SCHOOL MATHEMATICS NEWLETTER

ISBN 978-988-8370-22-1
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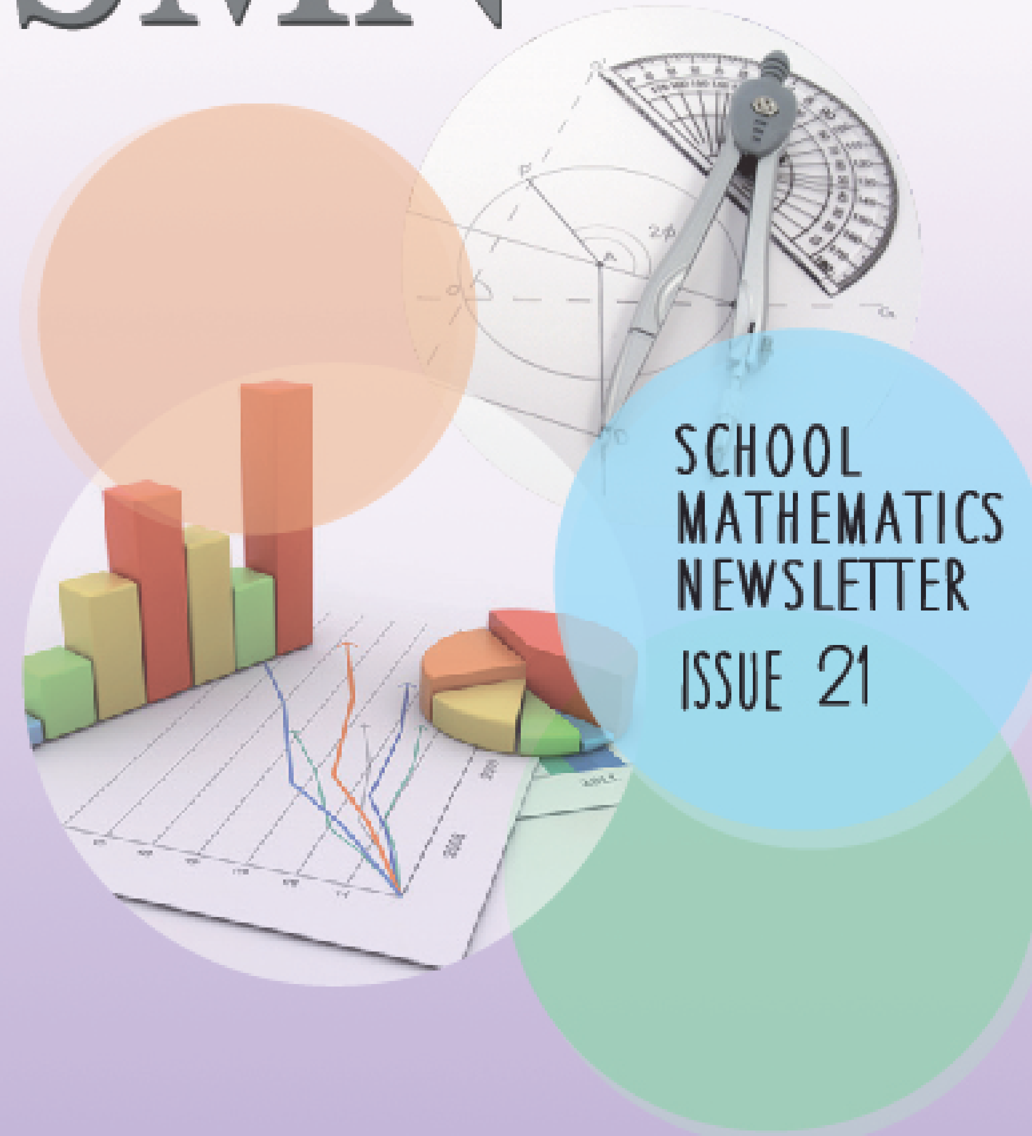
學校數學通訊 第二十一期

教育局

SMN



SCHOOL MATHEMATICS NEWSLETTER

ISSUE 21

Published by
Mathematics Education Section, Curriculum Development Institute,
Education Bureau, Government of the Hong Kong Special Administrative Region.
香港特別行政區政府教育局課程發展處數學教育組出版

ISBN 978-988-8370-34-4

School Mathematics Newsletter（SMN）

Foreword

The School Mathematics Newsletter（SMN）is for mathematics teachers. SMN aims at serving as a channel of communication for mathematics education in Hong Kong. This issue includes articles written by academics, principals and teachers. Most articles are about STEM education, its concept and the school-based implementation strategies. Some articles are about the learning and teaching of mathematics. Learning mathematics through technologies is also included in the second to last article, written by Dr TAN. The last article, “Magic Squares and Algebra” is contributed by Dr Mark Saul. He tactfully matches two ideas together. I hope all the readers can get some fascinating insights in mathematics education.

SMN provides an open forum for mathematics teachers and professionals to express their views learning and teaching in mathematics. We welcome contributions in the form of articles on all aspects of mathematics education. Please send all correspondence to:

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We extend our thanks to all who have contributed to this issue.

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羅浩源

香港中文大學課程與教學學系

引言

STEM 就是科學（Science）、科技（Technology）、工程（Engineering）及數學（Mathematics）四門學習領域的縮略詞。他們在排列上的次序，不表示其在學習上的優次，而只是有助記憶提取和溝通上的一種表達。

在香港推動 STEM 教育的目的乃通過學校課程在科學、科技及數學各範疇的持續更新來保持香港本身的國際競爭力（香港教育局課程發展處（2015））。這種思維，與美國在其本土以 STEM 教育來提升其競爭力[1]，從而滿足到以「美國優先」的觀念同出一轍（見 National Research Council, 2011）。但不同的是，香港學生在數學和科學上的表現較美國的為佳（PISA 2015：香港和美國在數學上的排名分別是第 2 名和第 40 名；而在科學上則分別是第 9 名和第 25 名）。從國際評估的角度看，香港學生在數理科的表現還是具備一定的優勢。但究竟怎樣利用這個「優勢」來達致在香港推動 STEM 教育所期盼的目標？這一個目標應涵蓋通過跨科在學習上的連繫，從而進一步提升學生在學科知識、共通能力、價值觀與態度方面的表現。從數學教育的角度看，STEM 教育的推行會為數學的教與學帶來不少衝擊和挑

[1] 世界經濟論壇發表《2017-2018 年度全球競爭力報告》，香港排名第六位而美國第三位 。

戰。美國教育學者 Nataly Chesky 和 Mark Wolfmeyer 皆曾在學校任教數學，他們在 2015 年所出版的書《STEM 教育的哲學：一個批判性研究》的序言（頁 xi）中提出了這樣的質疑：「我們不曉得數理科的教學是否應該視為進一步加強國家經濟能量所需的科技與工程技能的一項功利活動。而我們並不相信孩童的幸福和成就可與他們具備在本土或全球化競賽中勝出的能力畫上等號。」他們的觀點，值得我們細味和深思。

數學在 STEM 教育中的定位

STEM 的本質是跨學科的一種結合，但「真正的整合」（authentic integration）包含四個特徵（Treacy & O'Donoghue, 2014）：

（一）知識的發展、綜合與應用（knowledge development, synthesis and application）；

（二）產生高階學習的聚焦探究（focused inquiry）；

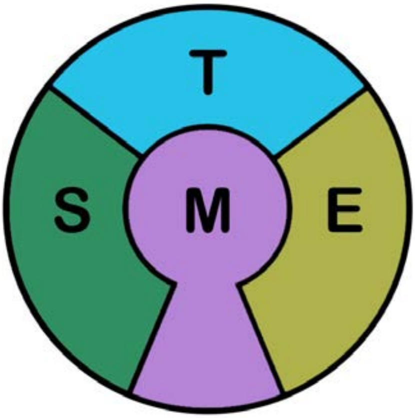
（三）符合現實世界情景的應用（real-world application）；

（四）豐富的作業（rich task）

這種特質的結合對數學科的教與學不僅是一個極大的挑戰。STEM 教育的實踐，一般會強調「以問題為本的學習」與「專題為本的學習」、「科學探究」與「工程設計學習模

式」，但往往卻忽略了怎樣以數學概念來帶動其他相關範疇學習應有的重視（Fitzallen, 2015）。

在中小學的基礎課程中，數學是一個重要的必修課目。在教育推行的過程中，我們需強調數學本身應有的角色。數學本身既是窺視 STEM 的鑰匙孔，亦是打開 STEM 之門的鑰匙（見圖一）。



圖一：M（數學）在 STEM 的角色

若要 STEM 教育更有效地施行，便應從學生本身的生活體驗中發掘更多數學的元素，從而進一步提升理解數學效用的層次。若我們以數學作為起點去推動 STEM 教育，可利用下列的學習模式來設計課堂活動。

M：以數學語言理解生活經驗的意識，從而進一步界定探究的問題

E：從問題出發，設計解難所需模型

T：協商探討為解決問題創造所需工具（包括電腦軟件的使用）

S：利用實驗進一步探索相關的自然環境現象

總結

以數學作為起點來配合 STEM 教育的推行無疑是一個很大的挑戰。從教師教育方面看，教師須具備更強的教學意識來引導學生以數學作為連繫科學、工程和科技各範疇所需的語言。為了促進學生積極參與 STEM 活動，教師須在課堂上創造更廣闊的討論空間，從而鼓勵他們多些從生活經驗中提出有意義的問題，讓學生產生把問題「STEM 化」（思維模式已不再局限於組成 STEM 各自的領域)的過程，然後進一步提升設計、解難、探索和創新的能力。

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2. 校本 STEM 教育經驗分享

關子雋老師、簡嘉禧老師
香港聖公會何明華會督中學

一、引言

近年全球各國均在推行「創客教育」、「STEM」學習，教育局亦大力推展有關計劃，藉以提升本港同學的跨學科自學與應用的能力，從而能夠發揮自己的潛能與創意。

為了配合 STEM 教育的教育趨勢，本校亦嘗試實踐一些「STEM」跨學科學習計劃，包括：全像投影器的制作、水火箭車的制作、植物生長環境的分析等。在校本的推展經驗當中，大部分教案也是以科學科的課題為起點，加入其他科目的內容，最後擴展為完整的學習活動。一個 STEM 學習計劃的誕生，除了需要不同科目老師有共同理念，亦要為實踐教案投入不少的設計心思與課時，所以整個教案從構思、討論、落實到最後檢討成效著實得來不易。接下來將會介紹本校的其中兩個「STEM」學習活動供各位老師參考，期能收拋磚引玉之效。

二、巧匠樂器

首個要介紹的是與教育局資優教育組合作的教學活動…「巧匠樂器」。整個學習活動是期望同學們能夠在制作樂器的過程中，讓同學能夠把理論層面的知識，應用到設計、探究以及實作的層面上。而活動分為兩個部份：首部份主要讓同學們學會聲音的各種不同特性，包括聲音的傳播媒

介、聽頻範圍等，對象為全體中二同學；而第二部份是探究活動，目的讓同學們學會數學與聲音的關係，然後著手製作屬於自己的樂器。由於課時有限，所以此部份採取了抽離式教學，由課任教師選取合適的中二級同學參與此部份學習活動。

第一部份是整班進行的學習活動。由於聲音的特性是屬於中二級綜合科學科的內容，所以本校選取了中二級推行是次的「STEM」活動。在中二級綜合科學科的課程中，關於聲音的特性的篇幅並不多，過往本校教師只消二至三堂課已經完成有關的教學。為了配合本次的「STEM」學習活動，本校教師參考了資優教育組同工的意見，特意設計了多個有趣的實驗活動予同學，額外多花了四個課節完成有關的課題。其中一個同學最喜歡的實驗是讓同學咬著一個接連到電話的馬達，然後播放音樂。馬達的震動能藉由同學的牙齒傳到耳朵中，但由於電話並沒有接連擴音器，所以其他同學不會聽到任何音樂。同學從中學習固體為聲音的傳播媒介之一。而另一個有趣的實驗是由本校電腦科教師設計並實行的。為了測試同學們的聽頻範圍，本校電腦科教師教導了同學利用「Arduino」部件接連蜂鳴器，並藉由簡單的編程使之發出不同頻率的聲音，以此測試自己的聽頻範圍。此部份為固有的科學科課程，但藉著推行「STEM」學習的機會，本校教師能為固有的教案注入新元素。

同學咬著接連電話的馬達，進行有關聲音傳播媒介的實驗。

同學正利用把「Arduino」部件接連蜂鳴器。

第二部份為抽離式的學習活動，讓同學探究聲音與數學之間的關係。同學於先前的學習活動中已經學會聲音與頻率之間存在某種關係。但「存在某種關係」對於需要製作樂器的同學來說是並不足夠的。同學們需要找出不同音階與頻率之間的關係。為了有效引導同學，教師要求同學們利用平板電腦量度不同音階的頻率，讓同學理解到音階的頻率之間是等比的關係，然後同學們便可著手製作樂器。本校同學選擇制作一件較大型的水管樂器。雖然同學於此階段已學過音階與頻率的關係，但是同學們還需要探究水管長度與頻率之間的關係。本校教師先著同學們隨意切出七支水管，然後量度它們所發出的頻率。把這些數據記錄於 EXCEL 之中，再配上合適的趨勢線，同學便能藉此進行頻率與水管長度之間的「互換」。在教師的協助與指導下，同學們需要學會利用工場內的鋸、打磨器等的工具，切割所需的水管長度。最後為整套樂器上色，便完成屬於同學自己的樂器。

同學們正量度所需的水管長度。

同學與製成品合照。

三、自製 PM2.5 測量儀

環境保護署每天均會公佈各區不同監察站的空氣污染數據，但卻未有提供特定位置的數據。因此，我們自行製作一個 PM2.5 的測量器，以便監察社區的空氣污染問題。

我們使用灰塵感應器測量空氣中的「灰塵」，並利用 ARDUINO 電子板把灰塵感應器的電壓轉化為讀數，並利用手提電腦用作顯示屏及供電(圖 1)。測量器製作完成後，我們利用香港中文大學太空與地球信息科學研究所借出的專業儀器，應用迴歸分析以測試我們的測量器，結果兩組數據的相關系數高達 0.939(圖 2)。在初中的數學科，學生已學習直線方程的概念。然而，不少學生對直線方程的應用所知不多。在科學上，我們經常需要研究不同變量的關係。我們一般先假定變量之間存在一定關係，應用迴歸分析測試各種數學模型與測量值的吻合程度(相關系數)。迴歸分析便是直線方程的延伸應用。當數學模型建立後，我們便能以一個變量的數值推斷另一變量的數值。

	專業儀器讀數	自製測量器讀數 (電壓)
誠明館外	8.93	29.6
新亞餐廳廚房外	9.47	43
圖書館外	8.77	17.3
新亞書院停車場	8.93	35.7
學思樓外	7.93	10.3
合一亭外	7.67	6.1

圖 1

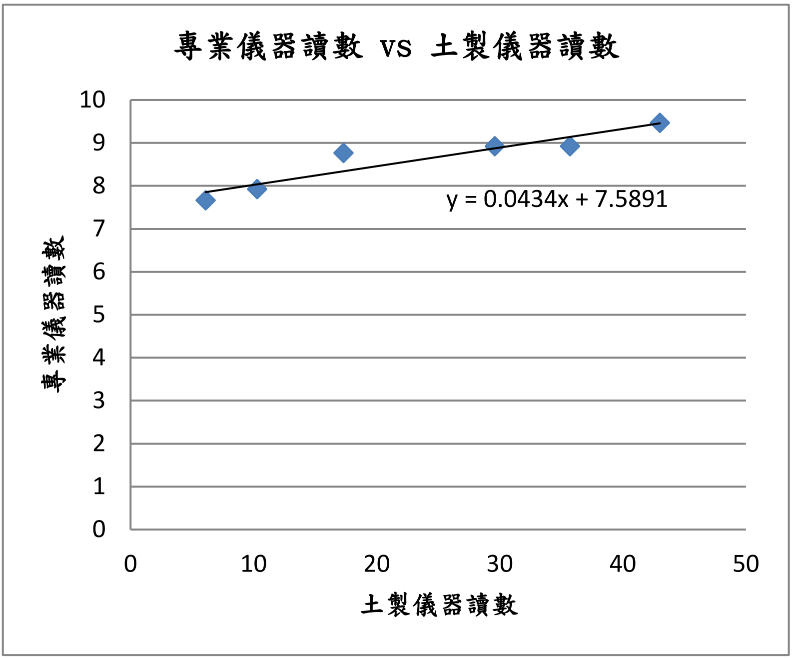
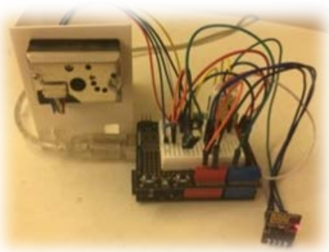


圖 2

自製 PM2.5 測量器-原型機

自製 PM2.5 測量器第二代

自製 PM2.5 測量儀的校準結果良好，然而仍有不少改進空間，最大的缺點便是攜帶不便。第二代測量儀加入了顯示屏及獨立電池，令測量器可脫離手提電腦獨立運作，方便攜帶。同時，我們亦利用立體打印製作外殼令作品更美觀。我們又在第三代中取消了顯示屏以節省電力，並加入一個 Wifi 模組，數據可實時上載至雲端伺服器，亦可下載為 Excel 格式編成的數據，方便分析。



自製 PM2.5 測量器第三代

數據實時上載至雲端伺服器

最後，我們利用自製 PM2.5 測量器，到學校社區不同的位置量度 PM2.5 數據，發現部份地點的空氣質素未符合世衛標準。

地點	緯度	經度	PM2.5
學校旁達摩廟	22.314288	114.234527	48.01
秀茂坪道	22.318059	114.235268	35.67
聯合醫院改建工程地盤	22.322799	114.227651	31.58
安達村	22.319369	114.235638	31.52
梁式芝書院正門	22.315149	114.233465	21.61
秀茂坪天主教小學正門	22.317592	114.236907	15.68
曉光閣	22.317858	114.230856	15.65
秀明小學正門	22.319581	114.232816	12.34
秀義樓	22.322566	114.230341	11.23
秀茂坪公園	22.323413	114.229219	10.04
曉麗商場	22.320654	114.229381	9.15
秀茂坪商場	22.319403	114.231727	8.52
學校一樓	22.314273	114.233578	7.59
秀茂坪村籃球場	22.318091	114.234031	6.15

四、結語

推動 STEM 跨學科學習活動實在有賴不同學科部門的同工一起嘗試，從不同的教學活動中汲取經驗並改良已有的教案。期望將來能夠從不同平台中與各老師作更多的交流，一起推動 STEM 學習風氣。

3. 李炳學校的「STEM 教育」

林嘉康校長
九龍婦女福利會李炳紀念學校

在李炳學校推行「STEM 教育」的由來

「STEM教育」的名稱可追溯到2015年，當時教育局開始推行「STEM教育」，向全港每所小學撥出10萬元，至於如何使用及推行模式等問題則由學校自行決定。因此，不少坊間的資訊科技教育機構均即時推出不少與STEM有關的項目及活動。恰巧，本人於2016年5月，參加了由教育局主辦的小學校長領導課程，當中包括到北京參觀「STEM教育」的發展。整個旅程令我印象最深刻的是清華大學為推行「STEM教育」所建立的「群體創新空間」。他們推行的CAME(Computer Aided Manufacturing for Education)課程，以平板(包括木板/亞架力板)為主要材料，配合電子材料，學習難度降低了，但又能做出三維作品，真正體現了學生的創意。正正是以上的原因，吸引了我依據CAME的方向在李炳學校推行「STEM教育」。

為推行「STEM教育」定位

要推行「STEM教育」的模式有很多，既然如此，我們必須為推行「STEM教育」定位。經討論後，我們認為「STEM教育」其實是希望幫助學生擺脫只讀書卻無法動手的桎梏，將創客文化與教育結合，基於學生興趣，以專題學習的方式，使用四周隨手可得的不同工具及材料，再配合傳統及先進的設備，藉此來培養學生跨學科解決問題能力、團隊

協作能力和創新能力的一種素質教育。學生可以隨時就生活及個人的需要，即時走到電腦前設計，然後拿起一些工具或使用一些設備做後期製作，最後成功解決問題。基於以上的原因，我們決定先從設備及教師培訓着手。

李炳學校推行「STEM教育」的計劃

「工欲善其事,必先利其器」,為了實踐CAME課程的理念,首要的工作是運用了七萬多元買了一部鐳射切割機，然後利用了暑假的時間作教師培訓,學習如何操作鐳射切割機。由於是首年推行「STEM教育」的關係，因此，目標主要以加強老師的信心及培養學生興趣為大前提,有關計劃如下:

- 成立STEM小組，並進行校本教師培訓(2016/17)
- 利用課外活動時段，成立「非常學生之STEM小組」，教授學生操作「鐳射切割機」、「6合1工具裝置」及「Rhinoceros繪圖軟件」，並利用有關器材及軟件製作各樣名牌、獎座、各科教具及小型裝置(2016/17)
- 進行跨科專題研習，利用所添置的器材及軟件製作不同物品：
    - 小五常識科、數學科及視藝科(製作燈座)(2016/17)
    - 小五及小六常識科、數學科及視藝科(2016/17、2017/18)
    - 小四至小六常識科、數學科及視藝科(2016/17、2017/18、2018/19)

- 於全方位學習日進行校本科學活動/比賽(2016/17、2017/18、2018/19)
- 參與校外科學活動/比賽，提升學生對STEM教育的興趣(2016/17、2017/18、2018/19)

如何在數學科推行「STEM教育」

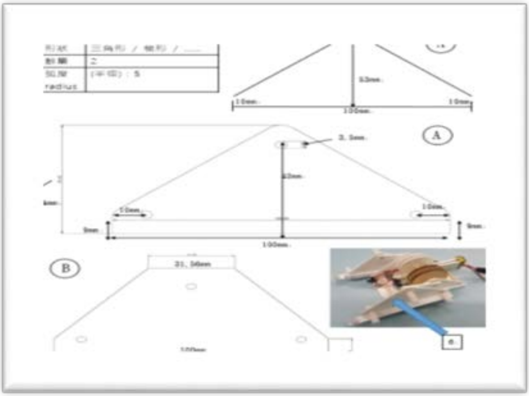
「STEM 教育」包含「SCIENCE(科學)」、「TECHNOLOGY(科技)」、「ENGINEERING(工程)」及「MATHEMATICS(數學)」，因此，我們認為「STEM教育」必須以跨學科學習的形式去推行，這樣才能帶出「STEM教育」的精髓。至於如何在作品或專題中顯示出數學的元素，本校會從幾個方向着手。

方向一：運用材料及工具

由於學生要利用「鐳射切割機」及「6 合 1工具裝置」去切割不同的厚度及闊度的木板及膠板，然後進行裝嵌，因此，對於尺寸的要求是很嚴格。學生除了需要運用他們慣常使用的直尺來進行量度外，還會使用到卡尺等工具去量度，務求量度出更準確的尺寸，減低裝嵌時的出錯。而這些工作，正正是在訓練學生在數學科中的實作技能，當他們在量度時出現了錯誤，作品便欠完整，甚至無法完成，學生因而要即時作出修

改，才能製成一個完整的作品。相比於學生只運用直尺及工作紙進行實作評量，此方法顯得更有趣及更有成效。

方向二：繪圖軟件的運用

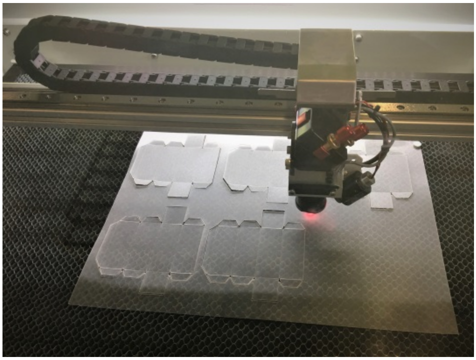


在運用「鐳射切割機」進行切割前，學生須在「Rhinoceros軟件」進行繪圖。在「Rhinoceros軟件」中，不少指令是與數學有關的，例如透過軟件的板面去學習「立體與平面的關係」、透過鏡面的原理去學習「對稱」、透過平均分配的原理去學習「角度」、透過繪畫圓形去學習「圓心、半徑與直徑」等，都是一些學生在學習繪圖軟件時可以學到的數學元素。學生因為在繪圖時的「需要」而去學習的數學知識，遠比老師按課題的教授，學生的「吸收度」會更深更廣。

方向三：讓學生親身感受STEM的好處

要讓學生明白到運用STEM的好處，最好的方法必定是讓他們親身感受。我校今年以「製作立體圖形」的課題入手，以往學生做此專題時，通常先在卡紙上繪圖，然後再剪出摺紙圖樣來進行拼砌。但往往在進行中，學生有時會因尺

寸問題而要重新製作，或要製作多個相同的立體圖形而要作影印及裁剪，浪費不少時間。我們在本年度針對問題作出修改，學生先利用「Rhinoceros軟件」繪



畫摺紙圖樣，然後運用「鐳射切割機」進行切割，學生除了可以在短時間內作出修改外，還可以透過「複製」的功能來製作多個立體圖形，減去不少製作的時間。

推行「STEM教育」的中期檢討

由2016年9月到現在，「STEM教育」約進行了一年時間，各項目均依據着預計的計劃進行，詳情如下：

- 成立了由五位老師組成的STEM小組，同時亦於課外活動小組成立了「非常學生之STEM小組」

- 在老師的協助下，「非常學生之STEM小組」利用「鐳射切割機」為全校特別室製作門牌；同時亦於本年度的運動會利用「鐳射切割機」製作特別的紀念品
- 以「製作燈座」為主題，在五年級進行跨科專題研習(常識科、數學科、視藝科及電腦科)

- 在2017年1月舉行的全方位學習日中，於高年級(小五

及小六)以STEM為主題，舉行名為「活學steam・活在teams之決戰李炳三道館」的學習活動，利用「鐳射切割機」、「6合1工具裝置」及「Rhinoceros繪圖軟件」等工具/軟件來製作遙控船、爬行機械人及乒乓球發射器。此活動有幸獲得星島日報及東方日報報導，詳情可SCAN以下的QR CODE

- 本年度「非常學生之STEM小組」參加「獅子盃科學科技比賽」及「**第二十屆「常識百搭」小學STEM探究**」，並在「獅子盃科學科技比賽」中獲得兩個銅獎及「最巧妙設計獎」

門牌製作

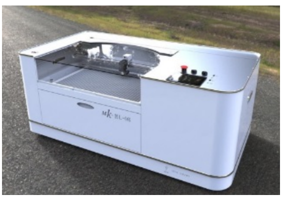
運動會紀念品

燈座

東方日報報導　　　　　　星島日報報導

展望未來

回顧過去的一年，本校在推行「STEM教育」上確實下了不少功夫，成績亦初見成效。在2017/18年度，本校STEM小組已為推行更全面的「STEM教育」做好準備。除了依照原定計劃進行不同的學習活動外，李炳學校亦會與坊間的資訊科技公司合作，於校內電腦室建立「STEM教室」，還會擺放以下設備，供本校老師及學生使用。希望在各項新設備的配合下，可以進行更多不同形式的跨學科學習，令創客文化、精神及技巧真正能融入教學中，從而令學習變得更有趣更有意義。



鐳射切割機

SLA 3D 鐳射打印機

CNC 電腦數控雕銑車床

4. Functions & Relations: Reflection in Secondary School Curriculum

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Introduction

This article is a reflection of classroom teaching and learning experiences of functions and relations in the senior secondary curriculum. Three aspects will be discussed: functions, variations and some elementary functions including exponential functions, logarithmic functions and trigonometric functions.

1. Functions

(a) Terminology

In functions, quantities are assigned with specific roles such as independent and dependent variables. Students learn this concept already in junior form Integrated Science. The main difference in Science and Mathematics is that the independent and dependent variables in the former may not be quantified, e.g. the brands of battery, the types of animals.

Once we represent a variable y as a function of x, say $y = 2x - 1$, it seems that there is no difference with formulae in junior form in terms of algebraic manipulations, say change of the subject of the formula. Nearly all examples of change of subject to a certain variable in junior form exercises can give a unique form. If the formula involves only two variables, we can say that this is a bijective function. Nevertheless, senior form students should understand that not all functions are bijective, or in simple terms, not all relation between two variables can be expressed with a single subject, say $x^2 y = xy^2 - 1$.

(b) Definition

A common definition of function in most of the senior secondary mathematics textbooks is as follows: "If each value of *x* determines exactly one value of *y*, then the dependent variable *y* is a function of the independent variable *x*."

Regarding the above definition, it is natural to ask for the meaning/significance of "one *x* gives one *y*", or "one input gives one output". We may consider the cases violating the definition, such as "one input gives two output", or "one *x* gives many *y*". Students may understand more about the definition when they learn inverse function at a later stage. It is not necessary to stress too much on this in the introduction. As a side track, students will learn "two inputs give one output" in the chapter "Linear Programming" later in senior secondary curriculum.

(c) Notation

Next, the notation of function also yields numerous concerns in learning. Introduced by Leonhard Euler (1707–1783), the notation $f(x)$ appears to be confusing to some students as if it is the product of *f* and *x*. Actually, when students represent a function graphically, they can express it as *y* in terms of *x* instead of using $f(x)$. It seems that the notation $f(x)$ is more useful in two other chapters instead of in the first chapter of functions.

In the topic "polynomials", the remainder theorem and the factor theorem and its converse are two important constituents. Let us recall the two theorems.

Remainder Theorem When a polynomial $f(x)$ is divided by $mx-n$, where m and n are constants and $m \neq 0$, then the remainder is equal to $f\left(\frac{n}{m}\right)$.

Factor Theorem & its Converse Let $f(x)$ be a polynomial, m and n are constants and $m \neq 0$. $mx-n$ is a factor of $f(x)$ if and only if $f\left(\frac{n}{m}\right)=0$.

Imagine that the two theorems are stated without the use of the notation $f(x)$. It turns out that the theorems look very clumsy.

Remainder Theorem When a polynomial whose variable is x divided by $mx-n$, where m and n are constants and $m \neq 0$, then the remainder is equal to the value of the polynomial when $x=\frac{n}{m}$.

Factor Theorem and its Converse It is given m and n are constants and $m \neq 0$. $mx-n$ is a factor of a polynomial whose variable is x if and only if the value of the polynomial when $x=\frac{n}{m}$ is equal to 0.

In addition, more conceptual questions about the two theorems can hardly be asked, as shown in the following examples of multiple-choice questions in the Hong Kong Certificate of Education Examination (HKCEE).

[HKCEE 2007 Q.40] Let $f(x)$ be a polynomial. If $f(x)$ is divisible by $x-1$, which of the following must be a factor of $f(2x+1)$? A. x B. $x-3$ C. $2x-1$ D. $2x+1$

[HKCEE 1994 Q.37]

$P(x)$ is a polynomial. When $P(x)$ is divided by $(5x-2)$, the remainder is R. If $P(x)$ is divided by $(2-5x)$, then the remainder is

A. R. B. $-R$. C. $\frac{2}{5}R$. D. $\frac{2}{5}$.

E. $-\frac{2}{5}$.

The function notation also demonstrates the beauty and symmetry of mathematics in the topic "transformation of graphs of functions". Let us recall the various types of transformation of graphs and the corresponding functional notation.

(In the following table, $f(x)$ is a function and k is a non-zero constant.)

	Translation	Reflection	Enlargement/ Reduction
Horizontal	$y=f(x+k)$	$y=f(-x)$	$y=f(kx)$
Vertical	$y=f(x)+k$	$y=-f(x)$	$y=kf(x)$

Consider the second row of the table. The transformations of graphs along the horizontal direction corresponds to the algebraic operation "in the bracket" of $f(x)$. An intuitive way of understanding the fact is that the variable x is “in the bracket”, and the x-axis is along the horizontal direction. Therefore, the transformations of graphs along the horizontal direction should be performed “in the bracket” algebraically. The same idea applies to transformation along the vertical direction. Such relationship between algebra and coordinate geometry is not easily set up without $f(x)$.

2. Variations

Another topic showing the relation between quantities is variation, which starts with direct & inverse variations in the curriculum. Indeed, the two variations are specific and simple examples of single-variable functions. They are natural extension of the topic “rate and ratio” in junior secondary Mathematics. In Mainland, they are studied in junior forms prior to functions in senior forms. So why do we not to introduce them earlier? Is it because joint and partial variations are not (necessarily) of single variable?

Next, we will make use of a multiple-choice question of variation to demonstrate how various topics in Mathematics are bridged together, and how it is related to mathematical modelling.

(a) <u>A case study of Multiple Choice Question</u>

[HKCEE 1995 Q.11]

x and y are 2 variables. The table below shows some values of x and their corresponding values of y. Which of the following may be a relation between x and y?

x	2	3	6	12
y	36	16	4	1

A. $x \propto \sqrt{y}$ B. $x \propto y$ C. $x \propto \dfrac{1}{\sqrt{y}}$

D. $x \propto \dfrac{1}{y}$ E. $x \propto \dfrac{1}{y^2}$

Students usually regard the topic "variation" as an "easy" one. The reasons are simple: the question types of variation in public examination seldom change, and the algebraic skills involved are not difficult to grasp. Yet, the multiple-choice question above appears to be hard to many students, as it is more demanding in terms of the understanding of the related concept.

The first question is "How to teach students to solve the problem?" By elimination, we know that choices A and B are impossible. It is because as the value of x increases, the value of y decreases according to the table. Thus, x cannot vary directly as y to a positive index. For the other choices, we have to go back to the essence of the topic: the variation constant.

Consider choice C. If $x \propto \frac{1}{\sqrt{y}}$, then we can let $x = \frac{k}{\sqrt{y}}$, where k is a non-zero constant. The description and the fundamental meaning of k is often overlooked. This is reflected in the marking scheme of public examination that no mark is deducted when omitting this part. However, it is indispensable in solving this problem, as we have $x\sqrt{y} = k$. That is, we can check the values of $x\sqrt{y}$ for each pair of x and y in the table and see if we obtain the same value. The same strategy is applied to choice D ($xy = \text{constant}$) and choice E ($xy^2 = \text{constant}$). The answer of this question is C, as $x\sqrt{y} = 12$ for all the 4 pairs of x and y.

<u>(b) Mathematical Modelling</u>

For a teacher who wants to extend students' mathematical thinking, the second question is "If the choices are not given, how can we find the functional relation between the 2 quantities x and y?"

There exist many different kinds of relation which can be satisfied by the values of x and y in the table. To start with, we may ASSUME a simple one: the "power relation" $x \propto y^n$, where n is a constant. Then we can let $x = ky^n$, where k is a non-zero constant. To find the values of k and n, we can substitute 2 pairs of values of x and y. After obtaining the relation with k and n known, we then substitute the other pairs of x and y to ensure that all the 4 pairs of x and y satisfy the same relation.

It is noteworthy about the assumption we made. If this question serves as a revision in Form 6, at that moment students have encountered other elementary functions as well. We may ask students to suggest other possible forms so that they can gain a better understanding of the features of various functions. For instance, as x increases and y decreases and $x, y > 0$, we may consider $x = ka^y$, where k and a are constants, $k > 0$ and $0 < a < 1$. Students can realize the importance of the restrictions on the functions.

Let us continue our discussion with the power relation. In reality, not all pairs of x and y can satisfy a single relation. There may be some outliers. Therefore, it is more desirable to plot a graph with the best-fit line. A natural choice of the graph is a straight line, and hence we consider logarithmic transformation as follows:

$$x = ky^n$$
$$\log x = \log k + \log y^n$$
$$\log x = \log k + n \log y$$

Plot the graph of $\log x$ vs $\log y$. The slope n is the index of relation and the vertical intercept $\log k$ is related to the variation constant. At this point, the usefulness of logarithmic transformation is demonstrated, and perhaps it helps to answer the question many students ask, "What is the USE of mathematics?" We can also see the interconnection between various parts of our Mathematics curriculum, which may seem to be formed by pieces of topics without linkage.

(c) Modelling beyond Mathematics: STEM?

Furthermore, the method of modelling in previous example is also applied in other science subjects. In HKDSE Chemistry, there is an Elective Part called "Industrial Chemistry". In this module, the topic "chemical kinetics" also relies on such kind of modelling strategy.

Consider a chemical reaction: $aA + bB \rightarrow$ products. It is ASSUMED that the rate of reaction depends on the concentrations of the reactants, i.e. $[A]$ and $[B]$ by the "power relation": rate $= k[A]^m[B]^n$, where k is the rate constant, and the indices m and n are called the orders of reaction with respect to reactants A and B respectively. Experimentally, we keep the concentration of, say B, constant, and vary the concentration of A. Thus, we have rate $= k'[A]^m$, which takes the same form as in the previous section.

For students taking Mathematics Extended Part, we may give enrichment of this law of mass action by considering the differential form $\frac{d[A]}{dt} = k'[A]^m$ and apply separation of variables for first order ordinary differential equation to obtain the integrated form for 2 cases: $m = 1$ and $m \neq 1$.

The determination of the order of reaction is not merely a mathematical model, as it may suggest the mechanism of a chemical reaction. The order is equal to the number of molecules/atoms/ions/free radicals involved in the rate-determining step (RDS) of the reaction. Two examples in the past Hong Kong A-Level Chemistry Curriculum are S_N1 and S_N2 reactions, namely nucleophilic substitution reaction of organic halogeno compounds.

In the era of STEM (Science, Technology, Engineering and Mathematics), many people tend to put the focus on the first three letters. The role of Mathematics is not obvious. Yet, scientific investigation and findings can hardly move one step further to prediction without the role of Mathematics as shown in the case of RDS. Another example is the subject Physics, which will take another article for the writer to further discuss.

3. Exponential, Logarithmic & Trigonometric Functions

(a) Domain, Co-domain, Range

Let us consider several sets involved in functions: domain, co-domain and range. They are introduced in the chapter "functions", in which only constant, linear and quadratic functions are discussed. In all these cases, the domain is the set of all real numbers. So what so special about domain? On the other hand, the term "co–domain" is stated in the syllabus. In secondary school level, while we consider functions (usually real-VALUED), it is a safe choice to choose the co-domain as the set of real numbers. Again, why bother mention this? In addition, the term "range" is not stressed in the syllabus. Yet in the section of "quadratic functions", students learn how to determine the optimum values of quadratic functions, and write the following: $y = (x-1)^2 + 3 \geq 3$. They are actually writing the range of function.

To further discuss the pedagogical roles of domain, co-domain and range, we investigate several elementary functions. Refer to the table below:

	Domain	Range
$y = a^x$	All real numbers	All positive numbers
$y = \log_a x$	All positive numbers	All real numbers
$y = \sin x^\circ$	All real numbers	Between –1 and 1, inclusive

	Domain	Range
$y = \cos x°$	All real numbers	Between –1 and 1, inclusive
$y = \tan x°$	All real numbers except $180n + 90$, where n is an integer.	All real numbers

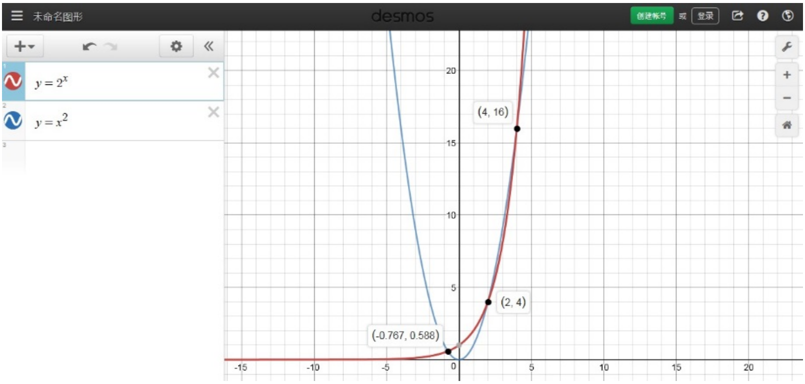
Students can realize that the set of all real numbers cannot be the domain of all functions, and the collection of all values of dependent variable, i.e. the range, may be a proper subset of $\mathbb{R}$ only. Also, note that in the first 2 rows, the domain and the range are interchanged. So the nature of "inverse function" is also established, though it is discussed in Mathematics Extended Part. Hence, in terms of teaching, it is suggested that the domain and the range should be further emphasized in the chapters of elementary functions after quadratic functions, and the term "co-domain" needs not be mentioned.

(b) Graphs of Elementary Functions and Graphical Method of Solving Equations

Many students are not interested in (some may be even afraid of) the graphs of various functions. This is rooted in the first encounter of graphs in junior form. When students learn graphical method of solving simultaneous linear equations in 2 unknowns, they are constantly reminded that the solutions read from the graph are inaccurate due to the precision of the keep and the fact that the point of intersection of graphs may not lie on the grid lines. However, the simultaneous equations, even of 1 linear and 1 quadratic in senior form, can be solved by algebraic method, and the answers obtained are in exact value. Students can never appreciate the graphical method in these circumstances and thus regard it as some add-on only.

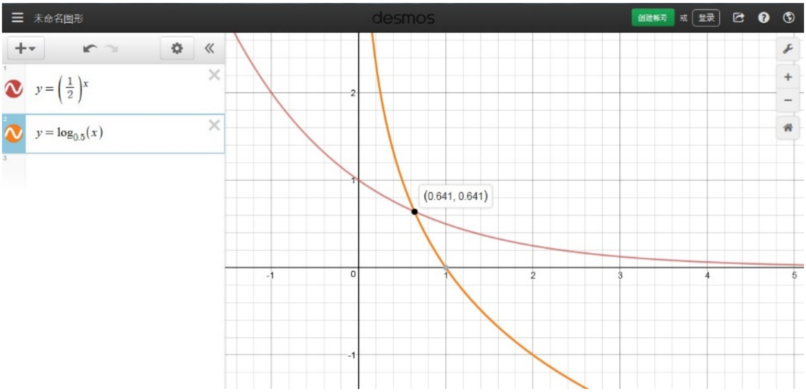
Indeed, we can convince our students the usefulness of graphical method in solving simple yet special equations like $2^x = x^2$. It is not difficult to see that $x = 2$ and $x = 4$ satisfy the equation. Yet, it is a much more demanding task to find other solutions, let alone to show that the number of solutions is 3. Let us plot the graphs of $y = 2^x$ and $y = x^2$, and find the points of intersection. With advanced information technology, there are many resources on the web that help students to plot graphs accurately. As seen from the graph below, a solution of $x = -0.767$ can be obtained. We can also explain that there are no more solutions by referring to the behaviour of the quadratic and exponential graphs for $x < 0$ and $x > 4$. Students thus can understand that the rate of increase of an exponential graph is larger than that of quadratic graph, and this is a starting point for students to investigate and to compare exponential graphs with polynomial graphs. In short, if we extend a single variable equation in *x* to graphs of functions with both *x* and *y*, we can obtain much more insights.

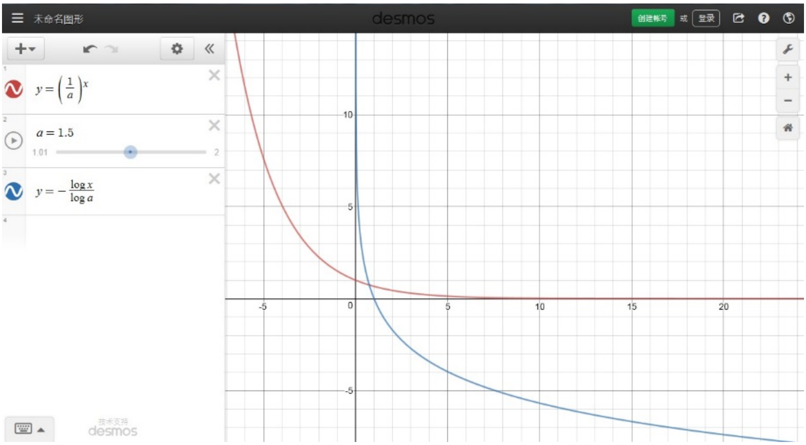
Concerning exponential and logarithmic graphs, students should know that the two graphs $y = a^x$ and $y = \log_a x$ (where $a > 0, a \neq 1$) are symmetrical about the straight line $y = x$. This can be related to the concepts of independent and dependent variables, and also the domain and the range.



Very often, when we compare the graphs of $y = a^x$ and $y = \log_a x$, the value of a is deliberately chosen to be in the range $a > 1$. In this case, there is no point of intersection. What if $0 < a < 1$, say $a = \frac{1}{2}$, how many solution(s) does the equation $\left(\frac{1}{2}\right)^x = \log_{\frac{1}{2}} x$ have? What is/are the solution(s)?

How about the general case, $\left(\frac{1}{a}\right)^x = \log_{\frac{1}{a}} x$, where a is a constant and $a > 1$? We start from studying some particular cases to derive a more general pattern, and here the graphs of functions play a vital role.



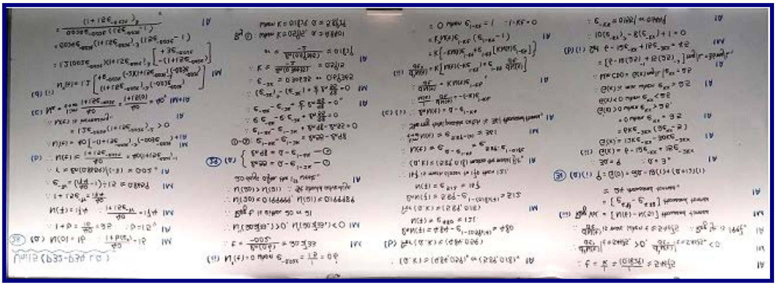


5. Inspiring Experience from Self–Preparation of Teaching Handouts

IP Ka–fai Gavin
Queen Elizabeth School

As a proactive mathematics teacher in this generation, I am always enthusiastic about reviewing the latest pedagogy and innovative ideas for educational development in Hong Kong. Since I began my teaching profession at the turn of this century, I have been devoting aggressively to edit my own teaching handouts for all forms to supplement the teaching, and my student feedback and learning outcome were mostly inspiring.

I would edit my own set of colored teaching handouts without using the textbooks. Each handout consists of theoretical part introducing and explaining the theorems and formulas concerned, followed by tailor–made examples of different objectives and formats with detailed problem solving analysis before abundant exercises of varied difficulties are assigned for student practices in lessons and after school. The effectiveness is even better in senior forms due to their higher maturity on guided, self–directed learning and their more willingness in doing pre–lesson assignments before thorough discussions of step–by–step solutions in lessons.



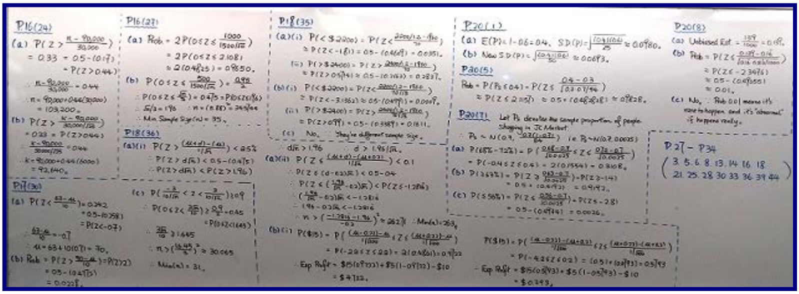


Figure 1 and 2 – Two completed whiteboards of my Mathematics Module 1 lessons.
All students have to do pre–lesson work before class attendance every time.

If you have the habit to prepare own teaching materials for students, you may share a similar feeling that the biggest problem is unlimited time consumption. I would like to share my own experience in preparing the handouts. I need to spare most of my long holidays and vacations to think, to draft, to calculate, to edit and to prepare the written solutions for each handout. The drive, determination and effort exerted are beyond description and I was often asked by my family, colleagues or alumni why I kept on doing so.

Who does not want to enjoy a long vacation truly without schoolwork? In my opinion, however, self–preparation of teaching handouts has the following advantages to facilitate our teaching that outweighs the costs of additional time and efforts in preparation. These advantages are what motivate me to prepare my own instructional handouts for over 10 years' time.

1. Cater for Learner Diversity

I believe most teachers would agree with the increasing diversity of students when our school banding was reduced from five to three in 2001. This makes the learner difference increase even the majority of schools have grouped students with similar ability level into the same class. As professionals, we commit to achieving equity for all students and believe they are capable of making a difference in learning. To help our students perceive their tasks as meaningful, the best solution is that we prepare and provide the tailor–made learning materials for them.

We should be aware of different abilities and learning needs of students to give corresponding assistance for their full development of multifaceted potential. Using our professional knowledge, tailor–made learning handouts can best suit the characteristics of your students by offering more user–friendly and appropriate materials. The majority of students would also appreciate the extra efforts and care the teachers show to them by preparing extra handout, which is beneficial to cultivating a good teacher–student relationship.

In my personal experience, it can definitely increase the student learning motivation and enthusiasm in attempting mathematics problems in the handouts. As the materials cater for their abilities, their job satisfaction will be higher which results in increased self–confidence accordingly. Under such favorable conditions, their academic results can be most likely guaranteed.

Some teachers have asked me if major amendment is needed every year to cater for students of different forms. It is not necessary except some minor changes of wording in the explanatory part and the level of difficulty of exercises offered in adjustment, if you are teaching in the same school. My experience is that over 90 % of the content could be kept, so it is always the first step being the hardest. As long as you persevere, you are bound to succeed in modifying the content slightly and continuously to suit students of different abilities.

2. Textbook Re–Edition of Publishers

The Education Bureau has launched a new policy since summer 2009, the same year as the start of our New Secondary School Curriculum (NSSC), in which the publishers were allowed to revise their editions only 5 years after the original editions were launched. Its introduction has a good intention to the public that the financial burden of parents could be reduced due to the price reduction of corresponding textbooks.

Nevertheless, the largest drawback would be the ineffectiveness of matching up the textbooks' written contents with the examination trend of HKDSE. The curriculums had slight changes in 2013 and 2015 for either Compulsory Part or Extended Modules, while the diversity of questions of HKDSE is clearly broader than the former HKCEE. My impression is that during the first several years of NSSC, the publishers include all potential contents in their textbooks, in which the depth of treatment are over the curriculum requirement.

Only 3 out of 6 publishers, who had launched S.4 – S.6 Compulsory Part textbooks, revised their editions in 2014. Moreover, it was a great shock that for Extended Modules 1 and 2, the numbers of new editions were even 1 and 2 respectively. The lack of investment and updates would render the textbooks obsolete for the public examination requirement.

Many authors are not frontline teachers and they may be unfamiliar with current teaching environment.

3. Limitations of Textbooks on Markets

The number for revised textbook editions is insufficient. Another drawback to me is that, the editorial style is too similar to HKCEE era that it has not catered effectively for the changing objectives of the new Mathematics curriculum and the public examination.

In my observation, the quality of questions shown in some textbooks is yet to be improved. For example, the exercises offered in some textbooks are very suitable for some lower–achievers according to their level of difficulty and the varieties; however, the higher–achievers would find it boring to tackle many similar–type problems that are easier than the public examination questions. When the textbooks are suitable for certain type of students, it is problematic that they cannot cater for the individual needs of other students of the same form.

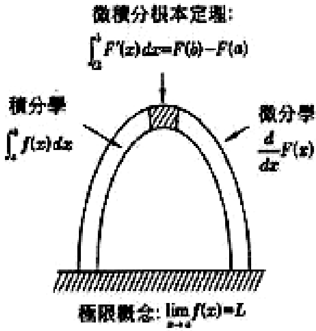
Experimental documentation and knowledge inheritance

I am always delighted to have a commendable opportunity to explore, to compare and to evaluate different methods that can be used effectively. The frontline teaching experience has also inspired my eagerness to analyse further on implementation of effective strategies that can optimize students' interest and learning outcome in Mathematics.

Unlimited innovative ideas had been generated in my mind at my different teaching moments, e.g., the introduction of specific mathematical concepts, the explanatory choices, the way of selection of examples and exercises, the highlight of problem solving strategies and skills, etc. I need to record them systematically and utilise them in my latter part of the teaching career. Preparation of own teaching handouts facilitates such experimental documentation and ensures the knowledge could be efficiently continued.

I want to emphasise that as frontline teachers, we are the ones who understand the students the most. We should strive to adopt pedagogies through which students learn best and most comfortably. Self–prepared teaching materials suit and cater for students' needs the most.

After introducing the advantages of preparing teaching handouts for students, I would like to share my personal experience in writing my latest handouts of Extended Module 1 as an illustration, using Differentiation and Integration, the most difficult component of the curriculum, for some in–depth discussion and brainstorming.



It is my honor and my pleasure to share and to discuss with all teaching colleagues for further development and improvement. Should you have any new idea, opinion or suggestions, please do not hesitate to reach me using the given electronic mail account.

Teaching the Theoretical Part

I used to search different textbooks before deciding my own approach in teaching the theory. We can also explain using some Pithy formulas, which rarely appeared in textbooks, to help students memorising theories with more fun and ease. The following is a part of my handout teaching Chain Rule (鏈式法則). Let us see if it helps our students.

Chain Rule (鏈式法則): If $y = g(u)$ and $u = h(x)$ are two differentiable functions, then the derivative of the composite function $y = g[h(x)]$ is given by $\frac{d}{dx}g\left[h\left(x\right)\right]=\frac{d}{du}g(u)\times\frac{d}{dx}h(x)$.

In general,

$$\frac{d}{dx}\left(k_1x^{n_1}+k_2x^{n_2}+\cdots\right)^n=n\left(k_1x^{n_1}+k_2x^{n_2}+\cdots\right)^{n-1}\left(k_1n_1x^{n_1-1}+k_2n_2x^{n_2-1}+\cdots\right)$$

Many students commit mistakes by forgetting differentiations of certain layers (inner functions). It is necessary to distinguish the outer and inner functions correctly in using Chain Rule.

Its pithy formula (口訣) is: 由外至內，逐層剝殼，再由最外層 D 到入最內層。

We may easily memorise the Chain Rule as Derivative of a Composite function = (Derivative of Outer Function) × (Derivative of Inner Function).

If you use the theoretical format of Chain Rule in explanation, some students may find it hard to understand and they often forget to differentiate the composite functions layer by layer for the right answer. To consolidate their memory and to maximize their chance of obtaining the right derivatives, I try to explain it using colloquial language with the help of my self–designed pity formula, and I found my students have really worked better in differentiating those advanced functions.

4. Introduction of Miscellaneous Problem Solving Strategies in the Examples

a) We have **Product and Quotient Rules** in which the students would use according to their preference, but our textbooks would usually present one solution format only, which has limited the students' choices and exposure. The following example, which is quoted from my handout, highlights three different problem–solving methods with detailed explanation in lessons so that the students can discuss and decide which method is most feasible for them.

Example: **Differentiate** $y=\left(\frac{x+1}{3x-1}\right)^4$ **with respect to *x*.**

(Method 1 – Use Chain Rule and Quotient Rule in Order)

$$\frac{dy}{dx}=4\left(\frac{x+1}{3x-1}\right)^3\frac{d}{dx}\left(\frac{x+1}{3x-1}\right)$$

(Chain Rule)

$$=4\left(\frac{x+1}{3x-1}\right)^3\frac{(3x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(3x-1)}{(3x-1)^2}$$

(Quotient Rule)

$$=\frac{4(x+1)^3}{(3x-1)^3}\cdot\frac{(3x-1)(1)-(x+1)(3)}{(3x-1)^2}$$

$$=\frac{4(x+1)^3}{(3x-1)^3}\cdot\frac{-4}{(3x-1)^2}=\frac{-16(x+1)^3}{(3x-1)^5}$$

(Method 2 – Use Quotient Rule and Chain Rule in Order)

$$y=\left(\frac{x+1}{3x-1}\right)^4=\frac{(x+1)^4}{(3x-1)^4}$$

$$\frac{dy}{dx}=\frac{(3x-1)^4\frac{d}{dx}(x+1)^4-(x+1)^4\frac{d}{dx}(3x-1)^4}{\left[(3x-1)^4\right]^2}$$

(Quotient Rule)

$$=\frac{(3x-1)^4\frac{d(x+1)^4}{d(x+1)}\cdot\frac{d(x+1)}{dx}-(x+1)^4\frac{d(3x-1)^4}{d(3x-1)}\cdot\frac{d(3x-1)}{dx}}{(3x-1)^8}$$

(Chain Rule)

$$=\frac{(3x-1)^4\left[4(x+1)^3(1)\right]-(x+1)^4\left[4(3x-1)^3(3)\right]}{(3x-1)^8}$$

$$=\frac{4(3x-1)^3(x+1)^3\left[(3x-1)-3(x+1)\right]}{(3x-1)^8}$$

(Taking Out Common Factors)

$$=\frac{4(x+x)^3(-4)}{(3x-1)^{8-3}}=\frac{-16(x+1)^3}{(3x-1)^5}$$

(Method 3 – Use Product Rule and Chain Rule in Order)

$$y=(x+1)^4(3x-1)^{-4}$$

(From Quotient to Product Format)

$$\frac{dy}{dx} = (x+1)^4 \frac{d}{dx}(3x-1)^{-4} + (3x-1)^{-4} \frac{d}{dx}(x+1)^4$$

(Product Rule)

$$\frac{dy}{dx} = (x+1)^4 \left[-4(3x-1)^{-5}(3)\right] + (3x-1)^{-4}\left[4(x+1)^3\right]$$

(Chain Rule)

$$= 4(3x-1)^{-5}(x+1)^3\left[-3(x+1)+(3x-1)\right]$$

(Taking Out Common Factors)

$$= 4(3x-1)^{-5}(x+1)^3(-4) \quad = \frac{-16(x+1)^3}{(3x-1)^5}$$

I prepared this example after exploring to different works of my students these years. For Method 1, we have the remaining inner function being (x + 1)/ (3x – 1) after applying Chain Rule in the first step. It is thus simple for further differentiation using Quotient Rule.

For Method 2, however, using Quotient Rule firstly would result in the degree of denominator being 8 and the numerator would contain complex terms of degree 4. Further differentiation is thus difficult and careless mistakes are more easily appeared. I guide the students to aware of their differences and agree that Method 1 is more preferable from its easier simplification compared to Method 2.

Method 3 provides an alternative by rewriting y as negative index and applying Product Rule for differentiation instead. This caters for students who are weak in using Quotient Rule but are eligible in using Product Rule, and I really have such type of students before!

It is always encouraging if the students can use their own approach to solve problems correctly. By the same time, we may widen their horizons and provide alternatives for their problem solving, so the students can compare and choose the best method themselves, and they can have more tools in their mind to tackle the problems.

b) For advanced learners, they can be capable to skip the **Substitution Method** at the early stage on **Indefinite Integrals**, while some others cannot abandon with the complexity of integrands. I used to introduce both methods at the very early stage for their choice and they would decide themselves when they are confident enough in integration without using the tedious Substitution Method. Let us see the example, which I quoted from my teaching handout below.

Example: Find the following indefinite integrals:

(a) $\int 3x^2 e^{x^3} dx$

(Method 1)	
Let $u = e^{x^3}$.	$\therefore \int 3x^2 e^{x^3} dx = \int du$
$\therefore \dfrac{du}{dx} = 3x^2 e^{x^3}$	$= u + C$
	$= e^{x^3} + C$
(i.e. $du = 3x^2 e^{x^3} dx$)	

(Method 2)

$$\int 3x^2 e^{x^3} dx = \int e^{x^3} dx^3 \ = e^{x^3} + C$$

(b) $\displaystyle\int \frac{(\ln x)^2}{x} dx$

(Method 1)	
Let $u = \ln x$	$\therefore \displaystyle\int \frac{(\ln x)^2}{x} dx$
$\therefore \dfrac{du}{dx} = \dfrac{1}{x}$	$= \displaystyle\int (\ln x)^2 \cdot \frac{dx}{x}$
(i.e. $du = \dfrac{1}{x} dx$)	$= \displaystyle\int u^2 du$
	$= \dfrac{u^3}{3} + C$
	$= \dfrac{1}{3}(\ln x)^3 + C$

(Method 2)

$$\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 d(\ln x) = \frac{1}{3}(\ln x)^3 + C$$

Note: Unless the question requires us using a specified substitution or the derivative of u with respect to x, we can opt out from showing clearly how x is changed to u.

Method 2 is catered for students who have developed the sense for integration in a faster way, without writing in details the conversion between two variables using method of substitution.

I like to address some comments after the example for comparing the pros and cons of two methods, and my students like to read and to deliberate which method of problem–solving suits them best. The note after each example is particularly helpful for those self–learners who read the handout before lessons and drill after school, which provides clearer explanation for them to understand before lesson attendance.

Effectiveness and Reflections of using Self–Prepared Teaching Handouts on Teaching

Due to limited space of this publication, I could not quote too much to elaborate the techniques in writing a well–organised handout and introducing numerous mathematical concepts for students using our innovative perspectives. It is never easy to edit a handout originating from your own style. My own experience tells me that the first time is harshest which starts from none. We can modify the existing handouts since then with lesser effort and you would find it comfortable doing so.

If you have the determination working out your first edition, you would be encouraged from the job satisfaction in preparation stage and the student appreciation in usage for revised editions in future. Most importantly, the handouts have summarized what you need to teach and what you want to say that facilitates your knowledge inheritance in and out of classroom.

My students, in which I use my own teaching handouts throughout the senior levels, obtained very remarkable results in 2017 HKDSE. Use Module 1 as an example for illustration, 66.7 % of my students achieved Level 5 or more while all 100 % students achieved Level 4 or more! This is such an amazing result compared to other local Band 1 schools and our own results of previous years, showing the effectiveness of self–preparation of teaching handouts to arouse students' interest in learning and to improve their academic performance. The similar good results apply on my other classes throughout my teaching career.

My upcoming target will be the incorporation of STEM elements into our normal teaching. Being a successful candidate of the Staff Interflow Scheme 2017, the six–month attachment in Mathematics Education Section, Curriculum Development Institute was not limited to experience sharing and knowledge transfer. The program fills me in on the latest educational policies and development thoroughly, and it offers me an occasion to reconsider seriously what directions I should take as an experienced teacher going forward after devotion for over a decade.

I was so honored to engage comprehensively in the “Seed” Project about “Exploration and Development of Effective Strategies for Promoting and Implementing STEM Education in Secondary Mathematics”. My main responsibility was to initiate and to design tailor–made STEM exemplars, with Mathematics as the key learning element in both junior and senior levels.

After playing an active role as the “designer” of many exemplars, I am confident to say that Mathematics can also play a major role in STEM education. I am enthusiastic to add the STEM concept into my self–prepared teaching handouts in future for widening students’ horizons and making the lessons more fruitful.

6. 數學自主學習：仿效 = 成效？

張建輝
東華三院邱金元中學

1. 有效自主學習模式的推行

當教育界開始回應自主學習之初，不少同工都會設計導學案，甚至到國內觀課，希望找到他山之石。有同工參觀示範課之餘，在另一些課室見到傳統講授之課堂，對自主學習之成效產生疑問。亦有學者指出國內模式不能全面模仿，而香港大多數導學案仍停留在填充式、答題式的工作紙。同時又指出，自主學習可以成功推行而有更佳效果，關鍵在於「課堂外的準備如何配合課內積極學習」。筆者相信上述學者的說法，跟從某種學習模式推行或會徒勞無功。有效策略（包括老師認知層面和具體推行層面），整個設計與學習重點之扣連，會為學習成果帶來出路。（註 1）本校遂於兩年前參加由教育局推行關於有效自主學習的種籽計劃。

2. 針對重點，用心設計

在兩年的種籽計劃中，本校考慮課題與生活應用、學生對課題的初步了解及掌握、學習過程的簡化，遂選取幾個初中數學單元，包括百分增減、折扣、指數、統計圖的誤用、概率簡介，統計簡介等課題。期望透過課堂活動、短片預習與反思，讓學生對該單元有初步概覽及概念，了解與日常生活的關係，鞏固所學。

3. 學習成效

學生於自主學習的課堂上，專注及參與明顯比一般課堂為佳。從本年度中三級及中二級問卷調查中，分別有 82.4% 及 62.7% 的同學表示喜歡課堂有課堂活動及短片至少一項，另兩級皆有 41.2% 表示喜歡課堂同時有短片及課堂活動。

此外，約 50% 的同學在學習過程中，會思想原因。約 20% 的同學在老師教學或家課遇困難,會思考原因及解決辦法。因此，學生學習的通達相當重要，自主的思考習慣，有待加強。

調查亦發現，分別有 52.9% 及 39.2% 的中三及中二同學，希望學習過程中有老師指導。希望獨立學習的，分別只有 26.5% 及 13.7%。此結果顯示,「杜朗口」的自主學習模式（註 2），未必完全滿足香港學生的學習需要。

在下學期考試，進行自主學習的單元，亦相對其他單元，有較佳的學習表現。

4. 要解決的問題

在推行之初，要先面對一些負面情意、一些課時加增的憂慮，並推行辦法之無助。是「摸着石頭過河」？抑或「抱着石頭投江」？

作為科主任，上述兩項皆不是。在安排自主學習活動上，中三級的概率簡介，兩課節的自主學習分組活動連預習短片和反思，已讓學生掌握理論概率和實驗概率的分別與兩者的關係，也令他們懂得製作「樹形圖」以解決簡單概率問題。中一級讓學生分組進行問卷調查，從選題，設計問卷，搜集、分析與表達數據，配合適當短片和預習反思工作紙，「一條龍式」的活動（其實是一個實驗），讓學生清楚一個簡單的統計程序，能力較佳的學生更能指出他組誤用統計圖表。整個過程，三至四堂，有減無增。

談到令不少參與計劃的同工困擾的短片製作，自行拍攝確實花了不少時間，但掌握後可以快一點了。況且，如教育電視及 YouTube，均有很多短片，我們從中選取與學習活動及重點扣連的段落，給學生預習觀看，還可重覆翻看至明白。

5. 展望將來

科主任只要帶領團隊踏出第一步，共同探索有效方法，讓老師多一種教學武器，提升學生學習動機與效能，多少能消除同工對過去成效不彰的新措施。是次種籽計劃推動的自主學習，多校參與、分享及觀摩，按部就班去推動，已有不錯的成效。只要適當及適量推行，而非一刀切過於頻密使用而令學生生厭，這一件好事將會得以持續發展與優化。

參考資料

註 1：趙志成(2014)。香港推行自主學習的探索。香港中文大學《教育學報》，2014 年，第 42 卷第 2 期，頁 143-153。

註 2：山東杜朗口中學自主學習的特點：

「三特點」：立體式、大容量、快節奏。

「三模塊」：預習、展示、反饋。

「六環節」：預習交流、明確目標、分組合作、展現提升、穿插鞏固、達標測評。

7. 打破常規：從一道數學面試題談起

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網絡媒體的流行，訊息傳播來得也特別的快。筆者不久前便收到一個有關數學問題的資訊，因為涉及的是微軟公司的面試題，在網絡上也當牽起一點討論熱潮。這裡以它為引子談談數學的教與學。原問題是這樣的：微軟公司的面試官出了一道幾何題（如下圖 1）：一個直角三角形的斜邊長為 10 cm，斜邊上的高為 6 cm，面試者需要算出這個直角三角 ABC 的面積。

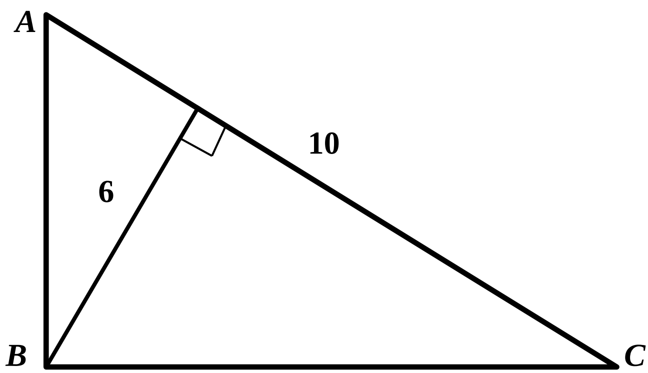


圖 1.面試題（來自 http://www.dailymail.co.uk/femail/article-3594527/Is-trickiest-intervie）

新聞報到不少面試者依據圖形數據，滿懷信心地回答 30 cm^2（即底×高÷2＝10×6÷2＝30）時，卻被指答錯了，最後不獲聘用。主考官最後揭曉答案：「考題的三角形根本不可

能存在，因為高度最多只可以是斜邊的一半才能成三角形，以此題為例，高度最多只可能是 5cm，絕對不會是 6cm」。

聽到答案的面試失敗者，恐怕會有些後悔，怎麼能在一道小學問題上犯了「常識性的錯誤」。你可以當作是則笑話來看這個面試故事，作為數學教師，我們不妨多想一層，對於我們的教學還是有不少啟示的。如此的常規想法（見到數字就進行運算），會不會在我們的課堂也很常見呢？我們的學生是否也會「不小心」犯這樣的錯誤呢？事實上，有學者曾在小學做過類似的實驗，問學生一道文字題的答案：「一條船上有 75 頭牛，34 頭羊，問船長幾歲？」不少學生給出的答案是 75-34 = 41 歲！在答案荒謬的背後，或許我們也要為學生的思維定勢感到擔憂。會否是我們的教學太著重計算而忽略了對問題背景的思考？會否我們的學生太熟練機械的計算，而缺少對問題的判斷？

另外，為何主考官揭示問題的答案指出三角形的高最多是 5cm？這就需要用到數學知識來回答了。我們也可以從代數的角度來思考這個問題。比如，在圖 2 中，若直角三角形 ABC 中，$BC = a, AC = b, AB = c, BD = h, AD = x$, b 和 h 需要滿足什麼條件，才可以計算三角形的面積呢？

由 $\Delta ABD \sim \Delta BCD$（AAA），我們有 $\dfrac{BD}{CD} = \dfrac{AD}{BD}$

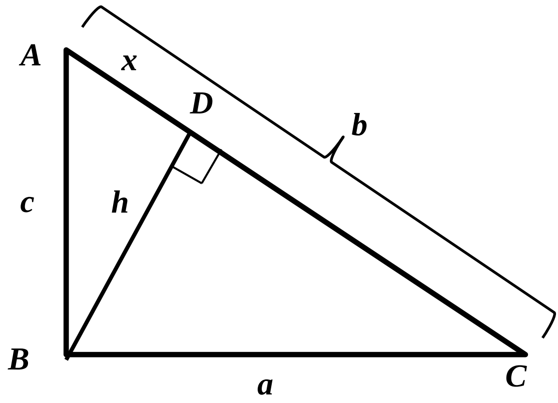


圖 2. 一般直角三角形 ABC

即 $\dfrac{h}{b-x}=\dfrac{x}{h}$，亦即 $x^2-bx+h^2=0$ 。

此方程若要有實數根，

則 $\Delta=b^2-4h^2\geq 0$

即 $(b+2h)(b-2h)\geq 0$

顯然，容易知道 $b+2h\geq 0$

所以，$b-2h\geq 0$ ，即 $h\leq\dfrac{1}{2}b$

因此，高 BD 的長度最大是斜邊長度的一半。

我們也可以從面試題目本身提供的數據來說明這樣的面積是算不出來，即這樣的直角三角形是不存在的。為簡單表示，不妨令 $AD=x$, $DC=y$, 則 $x+y=10$。

利用相似三角形的判定，不難得到 $\Delta ABD\sim\Delta BCD$（AAA）

因而有 $\dfrac{BD}{CD}=\dfrac{AD}{BD}$ ，即 $BD^2=AD\times CD$

將 $AD=x$, $DC=y$ 和 $BD=6$ 代入, 則有 $xy=36$。

綜合可得：$\begin{cases} x+y=10 \\ xy=36 \end{cases}$。

x,y 可以看成是一元二次方程 $t^2-10t+36=0$ 的兩根。容易知道，此方程是沒有實數根的（判別式 $\Delta<0$）。

那麼，能不能直觀一些弄明白問題的原因呢？平面幾何的知識告訴我們，任何一個直角三角形均內接於一個圓，圓的直徑即為直角三角形的斜邊，直角頂點在圓周上（如圖 3）。若求面試問題中斜邊 AC 上的高(h)的最大值，即為半圓周上的點到直徑的最長距離為半徑的長。不難得出，當 B 位於 AC 垂直平分線和圓周相交點時（即圖 3 中的 B_3），B_3O 即為所求距離，長度為直徑的一半。

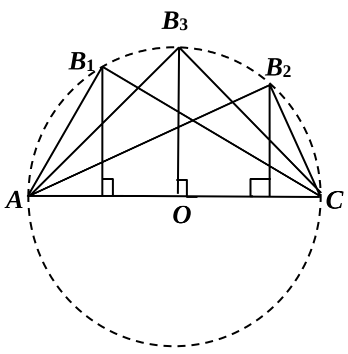


圖 3. 直角三角形 ABC 高的最大值幾何解釋

對這個解釋，我們可以更一般化地來理解。我們假設直角三角形斜邊(即 AC)的長度是定長,該斜邊上的高(即 BD)最長為斜邊的一半。

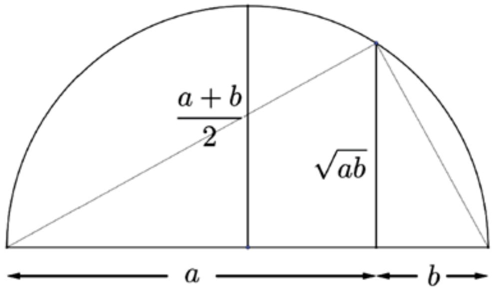
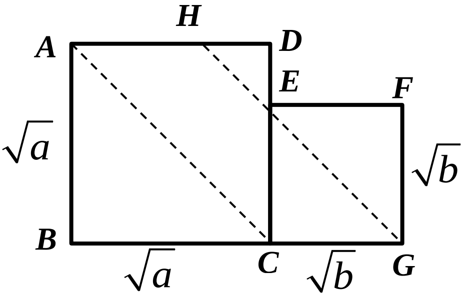


圖 4. 半圓中直角三角形 ABC 的斜邊上的高

要證明不等式 $\frac{a+b}{2} \geq \sqrt{ab}$ 並不難，比如利用 $(a-b)^2 \geq 0$ 的性質，可得 $(a+b)^2 \geq 4ab$（a，b 為非負實數），不等式兩邊開方即可得證（有興趣的讀者可自行完成）。

台灣有數學教師受到無字證明（proof without words）的啟發，透過畫圖，也說明上述不等式的結論(周伯欣，2016)。例如，我們考慮兩正方形, 面積分別為 a 與b, 不失一般性, 假定 $a \geq b$ 。因此，這兩個正方形的邊長分別為 $\sqrt{a}$ 和 $\sqrt{b}$ ，而且平行四邊形$ACGH$的面積為 $\sqrt{ab}$ 。



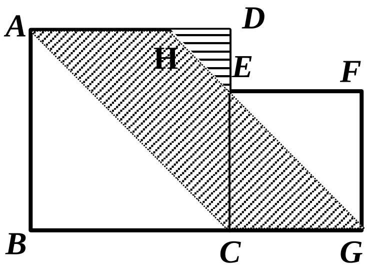


圖 5. 不等式$\frac{a+b}{2} \geq \sqrt{ab}$的無字證明

從圖中不難觀察得出 $\frac{a+b}{2} \geq \sqrt{ab}$ ，其中取等號的條件是 a=b，即兩個正方形一樣大小，直角三角形 HDE 的面積變為 0。

不等式$\frac{a+b}{2} \geq \sqrt{ab}$是一個有趣的結論，它是均值不等式中的一種算幾不等式(Arithmetic-Geometric Mean Inequality) $\frac{a_1+a_2+\cdots a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$ （當且僅當 $a_1 = a_2 = \cdots = a_n$ 時等號成立）在 n = 2 時的簡單版。它甚至跟小學的數學文字題有關聯：

（1） 用長為 20cm 的鐵絲圍成一個長方形，怎樣才能使得圍成的面積最大？

（2） 長為 100m 的籬笆圍成一個長方形的養雞場，問這個養雞場面積最大是多少?

問題（1）中的數字簡單，可以透過嘗試法找到答案。問題（2）就需要知道一般的規律了：兩個數的和一定時，差越小乘積越大；反之，乘積一定時，差越小，和越小。這規律背後的原因就是上面的表達式 $\frac{a+b}{2} \geq \sqrt{ab}$ 。不過，在香港最新的數學課程修訂意見稿中，「探究已知周界的圖形的最大面積」這一部分內容被建議刪除了（課程發展議會，2017）。

透過一道簡單的面試題，我們希望帶出其中蘊含著的數學知識和數學的學與教，哪怕一些課題在正規的學校課程中沒有出現，作為專業的數學教師，還是要能具備充足的數學知識和數學教學知識（pedagogical content knowledge），特別地，能夠瞭解學生對問題的可能的思維定勢和誤解，理解一些數學規律背後的原因，這樣在應對學生可能的發問時，也能做到遊刃有餘。

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2. 課程發展議會（2017）。《初中數學修訂課程內容（諮詢稿）與初中數學課程（1999）內容的比較》。2017 年 9 月 1 日取自 http://www.edb.gov.hk/attachment/tc/curriculum-development/kla/ma/CT_JS_tc.pdf

8. 何時才能注滿一池水？

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小時候，老師曾提出以下一道問題：

"有一水池，若將進水管打開，6 小時可以注滿；若將排水管打開，12 小時可以排空相同滿水的池。現若將進、排水管俱打開，需要多少時間才能將同一水池注滿？"

記得當時老師教我用算術方法求解：
設池水總容量為1，則注滿一池水所需的時間是：

$$\frac{1}{\frac{1}{6}-\frac{1}{12}}=12\text{（小時）}\cdots\cdots(1)$$

若用代數方法求解：

可設水池容積為V，單獨由進水管注水滿一池水，所需時間為t_1；單獨由排水管排完一池水所需的時間為t_2。當進、排水管都打開時，注滿一池水的所需時間T 的普遍式可表達為：

$$T=\frac{V}{\frac{V}{t_1}-\frac{V}{t_2}}=\frac{t_1t_2}{t_2-t_1}\cdots\cdots(2)$$

將t_1=6小時、t_2=12小時代入式(2)，即可得出式 (1) 的結果。

表面上看來，問題似乎已經得到了解決，但事情卻沒有這麼簡單，這是因為式 (1)、(2) 成立的先決條件是：

進水管的進水速度與排水管的排水速度必須是常量；換句話說，在進、排水過程中流量須保持不變，現實上，這是不可能的。

由於要能將水池內的水排完，排水管必須放在水池底部，排水速度是隨池內水位升高而增加、水位降低而減小，即排水速度是一個〔變量〕。因而式 (1)、(2)的解答是有商榷的餘地，最多只能算是極其粗略或近似的估計罷了。故此，這類數學題，用小學算術或初中代數根本是解不出來的。

以下，我們試試以另一角度來考慮此題的解答。

甲．單獨進水時間的計算：

設水池橫截面積為A，高度為H，且進水管放置在水池的頂部，則單獨由進水管灌滿池水所需的時間t_1為：

$$t_1 = \frac{AH}{Q_1} \cdots\cdots(3)$$

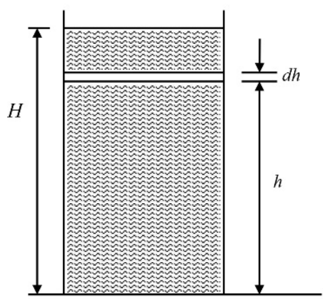

圖 1

式(3)中，Q_1 —進水管流量（米／小時）

乙・單獨排水時間的計算：

排水管放置在水池底部，且橫截面積為a。設在時刻t,水池高度為h；在時刻t，水池水位高度為$h+dh$；在dt時間內池水容積的增量應等于同時間內排水管排出之池水容積,故有:

$$\mu \cdot a\sqrt{2gh} \cdot dt = -A \cdot dh$$

$$\int_0^{t_2} dt = \frac{A}{\mu a\sqrt{2g}} \int_H^0 \frac{dh}{\sqrt{h}}$$

$$t_2 = \frac{A}{\mu a}\sqrt{\frac{2H}{g}} \cdots\cdots (4)$$

式 (4) 中，

t_2 — 單獨排完池水所需的時間

μ — 流量係數

g — 重力加速度（$g = 9.8$米／秒^2）

丙・進、排水管同時打開，灌水池所需時間的計算：

同理，如圖1，在時刻t，池水高度為h；在時刻$t+dt$，池水高度為$h + dh$，則有：

$$\left(Q_1 - \mu \cdot a\sqrt{2gh}\right)dt = Adh$$

$$\int_0^T dt = A\int_0^H \frac{dh}{Q_1 - \mu a\sqrt{2gh}}$$

$$T = \frac{t_2^2}{2t_1}\ln\left(\frac{Q_1}{Q_1 - \mu a\sqrt{2gh}}\right) - \frac{A}{\mu a}\sqrt{\frac{2H}{g}} \cdots\cdots (5)$$

將 $Q_1 = \frac{AH}{t_1}$ 、 $\mu = \frac{A}{at_2}\sqrt{\frac{2H}{g}}$ 代入式 (5) 並化簡，得：

$$T = \frac{t_2^2}{2t_1}\ln\left(\frac{t_2}{t_2 - 2t_1}\right) - t_2 \cdots\cdots (6)$$

式 (6) 即為進、排水管都打開時灌滿一池水所需的時間 T 的一般表達式。

若將 $t_1 = 6$ 小時、$t_2 = 12$ 小時代入式 (6)，則 $T = \infty$ ，然而，這結果與式(1)、(2)的解答相去甚遠。這說明在這種條件下灌滿水池需要極其漫長的時間。

為了說明代數解答式(2) 的近似程度，可採用如下的辦法：

將式 (2) 兩端除以 t_1，得：

$$\frac{T}{t_1} = \frac{\frac{t_2}{t_1}}{\frac{t_2}{t_1} - 1} \cdots\cdots (7)$$

以 $\frac{T}{t_1}$ 為縱坐標、 $\frac{t_2}{t_1}$ 為橫坐標作圖，即可得出 $\frac{T}{t_1} \sim \frac{t_2}{t_1}$ 的關係曲線。如圖2中曲線 I 所示，這是一條以 $\frac{T}{t_1} = 1$ 和 $\frac{t_2}{t_1} = 1$ 所表示的兩直線為漸近線的等邊雙曲線。

將式 (6) 兩端除以 t_1，得：

$$\frac{T}{t_1}=\frac{1}{2}\left(\frac{t_2}{t_1}\right)^2\ln\left(\frac{\frac{t_2}{t_1}}{\frac{t_2}{t_1}-2}\right)-\frac{t_2}{t_1}\cdots\cdots(8)$$

式(8)表示的曲線如圖2 曲線 II，這是一條以 $\frac{T}{t_1}=1$ 和 $\frac{t_2}{t_1}=2$ 兩直線為漸近線的超越曲線。

比較曲線 I 、II 可知，進排水管都打開灌滿一池水所需時間 T 的精確值較近似解要大。

在近似解答來看，似乎只要 $\frac{t_2}{t_1}>1$（即單獨排水時間大於單獨灌水時間），就可以在有限時間內灌滿水池。而實際上，從式(8)可知，只有當 $\frac{t_2}{t_1}>2$ 時，才有可能灌滿一池水。

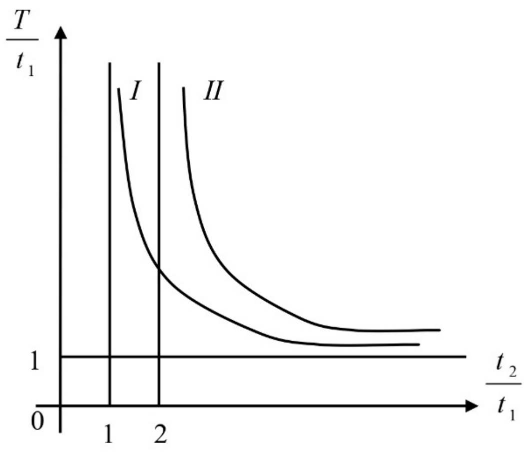

圖2

從圖2 還可看出：

近似解式 (7) 的誤差是隨比值$\frac{t_2}{t_1}$的增大而減小的。只有當$\frac{t_2}{t_1} \gg 1$（例如 $\frac{t_2}{t_1} > 8.3$）時，解答 (7) 才能有現實意義（此時相對誤差 < 5%）。否則，就只能用式(8) 來計算結果。

9. Julia Robinson Mathematics Festival in Hong Kong and Mathematics Circle Learning Technologies

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Abstract

The Julia Robinson Mathematics Festival

(http://www.jrmf.org) is one of the events in which students have the opportunity to develop their talent in mathematics by providing a wide range of problems, puzzles, and activities that are intriguing and accessible in a non-competitive atmosphere. The inaugural Julia Robinson Mathematics Festival in Hong Kong was recently held on 1st April 2017 at the Singapore International School (Hong Kong), during which a diverse audience of students, educators and people passionate about mathematics gathered with the goal of broadening the society's appreciation and support of mathematics. Personalized learning technologies that facilitate Math Circle Learning through Mathematics gamification was also introduced. The Julia Robinson Mathematics Festival in Hong Kong opens up new pedagogical ways to teach and learn advanced mathematics.

Introduction

The inaugural Julia Robinson Mathematics Festival in Hong Kong (www.algebragamification.com/jrmf) was held at the Singapore International School on 1st April 2017 in partnership with the U.S. management of the Julia Robinson Mathematics Festival and the American Institute of Mathematics [1]. The Director and Co-Director of the Festival organization are Dr. TAN Chee-wai and Mr. Jian SHEN respectively. Dr. Mark SAUL, who is the Executive Director of the Julia Robinson Mathematics Festival, Professor Tony CHAN of HKUST and Mr. Bernard NG of the Singapore International School (Hong Kong), support the festival. The organizers of the Julia Robinson Mathematics Festival in Hong Kong aspire to make mathematics accessible to every student of all abilities. The goal is to encourage students to focus on collaborative problem solving, as opposed to the competitive nature commonly founded in mathematics examinations and contests. In this way, students can enjoy the richness and beauty of mathematics without any anxiety.

Over the years, The Julia Robinson Mathematics Festivals (http://www.jrmf.org) has allowed young people to develop their talent in mathematics by providing problems, puzzles, and activities that are intriguing and accessible in a non-competitive atmosphere. Founded by Nancy Blachman, the name of the festival honors the memory of Julia Robinson, a mathematician recognized for her work in solving Hilbert's tenth problem, and the festivals seek to encourage more students to pursue mathematics [2,4]. The JRMFs began at Google in the San Francisco Bay Area in 2007, and have since expanded into many other cities around the world. The year 2017 marks the tenth anniversary of JRMF, and besides the JRMF in Hong Kong, JRMFs also took place in Taiwan and Mainland China in 2017.

At the JRMF in Hong Kong, the participants were a diverse audience of 243 students, aged 10 to 14, all of whom were passionate about mathematics and gathered for two hours to explore the joys and power of mathematics. The goal is to broaden the society's appreciation and support of mathematics. At the Festival, there were twenty tables, each with its own unique fun and challenging mathematical theme (such as mathematical origami, games and puzzles), and each table was staffed by a facilitator. Participants could roam around from table to table and choose to work at a table that they felt interested. The spirit of the Festival was that there was no pressure for the student to finish any task. The facilitators were faculty staff members and students from local universities, school teachers and practicing mathematicians in the industry. As students worked on mathematical problem sets, they were rewarded with a raffle ticket based on their persistence and collaborative learning attitude with their peers. Thirty raffle prizes such as Festival T-shirts, math games and books were given out at the end of the festival.

For the first time, mobile app software, that we have developed (described below) was used at the JRMF for students to develop a stronger intuition to the mathematical problems through observation and experimentation. Even after the Festival had concluded, the mobile app software could allow students (their parents and teachers) to relive the experience of the festival, and thus further encouraged collaborative problem solving among peers.

Mathematical Problem Set Design

The JRMF mathematical problem sets are carefully designed in a way such that they are initially easy and become progressively challenging. In this way, students can gain confidence at the beginning and also develop the right intuition to tackle the more challenging mathematical aspects at the latter part of each problem set. From a pedagogical viewpoint, the harder problems, even when unsolved, can in fact pique their curiosity. Hence, the intention is for the students to walk away with some sense of accomplishment and to have a new understanding of a mathematical problem, rather than to give superficial answers to the problem sets with poor understanding.

To get a feel of the difficulty level of the mathematical problem sets used in the Festival, one of the mathematical problem sets that involve notions of probability theory is shown below. We call it "A Number Game of Randomness":

I draw two cards from 13 poker cards, say, 1,2,…, KING=13, that are completely unknown to you, and hold one in my left hand and one in my right. You have absolutely no idea what these two numbers are. Which is larger?

You can point to one of my hands, and I will show you the number in it. Then you can decide to either select the number you have seen or switch to the number you have not seen, held in the other hand, as your final choice.

1. Is there a strategy that will give you a greater than 50 *percent chance of choosing the larger number, no matter which two cards I draw?*

2. Suppose I have infinite cards, labeled from 1, 2, 3, 4,….

Again, I randomly draw two cards from these cards, and I will show you the number of one of the two cards. Is there a strategy that will give you a greater than 50 percent chance of choosing the larger number?

This Number Game of Randomness is a deep mathematical problem with roots in information theory and probability theory. We observed that students who did not have any notion of probability theory could still logically deduce nontrivial insights to problems that looked deceivingly simple and straightforward. All other interesting mathematics tables had equally novel titles to pique the curiosity and interests of the students. Figures 1 shows the math tables for “Color Triangle Challenge” and “Modular Origami” respectively.

Figure 1: Math tables for “Color Triangle Challenge” and “Modular Origami” on the left and right respectively. The capacity of each table is ten students.

Below are two actual problem sets used at the Festival: the Tower of Hanoi and the Exploding Dots puzzle, also used at the 2017 Global Math Project. We devise the latter puzzle as one that connects with a past International Math Olympiad

(IMO) question towards the end – again highlighting our style of progressively difficulty levels in our problem sets.

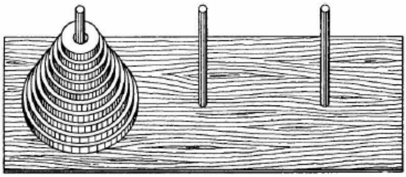
Julia Robinson Mathematics Festival

American Institute of Mathematics

JRMF at the Singapore International School (Hong Kong), 1 April 2017

The Tower of Hanoi - and Beyond

Legend has it that in Hanoi there is a tower of 64 disks of different sizes, initially all stacked on peg A as shown, and a group of monks working tirelessly to move the disks from peg A to peg B. Only one disk at a time may be moved, and at all times only a smaller disk may ever be on top of a larger (never a larger on top of a smaller). When the monks complete their task, the legend says the world will end.



1. Perhaps this is too easy, but how many moves will it take to complete the moving of the tower if there is only one disk?
2. How many moves will it take if there are two disks?
3. How many moves will it take for three disks?
4. How long will it take for four disks? Generalize. How many moves will it take for *n* disks?
5. If the monks never make a mistake, and can move one disk every second, 24 hours per day, how many years will it take for them to complete their tower of 64 disks ?

Julia Robinson Mathematics Festival

American Institute of Mathematics

JRMF at the Singapore International School (Hong Kong), 1 April 2017

Exploding Dots

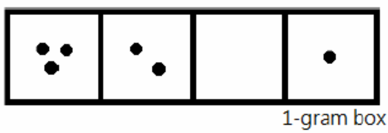
A special machine called the 2->1 machine consists of a row of boxes that stores dots. All the dots initially put into the machine weighs 1 gram, and are put into the rightmost box in the machine. Two types of operations are allowed:

Type 1: Remove a dot in a box, and put two dots in the box immediately to its right.
Type 2: Remove two dots in a box, and put one dot in the box immediately to its left.
Note, however, that the total weight of the entire machine does not change.
For example, the two machines below contain different numbers of dots, but the dots both weigh 3 grams.

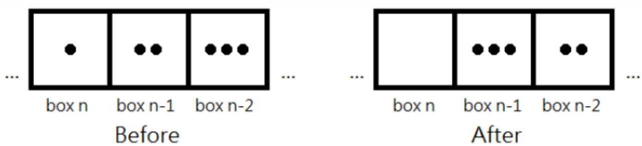
1. The dot in the rightmost box of this Exploding Dots Machine weighs 1 gram. Answer the following questions. Feel free to simulate this using the model on your table and count the total amount of grams using the right-most box only. Then, figure out the weight of each dot in different boxes.

a) How heavy are all the dots combined in this machine?

b) How heavy is one dot in each of the boxes below, counting from right to left?

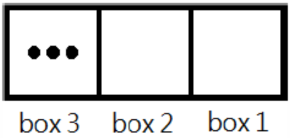


A new type of operation, called *swapping*, is added:
Type 3: With three boxes, remove a dot in *box n*, and exchange all dots in *box n-1* and *box n-2*.
For example:



There can be any number of dots in the second and third boxes, even zero.

4. Can swapping bring more dots to the machine? Only using the first two operations, the machine below, with 3 dots, can only form a maximum of 12 dots. Can you get 16 dots, using the swapping method as well?

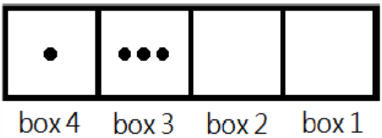


5. Imagine that *box n* in the machine above had 4 dots, instead of three dots. What is the maximum number of dots you can form then? What if *box n* had 5 dots? Show the pattern/relationship. Feel free to draw on paper if there are too many dots for the table.

Exploding Dots: Challenge Questions

In generalization, for the machine (with only 3 boxes) above, x dots in *box 3* will form $2x$ dots in box 2, and ultimately $2*2x$ dots in box 1. If you have not already reached this conclusion, do not attempt the following questions.

6. **[Challenge]** Imagine that another box with one dot, *box 4*, is added to the left of the machine. What is the maximum number of dots you can form with the machine below? What if *box 4* had 2 dots, 3 dots, or x dots?



7. **[Challenge]** Remove all dots from *box 4* except for one. Add a final box, *box 5*, to the left of the machine, and put 1 dot in there. What is the maximum number of dots you can form with this machine? Feel free to explore beyond these questions, and try to obtain different large numbers using the three operations allowed.

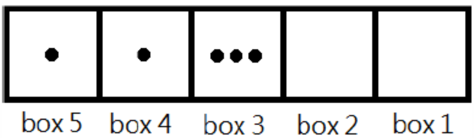


Figure 2: “Tower of Hanoi” Table with both the mechanical manipulative and the digitalization of the logical aspects of the puzzle in a mobile app software.

At some of the mathematics tables, we experimented with the use of technologies for learning mathematics. Certain logical aspects of mathematical problem sets were digitalized and transformed into a mesmerizing game (i.e., we could view “toy examples” in mathematics as literally a “toy” that could be manipulated or played with). This would enhance the appeal of the mathematical problem sets, when presented to the students, rather than the abstract elements. Certainly, abstract thinking as required in advanced mathematics often starts with curious observations and simple experimentation. In conjunction with the Festival preparation, various mathematics gamification (digitalization of the logical mathematical elements and turning them into small games at the entry level of a problem set) were incorporated into the mobile app software (described in the next section). In Figure 2, we see a young participant at the mathematics table for "Tower of Hanoi" experimenting with both a mechanical puzzle manipulative and digitalization of this puzzle incorporated into our mobile app software.

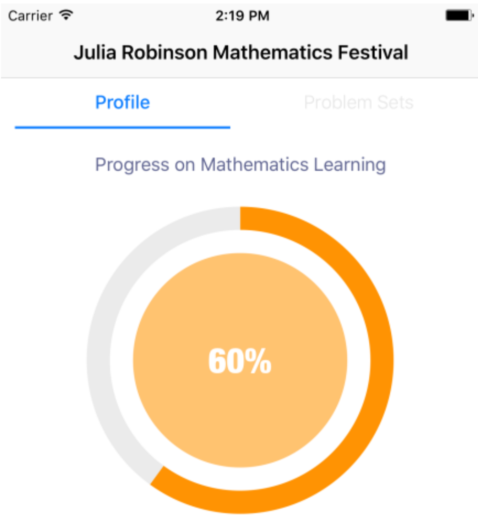
Mobile App Learning Software

The mobile app software JRMF App, used for the first time at the JRMF, allows students to develop a stronger intuition to the mathematical problems through observation and experimentation. The mobile app software is available for download at major mobile app stores, e.g. the iOS App Store and the Google Play Store [3]. A dedicated website for the mobile app software is available at: http://www.algebragamification.com/jrmf/app.

Our personalized learning technologies are based on mathematics gamification, which is defined as the process of embedding mathematical concepts and manipulations within puzzle-like instantiations. These software instantiations of mathematics gamifications can be delivered in several possibilities. We have chosen to deliver them by mobile app software that presents potentially strong opportunities for students to learn advanced mathematics in a systematic manner and usable even long after the festival concludes.

The benefit of this mobile app learning software for personalization learning is clear: through experimentation and curious observations, students acquire useful insights into the mathematical subject at hand that otherwise will not be obvious or found in traditional classrooms. The engaging game-like nature of mathematics gamification motivates students to peer-help one another as well as allows classroom teachers to use them as instructional tools for students to gain proficiency in mathematics at their own pace. As illustrative example, Figure 3, shows the opening interface that reports progress and navigates the user to some of the problem sets whose mathematical nature has been gamified. Figure 4 shows the "Tower of Hanoi" game whereby a user starts playing from an easy level of a single disc and then progresses through levels with an increasing numbers of discs. It also offers students to learn about the deep mathematics behind new mathematical games such as the Algebra Game developed in the Algebra

Game Project founded by the author [5] (going into the realm of number theory that otherwise is not apparent to a casual player).



Julia Robinson
Mathematics Festival

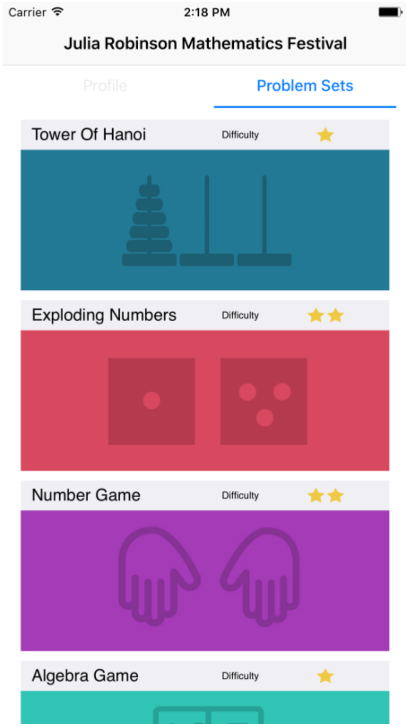


Figure 3: Opening interface of the JRMF App is freely available for download at iTunes App Store and Google Play Store.

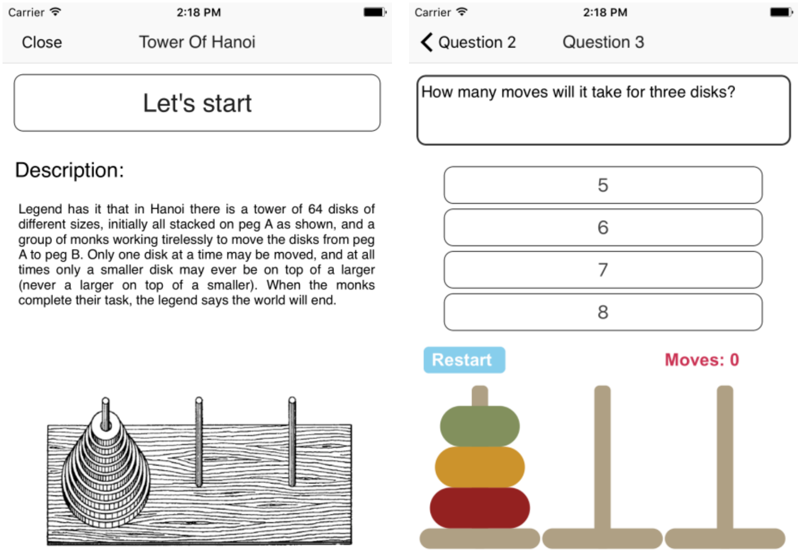


Figure 4: Tower of Hanoi problem set in the JRMF App: Problem background introduction and game-quiz format.

Conclusion

The Julia Robinson Mathematics Festival in Hong Kong has successfully reached out to many students and math enthusiasts not just in Hong Kong but also regionally (e.g., the U.S., Mainland China and Taiwan). We expect more of these global collaborative efforts to promote mathematics learning in intriguing and non-competitive ways. We showed the efficacy of mobile app software in Math Circle. We believe that personalized learning technologies can be created and tested for this purpose. At a meeting in August at the 2017 Mathematics of Various Entertaining Subjects (MOVES) Conference [5] in New York City with the JRMF founder, Ms. Nancy Blachman, (see Figure 5), we have discussed prospects of further collaboration to promote learning advanced mathematics in Hong Kong, in the Greater China area and beyond.

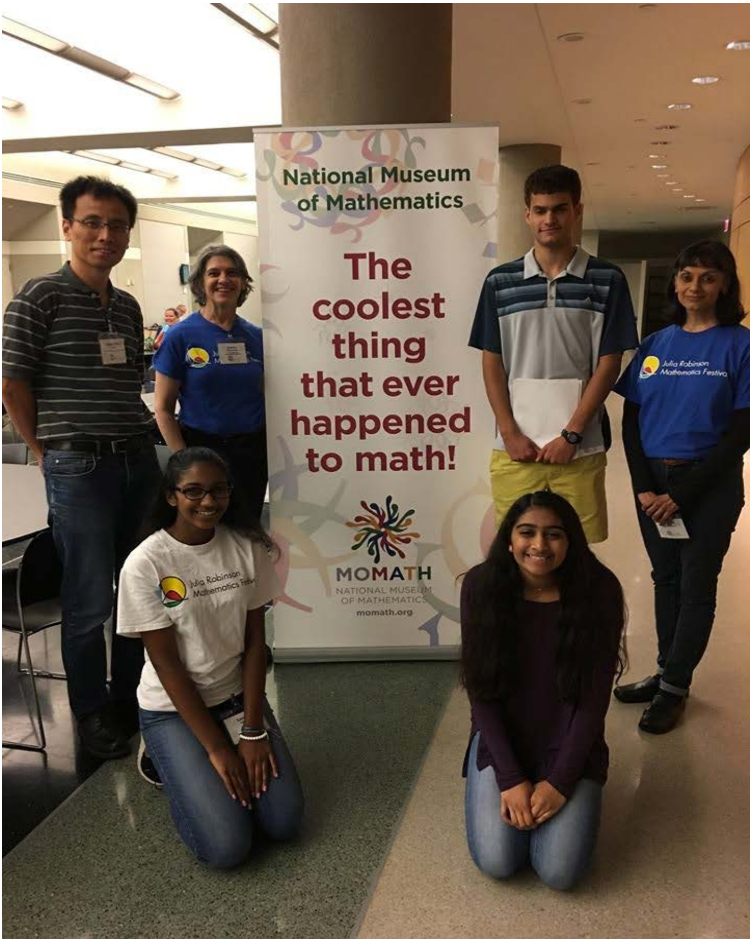


Figure 5: Meeting between Julia Robinson Mathematics Festival Founder Nancy Blachman and Prof. Tan at the 2017 Mathematics of Various Entertaining Subjects (MOVES) conference in August 2017.

References

[1] The Julia Robinson Mathematics Festival: http://www.jrmf.org

[2] The Julia Robinson Mathematics Festival (Hong Kong): http://www.AlgebraGamification.com/jrmf

[3] The JRMF App Webpage: www.algebragamification.com/jrmf/app. For immediate download of iOS version and Android version respectively, go to the website at: https://itunes.apple.com/us/app/jrmf/id1218671354 and the website at: https://play.google.com/store/apps/details?id=com.algebragamification.jrmfapp

[4] Nancy Blachman, History of The Julia Robinson Mathematics Festival, Mathematical Sciences Research Institute: https://www.msri.org/web/msri/pages/229

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10. MAGIC SQUARES AND ALGEBRA

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My readers are likely to have seen the classic topic of the title of this article. After all, it's thousands of years old. From the ancient Chinese Lo Shu to the well-known engraving of Albrecht Durer, magic squares have appeared in mathematical folklore, and even more general folklore, for ages.

In the classroom, magic squares are usually seen as brain teasers, or venues for practicing addition. In this article we will show how they can actually be used to introduce, and even develop, concepts from higher mathematics that will serve students well from elementary school through graduate education.

A magic square is a square filled with numbers such that the sum of each row, each column, and the two diagonals is always the same. We will limit ourselves here to 3×3 magic squares.

The first thing I do with magic squares is to have students construct their own. Now, there are well known rules for constructing magic squares. (The "knight's move" rule is perhaps the best known.) My goal—unbeknownst to the students—is not to find an efficient and general way to construct a magic square. Rather, I want to follow a path that will afford students the most discovery on their own.

Let us not be coy. Most readers will have seen the most common magic square:

2	9	4
7	5	3
6	1	8

We will have students construct this magic square themselves, and learn some valuable lessons on the way.

Activity I: The Rows

I give students cards, or slips of paper, with the numbers {1,2,3,4,5,6,7,8,9} written on them, and ask them to form three rows of three cards whose sums are all the same. Just the rows—we will get to the other conditions later. This is a simple task for even very young students. If they are not sure how to go about it, we can scaffold them. Just deal out three rows of three at random, and ask them which row has the largest sum, and which the smallest—which rows are ‘fattest’ and which are ‘thinnest’. Then show them that they can make the sums closer by ‘trading’ a high card in a fat row for a low card in a thin row. After a series of such trades, most students are able to achieve three rows with equal sums. Note that this construction is not quite algorithmic. Nothing is telling the student just which cards to trade. They have to use their intuition and try various combinations until they get it right. The goal is not to develop an algorithm.

There is a lesson to be learned from the three equal rows, by asking the students what the common sum of the three rows is. Most are not surprised that the answer is always 15. Somehow, 15 seems the ‘right’ number—not too big and not too small—to be common to the three rows. This intuition can be sharpened into a proof: the sum of all nine numbers is 45, so if they are put into three groups (rows or otherwise) with equal sums, that sum must be 15.

Not every student will think of this argument, but starting at about 10 years old, most will follow it. Moreover, we can find out, later in the discussion, if they have followed.

Activity II: The Columns

Having constructed three rows with equal sums, the next task is to make the column sums equal, while keeping the row sums equal. Students see, or guess, that the column sums must also be 15 (and some can restate the argument for this). Nevertheless, how to achieve it without disturbing the rows?

This is a bit tricky, and I usually end up giving students a hint: continue to ‘trade cards around’, but trade them only within a single row. That way the row sums are not changed, but we can even out the column sums.

There are some pitfalls in this step—the most ‘dangerous’ in our construction. Sometimes a student will get one column-sum to be 15, but cannot get the other two to ‘behave’. The problem is that the student’s 15-column is the wrong one. Holding it constant will prevent the other columns from falling in line.

Here is something we should know as teachers, but we cannot expect the students to know. Of the numbers {1,2,3,4,5,6,7,8,9}, there are only 8 subsets of 3 elements that add up to 15. This can be proven, for example, by a direct count[1]. In fact, these eight subsets are just the rows, columns, and diagonals of the magic square we want to construct. So if a student has a column that ‘should be’ a diagonal, it will block the other columns from falling into line. Such a ‘bad column’ (whose sum is 15) is not hard to recognize. It is a column with 5 in it, and two even numbers (we will prove this with the students later). Therefore, the pattern E5E (in any order) is the Signature of Death. When a student has a ‘bad column’, it is best for the teacher to intervene to get him or her unstuck.

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[1] Counting these subsets is a good exercise for advanced students. One nice argument is to observe that every subset must contain one number from the set {1,2,3,4} (or else the sum will be too large), and one number from the set {6,7,8,9} (or else the sum will be too small). We can make a 4x4 ‘table’ giving the third number in the subset, then cross out duplicates (as well as subsets with two ‘copies’ of the same number).

There is a concept to be learned here: the concept of *invariance*. We want a transformation (a ‘shuffling of cards’) which leaves the row sums invariant. It is usually worth introducing this term to students at this point.

Activity III. The Diagonals

With or without the teacher’s help, most students, even as young as 10 years old, can get the column sums to be the same. Their square is now ‘semi-magic’. To get a fully magic square, we need to get the two diagonals to add up to 15 as well. Students usually have trouble with this step.

We can give them a hint, and reinforce the notion of invariant, by showing them another transformation; one that leaves row and column sums the same. This transformation consists in ‘transporting’ a whole row from the top to the bottom of the array, or a whole column from the left to the right.

Here is an example of this transformation, which students generally do not invent for themselves.

$$\begin{array}{ccc} 3 & 7 & 5 \\ 8 & 6 & 1 \\ 4 & 2 & 9 \end{array} \quad \rightarrow \quad \begin{array}{ccc} 4 & 2 & 9 \\ 3 & 7 & 5 \\ 8 & 6 & 1 \end{array} \quad \rightarrow \quad \begin{array}{ccc} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{array}$$

An additional hint that some students might need is the question, “What number do you think should be in the middle?” Most students will guess that 5 must be in the middle. The reason for that can be the subject of discussion—we will return to it later on. However, for now a good guess is enough.

Having guessed that 5 must be in the middle, it is not hard to see that at most two moves of the type described will suffice to knock the configuration into 'magic' shape. Each move takes the 5 one place horizontally or vertically, and the 5 cannot be more than two such steps from the middle position. Students will even find that this hint guides them to see which moves are necessary.

With these hints, getting the diagonals to sum to 15, which might seem the most difficult part, is in fact the easiest.

Activity IV: Symmetries of a square.
Students will have different-looking magic squares. As they finish assignment III, I have at least three of them put their results on the board for all to see. I try to get, among the three chosen solutions, two that differ by a line reflection, and two that differ by a rotation.

I ask: How are these solutions the same, and how are they different? There are many answers to this question, and the discussion takes different turns, depending on what students see. We outline below one possible path, trying to touch on everything that may now come out. Nevertheless, in a classroom, the order will certainly be different.

The goal of the discussion is to move the class from looking at individual magic squares to the relationships among them: to the symmetries of the square that transform one solution into another.

One thing students notice quickly is that the same numbers appear in the same patterns in each solution. That is, the solutions all have the same rows or columns, just in different places. With this remark, I circle one row or column and ask how its position differs in the different solutions. Students quickly notice (if they have not already) that the other rows and columns 'move with' the one circled.

What emerges is that the entire square is rotated or reflected in a line to go from one solution to another. Students usually see the reflections ('flips') first. We can then elicit that there are four of them: horizontal, vertical, and two diagonal flips.

The diagonal flips are seen last, but it is not particularly difficult for students to see them. It is difficult, however, to name them. (We will need symbols for the various symmetries very soon, after students are made aware of them as object to manipulate mentally.) In addition, here things depend on local circumstance. When I teach in the US, I name the upper-left-to-lower-right reflection "Seattle to Miami" (or S2M), and the upper-right-to-lower-left reflection as "New York to Los Angeles" (NY2LA). A look at the map of the US will show

how this reminds students which diagonal is meant. Nevertheless, of course this must be adapted to local maps. Note that we are assuming that the map shows north on top, which is true of virtually all maps the students use.

The rotations are easier to talk about, but harder to visualize. Students may have to see all four of them (including the identity rotation, by zero degrees) specifically enacted on one particular magic square.

How do we get students to perceive each symmetry as an object in itself? They must play with it in their minds. An easy way is to ask repeatedly: How do we get from Square A to Square B? Since the solutions 'belong' to different students, it is not hard to get them interested in how their square can be transformed into their neighbor's square. With any luck, most of the transformations appear among the students' solutions. However, we can elicit the others if there is at least one rotation and at least one reflection.

A more advanced question is: starting with one solution, how many more solutions can you get? Students quickly focus on the shape of the square, using the 'magic numbers' simply as markers or labels for the corners. That is the focus they will need for the next set of insights.

A 'symmetry' of a figure is an isometry (rigid motion of the plane) which leaves the figure fixed. However, students may not need this formal definition. Even without a complete definition of symmetry or isometry, they will usually be able to see that a square has eight symmetries: four rotations (including rotation by zero degrees, which will take some explanation to the class) and four line reflections. Moreover, in future, when they are ready for a formal definition, they will be able to look back at this example and draw from it the intuitions they will need.

The eight symmetries of a square can be ‘applied’ to any one magic square to produce all eight variants. For some classes, I put these all up on the board. For others, I merely have them imagine the eight positions.

Note that this does not answer the question of how many magic squares there are whose entries are the numbers 1 through 9. It merely tells us how many there are derived from any one magic square. We will deal with the full question a bit later. Again, the order of exposition can easily vary from class to class.

Activity V: The Group Structure of Symmetries of a Square
The next step in getting students to focus on the symmetries is to talk about combining symmetries. This begins to develop the group structure of the symmetries of a square. How far this is taken in class depends on the goals of the instruction and the background of the students.

I start by taking three squares, for example:

2 9 4	6 7 2	6 1 8
7 5 3	1 5 9	7 5 3
6 1 8	8 3 4	2 9 4
(A)	(B)	(C)

I ask: How do you get from square (A) to square (B)? Students will see that it is a rotation of 90 degrees clockwise. (I notate this as r_{90}.)

Then: How do you get from square (B) to square (C)? Student will see that it is a reflection along the diagonal Seattle to Miami (f_{s2m}).

At some point, it is useful to note that we can easily tell a reflection from a rotation: A reflection leaves one of the rows, columns, or diagonals in place. A rotation does not. In this case, the diagonal 6-5-4 stays in place, so it is a reflection about that line.

Then the crucial question: How could we get directly from square (A) to square (C)? Students will see that the row 7-5-1 stays in place, so we get from (A) to (C) by a horizontal flip (f_h). We can write this as $r_{90} * f_{s2m} = f_h$. Writing the fact this way is an important step. When students can comprehend these symbols, they have made the transition from thinking of the effect of a symmetry on a particular square to thinking of symmetries as objects in themselves, which have properties and can be operated with. Here, the operation '*' means 'composition of functions'. However, I usually read it as 'followed by'. It may be important to have the students read it themselves this way (rather than pronouncing it 'star').

Students can then practice composing different symmetries. They will quickly discover the following facts:

a) Rotations compose easily: you just add the angles of rotation, and subtract 360 if necessary.

b) The symmetry r_0 is special. You compose it with any other symmetry, in any order (see below) and you just get the other symmetry. Therefore, it is like the number 0 for addition, or 1 for multiplication. It is an *identity* element.

c) Any reflection, composed with itself, is the identity.

d) Any reflection, composed with another reflection, ends up a rotation.

e) The order in which you compose symmetries makes a difference.

These facts build towards an even greater abstraction, the group structure of the symmetries of a square. This group (usually called D_4) is a non-commutative group of order 8.

Here are more advanced notes about these ‘facts’:

a) The rotations form a subgroup of the group of symmetries of a square. This subgroup has four elements, and is isomorphic to the additive group of integers modulo 4. Note that fact (c) states that the set consisting of any reflection and the identity also forms a (tiny) subgroup, of two elements.

b) Every group must have an identity element. Since this group is not commutative, we must ensure that the identity works from the left and from the right. This is easy to verify in the present case.

c) We say that a line reflection is its own *inverse*. Every element of a group must have an inverse, an element that takes us back to the identity. Students who are ready for this concept will be able to see what the inverses are for each of the rotations, as well as for the reflections.

d) This statement is a general, and little-known, property of line reflections. The composition of reflections in two intersecting lines is always a rotation. The angle of rotation is double the angle formed by the two lines. Some able students can be challenged to fill in the exceptional case when the lines are parallel. The composition of reflection in two parallel lines is a translation.

e) This note is particularly important, and worth eliciting from students even if they are not ready for the other advanced comments above. The operation of composition in this group is not *commutative*.

It is useful to make a further note about commutativity, by giving an example. For instance, take the example given above:

2	9	4		6	7	2		6	1	8
7	5	3		1	5	9		7	5	3
6	1	8		8	3	4		2	9	4
	(A)				(B)				(C)	

The example shows that $r_{90} * f_{s2m} = f_h$. Composing in the opposite order, students get:

2	9	4		2	7	6		4	9	2
7	5	3		9	5	1		3	5	7
6	1	8		4	3	8		8	1	6
	(A)				(B′)				(C′)	

This shows that $f_{s2m} * r_{90} = f_v$, which is not the same. The order of composition makes a difference. Students have not often seen a reasonable non-commutative operation. Certainly, subtraction and division are not commutative. However, the results of performing the operation in the 'wrong' order are predictable: it is easy to relate the result of one order to the result of the other. In the case of composition of symmetries, the change in order can give an unexpected result.

Further exploration will show that reflections do not commute with each other: changing the order of composition makes the resulting rotation go in the opposite direction. In addition, reflections do not commute with rotations: changing the order of composition has an effect that is difficult to predict.

All these notes build toward topics in group theory that generalize readily. For elementary or middle school students, it is enough to comprehend these few special cases of much more general results.

Activity VI: Uniqueness of the Magic Square

The title of this section is not quite correct. The magic square with entries 1 through 9 is unique up to symmetries of the square.

Nevertheless, we have not shown this yet. We have shown only that there are 8 squares derived from any one by symmetry. The following development shows that these solutions are all there are. I always make these observations, but I do not always prove them formally, (as we will do below) and I do not always note that they constitute a proof of uniqueness.

(A) The ‘magic sum’ must be 15.

Students will often construct a proof of this fact, but they may not be able to express it accurately. They get the idea that 16 is ‘too big’ and 14 is ‘too small’ as they experiment to get the rows to be the same—the very first activity in this sequence.

With a little scaffolding, they can be led to the argument that the sum of all the numbers from 1 to 9 is 45, so if they are separated into three sets whose sum is the same, each sum must be 45/3 = 15.

(B) The number 5 must be in the center.

Students will guess that this is true, and we have seen how they can use this guess to construct one magic square. A proof of this fact is more difficult. For students who do not know algebra, we can use the following argument:

There are four sets of three-in-a-row that include the center: horizontal, vertical and two diagonals. The sum of all these rows will be 4×15 = 60. Then we note that this includes each number exactly once, but the central number four times. Since the sum of all the numbers is 45, the difference 60 - 45 must give us three copies of the central number. This difference is 15, so the central number must be 5.

A bit of reflection will show that this argument says a bit more. Whatever the entries of a magic square, the magic sum is three times the central number. (Alternatively, the central number is 1/3 the magic sum.) This will be a useful remark in later developments.

Students who know algebra can be given this verbal argument, then challenged them to write it down using algebra.

(C) The even numbers must be in the corners.

Students will probably not see this quickly, but respond quickly when asked: "Which numbers are in the corners?" A proof of this statement is a bit subtle. One argument goes as follows:

Once we place 5 in the centre, there are four even and four odd numbers to place in the square. We will show that no even number can be in the middle square of a row or column. For example, suppose we put an even number to the right of the 5:

A B C
E 5 D (E for Even!)
F G H

Looking at the middle row, we see that *D* must also be even, since the three numbers in the middle row must sum to 15 (an odd number).

Looking at the first column, we see that one of *A* or *F* must be even (or else the sum of that column would be even). Without loss of generality, we can assume that A is even. Then, looking at the Seattle-to-Miami diameter, we see that *H* must be even.

This means that *A*, *E*, *D*, *H* are all even, so that the remaining entries must be odd. However, this gets us into trouble. Looking at the first row, this would mean that *A*+ *B*+ *C* is the sum of one even and two odd numbers, so cannot be 15.

This is a difficult argument for many students. Not only is it a proof by contradiction, in which they must imagine something that in fact they cannot get an example for, but the chain of inferences is rather long. I find it rare for students to come up with a proof like this themselves. However, it is not so difficult for them to follow, if laid out clearly. Later we will see how one can tell if they have comprehended the argument.

Note that statements (A), (B), and (C) can be proved with students independent of their knowledge of the symmetries of a square: the two topics can be developed independently of each other.

Note further that statements (A), (B), and (C), taken together, constitute a proof that the magic squares we have derived are the only ones possible with the numbers 1 through 9. Indeed, 5 must be in the center, and 2 must be in one of the corners. This puts 8 in the other corner, and the other two corners must contain 4 and 6 in some order. The placement of 5 and the even numbers determine the rest of the square, and it must be one of those we have already considered.

Activity VII: New Magic Squares from Old.

The first six activities here introduce concepts from geometry and group theory. This activity begins a new series, which introduces concepts from linear algebra.

I usually start this activity when students have worked the previous activities, but are fresh for a new start. I review the magic squares we have constructed, putting at least one on the board visible to students as they work. (This point will soon become important.)

Then I give them cards numbered 2 through 10 (rather than 1 through 9), and ask them to construct a magic square with them. There are in fact two interesting ways to do it. One way, the Short Way, is something we will talk about later (if the reader has not guessed it by now). Nevertheless, 90% of my students think of doing it the Long Way—which has its own virtues. The Long Way involves repeating the steps in construction in Activities I – III. The virtue in it is that the students are not repeating an algorithm. It is too early for the construction to be ossified in their minds as an algorithm. They are repeating a pattern of reasoning. In addition, recognising patterns of reasoning is an important objective of our teaching.

We can take advantage of their review of the reasoning—which happens naturally—by making conscious, or reviewing, the notion of transformation (of a 3×3 array) and of invariants. Often, these concepts will come alive with a second example.

As in Activities IV-VI, I put several examples of the completed magic square on the board, and ask how they are the same and how they are different. Of course, everything goes much more quickly than the first time. Students immediately see the rotations and reflections. These generally do not need reinforcement.

What does need reinforcement is the logic leading to the following questions:

1) What is the magic constant? Of course, it is the sum of the entries divided by 3. This is already a generalization of the previous result. (Advanced students will find this easy.)

2) What number is in the middle? This repeat of the reasoning gives us a chance to solidify the fact that the middle number—of any 3×3 magic square—must be 1/3 the magic sum, or 1/9 the sum of the numbers. (This note can be picked up on later, when students can be asked to show that the sum of nine consecutive integers is a multiple of 9.)

3) This time it is the *odd* numbers that end up in the corners, and even for advanced students, a proof may require some thinking. The pattern of reasoning is the same as in the earlier case.

4) If students don't see this already, I point out that statements (1)-(3), together with what we know about symmetries of a square, constitute a proof that the magic square we've constructed in unique, up to symmetries[1].

Then comes a dramatic moment in most classrooms. The Short Way of doing this problem is simply to add 1 to each entry of the original magic square. Most students do not see this at first, but see it immediately when pointed out. Their reaction is that they could kick themselves. I would encourage them to do so. Kicking oneself is both good physical exercise and a useful

[1] The degree to which students recognize and repeat these patterns of reasoning will show how much they assimilated them in the first set of activities.

habit of mind.

To increase the drama a bit, I try to make sure that one of the new magic squares I have chosen to put up is in the same position as an 'old' magic square (numbers 1 through 9), so that the Short Way is right in front of the students.

In working Activities I through III, we sometimes get students who complete the task long before others, or who have seen and remembered the solution. They simply reproduce a magic square with numbers 1 through 9 that they have seen before, without going through any reasoning. For these students, we can jump ahead to this activity, by taking the 1 in their magic square and replacing it with 10. (This is easily done if using playing cards for the numbers.) The square is of course no longer magic, and the challenge is to restore the magic. Replacing the 1 with 10 virtually assures that the student will not discover the Short Way this time around.

Activity VIII: Generalisations.

I then ask students the following questions, often without actually constructing the magic squares;

1) How would you make a magic square with the numbers {3, 4, 5….11}? The answer is not hard: add 2 to each entry of the square from Activity III.

2) How would you make a magic square with the numbers {101, 102…109}? Answer: add 100. However, see the next question.

3) How would you make a magic square with the numbers {100, 200, 300…900}? Students may just answer that they can put '00' after every entry of the old square. It is important to elicit from them that we are actually multiplying by 100.

4) How would you make a magic square with the numbers {3,6,9, . . . 27}? Answer: multiply each number in the original magic square by 3. (Some students may be able to jump from question (1) to question (4) immediately, without the scaffolding provided by questions (2) and (3).

This is a good place to stop and ask for generalizations. The typical generalizations students point out are:

a) You can multiply the entries of a magic square by any constant, and it retains its magic.

b) You can add the same number to all the entries of any magic square, and it retains its magic.

It is worth asking students what happens to the central number, and thus to the magic constant, in each case. For (a), the magic constant is multiplied the same number used for the whole square. For (b), the magic constant increased by three times the number used for the whole square.

However, we can go further, and the next step is a bit difficult. I ask students:

(4) How do you create a magic square with the numbers {101, 201, 301…., 901}?

They can do this easily, but the trick is to ask them *how* they do it. It is important to bring out is that they have done two things: multiplied the original magic square by 100, and then added 1. Then I ask them to make a magic square of the numbers {4, 7, 10, . . . 28}. This usually gives them pause. They recognize that the numbers form an arithmetic progression (they will say that they are 'equally spaced'). It is often useful to put the following chart on the board:

4	3
7	6
10	9
13	12
16	15
19	18
22	21
25	24
28	27

Students will quickly recognize the second column as multiples of 3 (the 'three times table') and then the first column as one more than the multiples of 3. Therefore, they can make the new magic square by multiply the original one by 3, then adding 1.

Now the lesson can take a whole new turn, about arithmetic progression. I teach this topic as a generalization of the 'Gauss' trick of adding 1 + 2 + 3 + ... + 100. This pedagogical technique is well known, and I will not describe it here. I just leave the reader with the thought that a door is open here for the development of a whole chunk of valuable mathematics, and will go on to another chunk of valuable mathematics.

The generalization we have come to now, which can be elicited from the class, is this:

One can make a magic square from any arithmetic progression *of nine elements. The square is unique, up to symmetries, and the magic constant is three times the central entry.*

(The statement about the numbers at the corners of the square is harder to generalize in a useful way. I usually do not.)

Activity IX: Introduction to Vectors and Vector Spaces

(Disclaimer: This section is less an activity than a lesson plan. It involves more intervention by the teacher than previous activities. Most of what is explored is convention and notation.)

Now another critical juncture: rephrasing the generalization above in a way that students can understand, but rarely generate for themselves. If we drop the condition that the entries of a magic square must be distinct (a tacit condition up to this point), then we can say that the following (silly) square is 'magic':

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

(Note that the middle number is still 1/3 the magic constant.) Then we can say that adding 1 to each entry of a magic square is just 'adding' two magic squares term-by-term, as if they were matrices:

$$\begin{matrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{matrix} \;+\; \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \;=\; \begin{matrix} 3 & 10 & 5 \\ 8 & 6 & 4 \\ 7 & 2 & 9 \end{matrix}$$

Then we can elicit from the students the following generalization:

Two magic squares can be added (term-by-term) and the resulting square is still magic.

I have students experiment to see that this works, with two squares that are not ‘silly’. For example:

$$\begin{matrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{matrix} \;+\; \begin{matrix} 3 & 10 & 5 \\ 8 & 6 & 4 \\ 7 & 2 & 9 \end{matrix} \;=\; \begin{matrix} 5 & 19 & 9 \\ 15 & 11 & 7 \\ 13 & 3 & 17 \end{matrix} \qquad (*)$$

Students should check (a) that the new square is in fact magic, and (b) that the magic sum is three times the central number.

And now some notation. We let letters and numbers stand for magic squares. Therefore, if the original magic square (in some fixed position) is M, then we can create M+1, M+2, M+100, etc. by adding a constant to each term. This is a very natural notation, which students accept readily. Note that we are using the symbol ‘1’ to denote the magic square all of whose entries are 1, and the symbol + to denote addition of squares, not just numbers. Students do not usually pick up on this subtlety.

We can also write $2M$, $3M$, $100M$, etc., for the squares whose entries are those of M multiplied by a constant. (Note that here '2' is the number two, while in the last paragraph, '2' is the 'silly magic square', all of whose entries are 2. Luckily, most students simply do not notice this slight inconsistency in notation.

Finally, we can write $2M + 3$ for the square obtained by multiplying M by 2, then adding 3. So we have a whole set of magic squares, all of the form $aM + B$. The point of the next few activities is to give this set a structure, the structure of a vector space.

I start by looking at example (*) above and rewriting it in the new notation:

$M + (M+1) = 2M + 1$

(Advanced students may actually have recognised the square on the right already, and written it down in $aM+B$ form.)

Students can then practice the new notation, using the following sort of question:

What is $(2M) + (3M-1)$? Answer: $5M - 1$. It is useful for them to write the magic square specifically, after they have done the algebra.

The other way to practice this notation is to give students actual magic squares to add, and then ask them to write the process algebraically. I usually do not go this far, although an advanced class—which is familiar with arithmetic progressions—might find such questions useful.

Now some more terminology, which consolidates what we have been talking.

When we multiply every entry in a magic square by a given number, we say that we are *scaling* the square. There is actually a metaphor here: rather than using the number '1' to 'measure' the entries of the square, we are using the multiplier as a measure. Instead of '2' in the upper left-hand corner (in the square M above), we have $2m$, where m is the multiplier. Therefore, it is like a scale on a map, where every centimeter or inch must be multiplied by a particular number to give the true distance.

Now we can say that magic squares M and N can be added and scaled. In algebra, if M and N are magic squares, then $aM + bN$ is also a magic square, for numbers a and b.

The name for a type of quantity that can be added and scaled is a *vector*. Some students may have heard this term before, and perhaps even studied it, say in physics—depending on their grade and achievement level. It is a term that grows with the student. At first, (usually) it is ‘a quantity which has magnitude and direction’, an arrow that you can add and scale. Later it is seen as an ordered n-tuple, which generalizes the dimension but not the algebra. Finally, it is seen as an element of a set with three operations obeying certain axioms. That is the ‘mature’ mathematical definition, and is a fruitful generalization of the ‘magnitude and direction’ idea.

The intent of these exercises is to provide an example of a vector space which is not abstract, yet does not (immediately) look like ordered n-tuples.

The next activity is intended to elicit the notion of a *basis* for a vector space, and of its *dimension*.

Activity X: Dimension of the Vector Space of Magic Squares. (I do not usually introduce the term ‘vector space’ now. Just ‘vector’.)

Take a typical magic square, for example:

$$\begin{matrix} 3 & 10 & 5 \\ 8 & 6 & 4 \\ 7 & 2 & 9 \end{matrix}$$

Ask: If I erased the 10, could you reconstruct the magic square?

This is easy, and students readily agree that they can. This means that if they were given the magic square with the 10 missing, they would know what belongs there. Then erase the 4, and ask the same question. Again, students see that even with the 10 and the 4 erased, the magic square can be reconstructed.

Finally, we can challenge students: How many entries can you erase, and then reconstruct the square? What is the minimal amount of information you need to specify which magic square this is?

This is a good activity for students to work on. There will be varying results, of course. The final answer, which some students will get, is that three (judiciously chosen) entries will determine the whole magic square, for example:

$$\begin{matrix} 3 & A & B \\ C & 6 & D \\ E & 2 & F \end{matrix}$$

As an example, let us reconstruct this square. The magic constant, from our discussion, is 18. So F = 9 and A = 10. We have

$$\begin{matrix} 3 & 10 & B \\ C & 6 & D \\ E & 2 & 9 \end{matrix}$$

This configuration easily gives us B = 5, which leads to D = 4.

Students can readily recover the rest of the square from this. Will any three entries in this position give us a magic square? I give students an example:

$$\begin{matrix} 6 & A & B \\ C & 5 & D \\ E & 2 & F \end{matrix}$$

Note that the 6 and the 5, in position, suggest the original square M. However, the 2 in the bottom row spoils it. Is there in fact such a magic square?

Again, repetition of reasoning pays off. The magic constant is 15, which gives $F = 4$. Then $E = 9$, so $C = 0$ (Surprise!). Then $D = 10$ and $B = 8$, and the rest follows easily.

$$\begin{matrix} 6 & 8 & 1 \\ 0 & 5 & 10 \\ 9 & 2 & 4 \end{matrix}$$

There are, of course, other orders in which to find the entries. Note that when we get to the last number (whichever number that is), there are two conditions to be satisfied, and they do not conflict. In fact, they will never conflict, no matter what the three original entries might be. This is ‘magic’ in itself.

Activity XI: Explaining the magic.

A little algebra will give the explanation. Here is a general solution.

x	$2y - z$	$-x + y + z$
$-2x + 2y + z$	y	$2x - z$
$x + y - z$	z	$2y - x$

Now I ask students if these are the only three entries which allow us to reconstruct the square. They quickly see that the symmetries of a square give eight such entries, with a corner, a middle entry, and the center.

There are other possibilities for three given entries to reconstitute the magic square. Advanced students can work these out algebraically. If the center entry is given, there is essentially one other possibility:

x	z	$-x + 3y - z$
$-2x + 4y - z$	y	$2x - 2y + z$
$x - y + z$	$2y - z$	$-x + 2y$

If the center value is not given, the task is more difficult. One must first derive the magic constant. In the case below (where x, y, z are given), we can let the constant be $3C$, so that the top middle entry must be $3C - x - y$. Then, from the middle column (which must add up to $3C$), we know that the center entry is $x + y - z$, so the magic constant C is three times this, or $3x + 3y - 3z$. The other entries follow readily.

x	$3C - x - y$	y
$2y - z$	$C = x + y - z$	$2x - z$

$2x + y - 2z$	z	$x + 2y - 2z$

Students can look at other configurations. A good question to ask is: Do three entries in the same row work?

Activity XII: A Basis.

The algebraic or arithmetic explorations above lead to the idea that a magic square is determined by three well-chosen pieces of information. In mathematical terms, this indicates that the vector space is of dimension 3.

More formally, if we are given three particular vectors, we can recreate all the others by adding and scaling[1]. Three such vectors are called a *basis* for the vector space. For example, for the vector space of points in (Euclidean) 3-space, the vectors (1,0,0), (0,1,0), (0,0,1) form a basis. By scaling these and adding the results, you can get any vector (a,b,c).

Bases are not unique. The 3-dimensional Euclidean vectors can also be generated from the vectors (1,0,0), (1,1,0), (1,1,1). The concept of basis leads to a coordinate representation of the vectors in a vector space, which is often useful in computations.

[1] To be precise, a basis is a minimal set with this property: no basis vector can be expressed as a linear combination of the other basis vectors. They are *linearly independent*. But this notion can be left unspoken for many classes.

But it is a central fact—we will not prove it—that if a vector space has a (finite) basis, then any two sets of basis vectors have the same number of elements. This number is called the *dimension* of the vector space.

Our explorations indicate that the dimension of our vector space of 3×3 magic squares is 3. (Warning: this does not imply that the dimension of the vector space of 4×4 magic squares will be 4.) We can confirm this by finding three magic squares that form a basis.

One nice way to do this is to map all the magic squares onto the set of magic squares with magic constant 0 (or middle entry 0). We will call these the ‘zero squares’. This will expose the structure of the magic square.

Any magic square can be represented as the sum of a zero square and a constant square. And any such sum is magic. That is, there is a one-to-one correspondence between the set of all magic squares and the set of pairs, one of which is a zero square and the other a constant square.

Note that the constant squares are all scaled multiples of the square with all entries 1’s (the *unit square*). So if we can represent a zero square as the sum of multiples of two other squares, we have three squares that generate the whole set.

Having agreed to this—or even if they come to see it later on--students can be asked for the minimal information to generate a zero square. They quickly see that a zero square is *antisymmetric*: opposite entries (with respect to the center) have opposite signs.

Having noted this, we can ask the students to fill in this zero square:

a	b	
	0	

,

with two entries given algebraically. This shows that with two particular pieces of information, we can reconstruct the entire zero square. The result is:

a	b	$-a-b$
$-2a-b$	0	$2a+b$.
$a+b$	$-b$	$-a$

We call this magic square S, and will use it below to get a basis.

Nevertheless, we can also reconstruct a zero square with two other pieces of information. For example, another possible representation is:

$$\begin{array}{ccc} a & -a-b & b \\ -a+b & 0 & a-b. \\ -b & a+b & -a \end{array}$$

(Here the upper left and upper right corners are given.)

Each of these, for suitable *a* and *b*, can represent any zero square. Another way of saying this: for any zero-square, we can find suitable numbers *a* and *b* so it is represented by one of the squares above.

As an exercise, we can ask students to start with the magic square from activity III, construct its related zero-square, and then find *a* and *b* that puts it in the form of *S*.

There are other ways to choose two pieces of information that determine a zero square. Advanced students can be asked to count all such possibilities. They are given below. There are only four possibilities, up to symmetry. Counting them is a good exercise in symmetries of a square, and filling in the representations is an exercise in algebra. Note that we must not be given two opposite entries: each determines the other, and that makes only one piece of information.

a	b		a		b	a				a	
	0			0			0	b		0	b

And now the square *S* above gives us a basis for our vector space. If we start with our first zero-square:

$$\begin{array}{lll} a & b & -a-b \\ -2a-b & 0 & 2a+b \\ a+b & -b & -a \end{array}$$

and set $a = 1$, $b = 0$ we get

$$A = \begin{array}{rrr} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{array}$$

Then we set $a = 0$, $b = 1$ to get:

$$B = \begin{array}{rrr} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{array}$$

For any zero square, we can find the numbers a, b so that $aA + Bb$ is this magic square. In addition, the representation is unique. In formal language (which I do not always use), the subspace of zero squares (for it is a subspace: it is closed under addition) is of dimension 2.

How does this give a basis for the full set of magic squares? Well, we can take the unit square C, consisting of all 1's, and add a suitable multiple to get any magic square at all. That is, the three magic squares A, B, C form a basis for the vector space of magic squares.

Students can be challenged with numerical examples. Alternatively, they can be asked to construct another basis, reviewing the reasoning of the discussion above and starting with another set of two entries a, b, in different positions.

I have not introduced the terms *linear combination*, or *linear independence* in this development. I usually leave that for a more formal course. This discussion builds an example that looks very different from the usual. When formal language is introduce, students will have something in their minds that it describes.

There is even more that we can do with this construction. Once we have a basis, we can use the coordinates (a,b,c) in the representation $aA + bB + cC$ to specify a given magic square. Then we can visualize the space as 3-dimensional, and even define a metric (distance between two magic squares) or norm (distance to zero, a sort of 'absolute value'). We can visualize the sum of two magic squares as addition of 'arrow' vectors, and scaling as, well, scaling of the arrows.

This is all an exercise is re-formulating a situation which is suitable (and usually easy) for advanced students. For these students, we can also go back to the symmetries of a square and introduce an equivalence relation. For each vector (magic square), there are seven other 'equivalent' vectors that represent the squares obtained by symmetry. What is the geometry of this situation? Where do those other vectors lie? In some (simple) cases, a magic square may itself be symmetric. What then happens to the ‘symmetric’ vectors? How far one goes with these explorations depends on the class and on its learning goals.

Activity XIII (Epilogue): Magic squares and tic-tac-toe

I am tempted to call this section ‘dessert’ (for the meal). I use this activity at any time in the sequence. Students enjoy it and remember it whenever it comes up.

I have students play the ‘Fifteen Game’:

(a) Write the numbers 1 through 9 on the board in a straight line.

(b) The first person “takes” a number by crossing it out and writing it on his side of the board.

(c) The winner is the first person who makes a sum of 15 with exactly 3 numbers.

For example, after three turns of play, suppose Adam has collected the numbers {3,8,1} and Betty has collected the numbers {6,9,2}. Neither has won, although Betty’s numbers 6 and 9 add up to 15. Betty would need three numbers with this sum to win.

Nevertheless, suppose it is Adam's turn. The 4 is left on the board. If Adam takes it, he will win, because 3 + 8 + 4 = 15.

The first time they play it, I play an 'exhibition' game in front of the class, against a brave student. I tell the student that she or he will not lose. (In addition, I rig the game so that the student wins.) This is to clinch their understanding of the rules, which are not easy to integrate just from reading.

Then I have two students play in front of the class, with the class calling out advice. I do this at various odd times during the sequence of lessons. The activity need not be done all at once.

Eventually, and sometimes over several weeks, students see the following:

1) When players get good, most games end in a draw. There are no more numbers left to choose, and no play has three numbers summing to 15.

2) The strategy in play often consists of 'blocking' the other player.

3) The number 5 is key, as well as the numbers that add up to 10.

Sometimes students actually say, “It’s like tic-tac-toe”. I ask them how, but they usually cannot verbalize the similarity. It is just the feeling they get in playing the game.

Nevertheless, in fact this game is exactly tic-tac-toe, played on a magic square! Eventually I show them this, by recording the moves of a game as x’s and o’s on a magic square. As two students play, I record their moves off on the side by crossing out (x) the first player’s choices, and circling (o) the second player. No words are necessary.

The underlying mathematical idea here is that of *isomorphism*. The two games are really the same game in two different guises, or notations. Isomorphism is a major concept pervading all of mathematics. The concept is expressed in the formal language of functions and operations, but that is not the point of this lesson. It is a bit difficult, but can be done, to define the game as a mathematical object, and the isomorphism as a function on the object.

What is the point then? Well, to have fun, and at the same time lay the intuitive groundwork for further understanding. In my view, both these things should happen as often as possible in any mathematics classroom.