等角螺線"

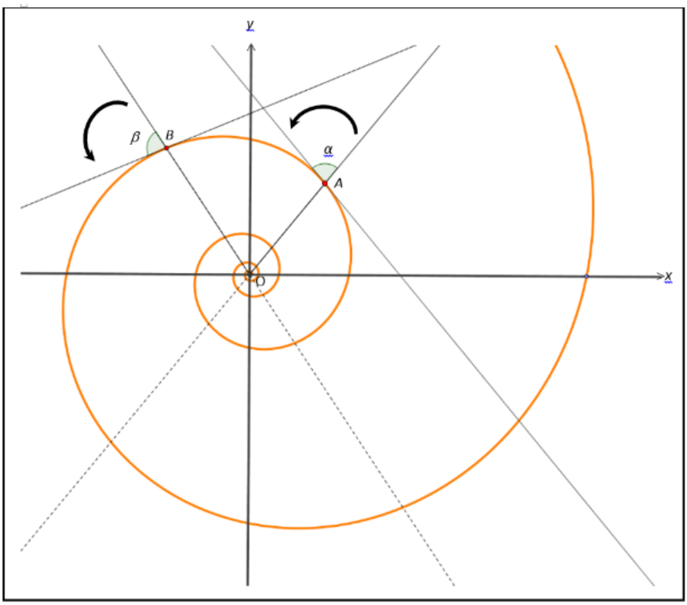
驟眼一看，"等速螺線"的闊度像保持不變而"等角螺線"的闊度則越來越大。可是後者為何稱作"等角螺線"，何來角相等，那麼非要從數學入手不可！

"等角螺線"是由極坐標方程定義：$r = ae^{b\theta}$，其中 a 是大於零的常數。r 是曲線上點 (x, y) 和原點 $(0, 0)$ 之間的距離及 θ 是曲線上點 (x, y) 和原點 $(0, 0)$ 的連線與 x 軸正方之間以反時針方向量度的角，以弧度為量度單位，方便其後微積分的處理。同時為方便討論，亦只考慮 b 是大於零的常數及 $\theta \geq 0$。其實"等角螺線"的極坐標方程只是一條軌跡方程。它指出曲線是由移動點受 $r = ae^{b\theta}$ 的限制下"走"出來的路徑。

以下是用 GeoGebra 繪畫 $r = 0.5e^{0.2\theta}$ 的圖像。



圖三



圖四

原來數學家發現這類曲線有著一種特別的性質：設 L 是穿過原點 $(0,0)$ 的任意直線。當 L 與該曲線相交，其相交點的切線與 L 之間的夾角保持不變。如圖四，在曲線上任意兩點 A，B，我們有 $\alpha=\beta$。為什麼呢？讓我們用微積分作證明。

設 (x,y) 是“等角螺線”上任意一點 A 的坐標。
設 α 是穿過原點 $(0,0)$ 和曲線上點 A 的直線以反時針至該點切線的夾角。可參考圖四。
已知“等角螺線”的極坐標方程是 $r=ae^{b\theta}$，其中 a，b 是大於零的常數，$\theta\geq 0$ 並以弧度為量度單位。

$$(x,y)=(r\cos\theta, r\sin\theta)$$
$$(x,y)=(ae^{b\theta}\cos\theta, ae^{b\theta}\sin\theta)$$
$$\frac{dy}{d\theta}=a\left(be^{b\theta}\sin\theta+e^{b\theta}\cos\theta\right)=ae^{b\theta}\left(b\sin\theta+\cos\theta\right)$$
$$\frac{dx}{d\theta}=a\left(be^{b\theta}\cos\theta-e^{b\theta}\sin\theta\right)=ae^{b\theta}\left(b\cos\theta-\sin\theta\right)$$

現在證明 α 是一個常數。

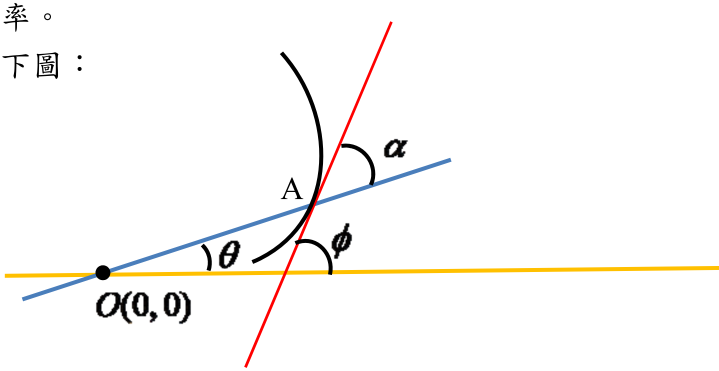
設 $0\leq\theta<\frac{\pi}{2}$，$A$ 有三種情況要考慮(至於 $\theta\geq\frac{\pi}{2}$ 的情況，留待讀者自行證明) 。

情況一： $0 \le \theta < \tan^{-1} b$ ，即 $b - \tan\theta > 0$ 。

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{ae^{b\theta}(b\sin\theta + \cos\theta)}{ae^{b\theta}(b\cos\theta - \sin\theta)} = \frac{b\sin\theta + \cos\theta}{b\cos\theta - \sin\theta}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{\cos\theta}\left(\frac{b + \frac{1}{\tan\theta}}{b - \tan\theta}\right) > \frac{\sin\theta}{\cos\theta}$$ ，即切線的斜率大於 OA 的斜率。

看下圖：



圖五

我們有 $\alpha = \phi - \theta$ 。

$$\begin{aligned}
&\tan\alpha \\
&= \tan(\phi - \theta) \\
&= \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta} \\
&= \frac{\frac{dy}{dx} - \tan\theta}{1 + \left(\frac{dy}{dx}\right)\tan\theta}
\end{aligned}$$

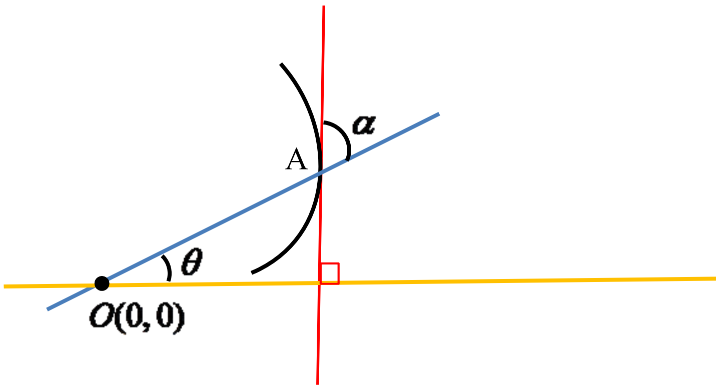
$$\begin{aligned}
&= \frac{\frac{b\sin\theta + \cos\theta}{b\cos\theta - \sin\theta} - \tan\theta}{1 + \left(\frac{b\sin\theta + \cos\theta}{b\cos\theta - \sin\theta}\right)\tan\theta} \\
&= \frac{b\tan\theta + 1 - b\tan\theta + \tan^2\theta}{b - \tan\theta + b\tan^2\theta + \tan\theta} \\
&= \frac{1 + \tan^2\theta}{b(1 + \tan^2\theta)} = \frac{1}{b}
\end{aligned}$$

$$\alpha = \tan^{-1}\frac{1}{b}$$

情況二： $\theta=\tan^{-1}b$，即 $b-\tan\theta=0$。

我們有 $\frac{dx}{d\theta}=0$。但 $\frac{dy}{d\theta}>0$，由此，可知曲線在這點 A 的切線是平行於 y 軸的。

看下圖：



圖六

我們有 $\alpha+\theta=\frac{\pi}{2}$。

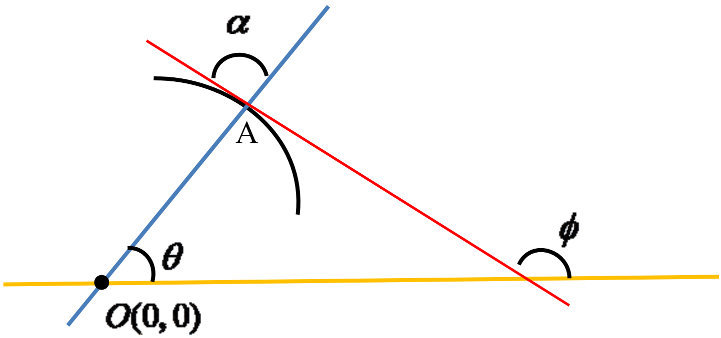
$\tan\alpha=\frac{1}{\tan\theta}=\frac{1}{b}$。即 $\alpha=\tan^{-1}\frac{1}{b}$。

情況三： $\tan^{-1}b<\theta<\frac{\pi}{2}$，即 $b-\tan\theta<0$。

$$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{ae^{b\theta}(b\sin\theta+\cos\theta)}{ae^{b\theta}(b\cos\theta-\sin\theta)}=\frac{b\sin\theta+\cos\theta}{b\cos\theta-\sin\theta}$$

$$\frac{dy}{dx}=\frac{\sin\theta}{\cos\theta}\left(\frac{b+\dfrac{1}{\tan\theta}}{b-\tan\theta}\right)<0$$ ，即切線的斜率為負值。

看下圖：



圖七

我們有 $\alpha=\phi-\theta$ 。

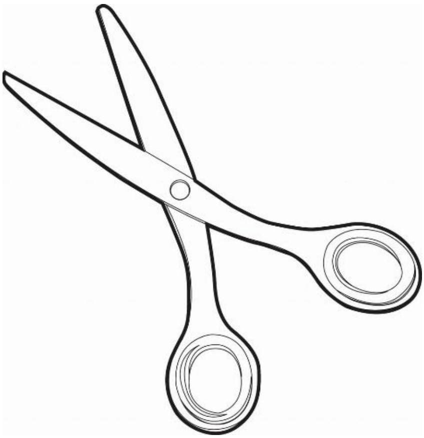
跟情況一相同，得 $\alpha=\tan^{-1}\dfrac{1}{b}$ 。

因此，綜合以上三種情況：

$\alpha=\tan^{-1}\dfrac{1}{b}$ ，並且 α 是一個常數銳角。

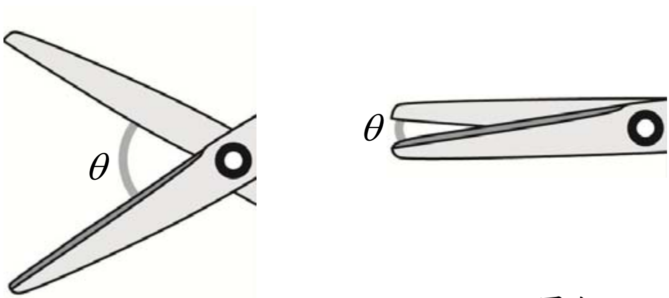
可能你會覺得從純數的角度，這類曲線有著一種漂亮的氣質，比“等速螺線”好看！那麼，故事是否就此完結呢？

現在轉一轉場地，看看我們日常生活常用的剪刀：



圖八

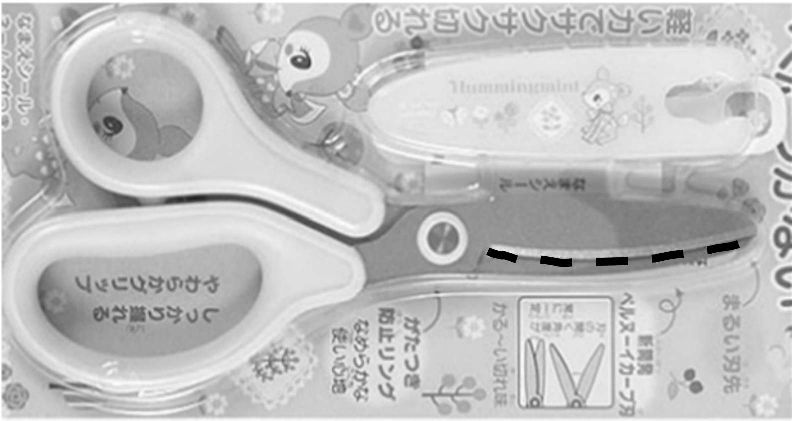
從圖九中，剪刀的“V”形刀口的夾角θ在開始的時候很大，其後越剪，θ就越小。



圖九

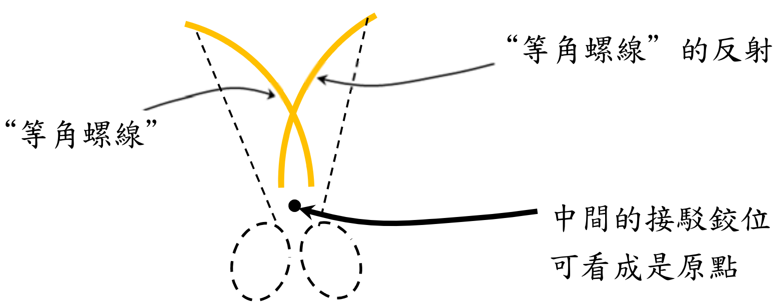
你是否覺得當θ越小，剪刀會變得不太“好剪”呢？若果能保持夾角θ不變，是否使剪刀變得比較“好剪”呢？可是刀口是直線形狀的，無論怎樣，夾角始終會變，何來不變？我們知道幾何圖形不單有直線還有其他，如曲線等。若將刀口的形狀由直線換成曲線，是否存在某類曲線可使刀口

依照這種形狀能使夾角 θ 保持不變呢？答案是肯定的。原來在 2012 期間熱賣一款由日本公司研發的剪刀，名為 Fit cut Curve scissors (註 3)，它正好能滿足以上的要求：

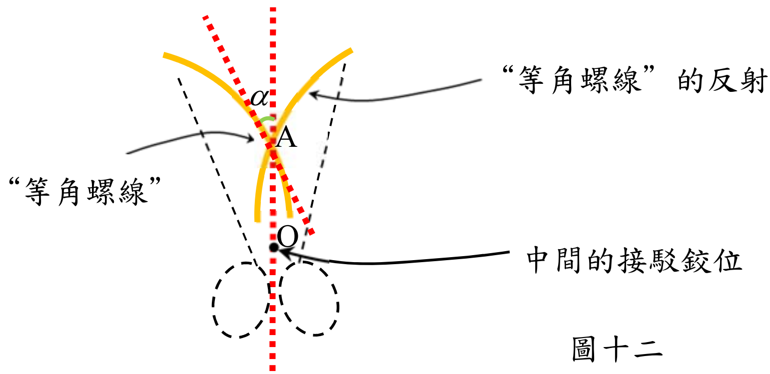


圖十

這款剪刀的刀口曲線正是“等角螺線”和它的反射(見圖十一)，能將數學活用在這種日常用品中，真是十分有創意的。話雖如此，但為何這類曲線能使刀口的夾角 θ 保持不變呢？我們可以考慮下圖：



圖十一



圖十二

由"等角螺線"的性質，知道連接 OA 的直線與曲線在 A 點的切線的夾角 α 是常數 (見圖十二)。運用反射對稱的原理，另一面的夾角亦等於 α。因此，無論怎樣剪，剪刀的刀口夾角等於 2α，一個不變的角度。前面已經指出，$\alpha = \tan^{-1}\left(\frac{1}{b}\right)$。因此，不同選擇的 b，可製造出不同的"等角螺線"，同時亦產生不同的刀口夾角 2α。那麼，是否不同夾角可使剪刀的切割效果造成差異？我們可以從那生產商的資料提供，刀口夾角等於 30° 是一個好的選擇，你同意嗎？(註 4) 相信這是一個數學與科學共同研究的好問題。

註 1：數學教育組於 2007 年成立數學歷史研習小組，其目的是建立一個教師學習圈以深入探討如何將數學歷史融入數學的學與教內。承蒙蕭文強教授及梁子傑教師積極支持，數學歷史研習小組歷年來於教育局專業發展研討會中與教師分享不同的數學歷史課題。現時小組成員包括二十多位數學教師及教育局

課程發展主任。

註 2 ：參考 https://en.wikipedia.org/wiki/Jacob_Bernoulli

註 3：參考 https://bungu.plus.co.jp/product/cut/fcurve/

註 4：參考

http://stasrv.bungu.plus.co.jp/product/cut/fcurve/index.html

12. Reflection of 40th Annual Conference of the International Group for the Psychology of Mathematics Education (PME 40)

CHAN Sau-tang

I had an opportunity to participate in PME 40 annual conference this summer time. The annual conference took place in Szeged of Hungary from August 2 to 7, 2016. The conference comprised a series of activities which were related to mathematics education. These included research forums, working sessions and discussion groups, research report presentation sessions, and poster presentation sessions. During those sessions, I could learn from recent research findings, exchanged ideas and experience with overseas educators and professionals on the current trend of curriculum development in mathematics. I could also gather professional views on the contemporary theories and practices on mathematics education so as to pave the way forward in reviewing the Hong Kong mathematics curricula and providing support to mathematics teachers. It was a precious time for me.

When we talked about Mathematics Education in Hungary, what was the first thing getting into your mind? Was it a Rubik's Cube? It was a famous toy designed by a Hungary's professor. But I would like to talk about another thing in this article. When I was a secondary school student, once upon a time, I struggled for a long period of time in solving a mathematical problem. I asked my teacher for help. I was deeply impressed by my teacher's clever technique of tackling the problem. Then, I asked my teacher why he could do it in that way. He thought for a while and said that he did it by experience. Practice made perfect. The skill at choosing an appropriate strategy was best learnt by solving various problems day by day. He might be right, but was it possible to teach problem solving?

The theme of the PME 40 conference was "Mathematics Education: How to Solve it?" I guessed all of you knew the reason why the theme was set in that way. ***How to Solve It*** (1945) was a famous book written by Hungary mathematician George Pólya. In the book, he identified four basic principles of problem solving. Problem solving was difficult to most people because there was no single procedure that worked all the time. Problem solving required practical knowledge about the specific situation. George Pólya hoped to set up a framework for thinking about problem solving. I would like to recap his idea with illustrative examples in this article.

Polya's four-step principles to problem solving included *understanding the problems*, *devising a plan*, *carrying out the plan* and *looking back*. To understand the problems, Polya stated that we needed to consider the terminology and notation in the problem. What sort of a problem was it? What was the known or unknown? Were the conditions sufficient to determine the unknown, insufficient, redundant, and contradictory? Could you restate the problem? Could we think of a picture or diagram that might help us understand the problem?

At the second stage, we needed to devise a plan. We might ask ourselves a series of questions: Had you seen that before? Did you know a related problem? Was there another problem with the same unknown? Was there a related problem solved? The possible strategies were to draw a picture, solve a simpler version of the problem, guess and check, trial and error, look for a pattern, make a list, use direct reasoning, work backwards, use a formula, etc.

We needed to carry out the plan at the third stage. We followed our plan step by step, given that we had the necessary skills. We should check each step and make sure that it was correct. If the plan did not seem to be working, then we should start over and try another approach. We should keep trying until something worked.

The final stage was looking back. We needed to check our potential solution. Did we answer the question? Was our result reasonable? Was the result unique? Was there another way of doing the problem which might be simpler? Could the problem be generalised so as to be useful for future problems?

I would like to tackle the following problems with Polya's

Problem Solving Techniques.

Problem 1:

Prove that no number in the sequence
11, 111, 1111, 11111, … is the square of an integer.

Hints:

Do you understand the meaning of "the square of an integer"?
We can try the first two terms, where $3^2 < 11 < 4^2$, $10^2 < 111 < 11^2$. It seems the statement is correct.
What properties of square numbers do you know?

Can you restate the problem? The problem can be restated as "Find a perfect square k of the form $1 + 10 + 10^2 + ... + 10^n$, where n is a positive integer."

Solution:

If k is a number in the sequence, it can be written as $k = 11 + 100a = 4(25a + 2) + 3$ where a is a non-negative integer. We know that square numbers can be represented as the form $4n^2$ or $4n^2 + 4n + 1$. When it is divided by 4, the remainder is either 0 or 1, but the remainder for k is 3, hence it is not the square of an integer.

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Problem 2:

Peter has 10 pockets and 44 coins. He wants to put his coins into his pockets so distributed that each pocket contains a different number of coins. His pocket can be empty.

(i) Can he do so?

(ii) Generalise the problem, considering k pockets and n coins. When $n = \frac{(k+1)(k-2)}{2}$, what will happen?

Hints:

Do you understand the question? To solve a simpler version of the problem, we may consider 3 pockets only. What is the minimum number of coins that we need to put into the pockets so that each one contains a different number of coins. The answer is $0+1+2=3$. The problem can be re-stated as " What is the minimum number of coins that can be put into the 10 pockets so that no two different pockets contain the same amount?"

Solution:

To fulfill the requirement, the least number of coins that we need = 0 + 1 + 2 + 3 + 4 +5 + 6 + 7 + 8 + 9 = $\frac{10\cdot(0+9)}{2} = 45$.

As Peter has only 44 coins, he cannot do so.

For the general case, the problem has a solution when n is equal to or greater than $0+1+2+3+...+k-1=\dfrac{k(k-1)}{2}$

The problem has no solution when n is equal to $\dfrac{k(k-1)}{2}-1=\dfrac{k^2-k-2}{2}=\dfrac{(k-2)(k+1)}{2}$.

Problem 3:

Of all the triangles with fixed perimeter, which one has the largest area?

Hints:

What is the meaning of "perimeter"? What is the relationship of the length of different sides of a triangle with its area? We may let the perimeter be 12 cm, draw the triangle with empirical values and calculate its area. Draw the triangles with the sides (4, 4, 4), (3, 4, 5), (3.5, 3.5, 5), etc. Do you have some idea about the answer?

Do you know Heron's formula? Do you know the Inequality of arithmetic and geometric means?

Solution:

Heron's formula states that the area of a triangle whose sides of lengths a, b and c is $A=\sqrt{s(s-a)(s-b)(s-c)}$ where

$$s = \frac{a+b+c}{2}$$

Suppose we have triangle of different sides *a*, *b* and *c*,

then $A^2 = s(s-a)(s-b)(s-c)$

Since G.M. $\leq$ A.M.

We have $\sqrt[3]{(s-a)(s-b)(s-c)} \leq \frac{(s-a)+(s-b)+(s-c)}{3}$

$$(s-a)(s-b)(s-c) \leq \left[\frac{(s-a)+(s-b)+(s-c)}{3}\right]^3$$

$$(s-a)(s-b)(s-c) \leq \left[\frac{3s-(a+b+c)}{3}\right]^3$$

$$(s-a)(s-b)(s-c) \leq \left(\frac{s}{3}\right)^3$$

Since s is fixed, we maximise $\sqrt{(s-a)(s-b)(s-c)}$ which is

$$\leq \frac{s^{\frac{3}{2}}}{3\sqrt{3}}$$

For equality to hold, we have $s-a = s-b = s-c$, so $a = b = c$

So, $A^2 = s(s-a)(s-b)(s-c)$

$A^2 = s(s-a)^3$ where $a = b = c$

$$A^2 = s\left(s - \frac{2s}{3}\right)^3 \qquad \text{where } s = \frac{a+b+c}{2} = \frac{3a}{2}$$

$$A^2 = s\left(\frac{s}{3}\right)^3$$

$$\text{Max } A = \frac{s^2}{3\sqrt{3}}$$

Equilateral triangle has the largest area under the given condition. We can also solve the problem with Calculus.

To conclude, Polya's four-step approach to problem solving could train us to become better problem solver. But, it was not so easy to make sense of the problems by selecting appropriate examples. Detailed training was needed.

Reference:

George Pólya and Jeremy Kilpatrick (1974). *The Stanford Mathematics Problem Book: With Hints and Solutions*. Teachers College Press, New York