SCHOOL MATHEMATICS NEWSLETTER

ISBN 978-988-8370-89-4

政府物流服務署印

學校數學通訊

第二十三期

教育局

SMN

SCHOOL MATHEMATICS NEWSLETTER

ISSUE 23

Published by
Mathematics Education Section, Curriculum Development Institute,
Education Bureau, Government of the Hong Kong Special Administrative Region.
香港特別行政區政府教育局課程發展處數學教育組出版

版權

©**2019** 本書版權屬香港特別行政區政府教育局所有。本書任何部分之文字及圖片等，如未獲版權持有人之書面同意，不得用任何方式抄襲、節錄或翻印作商業用途，亦不得以任何方式透過互聯網發放。

ISBN 978-988-8370-89-4

School Mathematics Newsletter（SMN）

Foreword

The School Mathematics Newsletter（SMN）is for mathematics teachers. SMN aims at serving as a channel of communication for mathematics education in Hong Kong. This issue includes various articles written by academics and teachers, related to the current hot topics in education, e.g. STEM education, e-learning and the practical application of artificial intelligence, etc. Other articles involve different areas, including suggestions of effective strategies in learning and teaching of mathematics on specific topics; strategies to facilitate the teaching of Compulsory Part with Module One and story about the number e. I hope all the readers can get some fascinating insights in mathematics education.

We would like to take this opportunity to thank all the authors in this issue. Without your support, it would not be possible to publish SMN issue 23.

SMN provides an open forum for mathematics teachers and professionals to express their views learning and teaching in mathematics. We welcome contributions in the form of articles on all aspects of mathematics education. Please send all correspondence to:

The Editor, School Mathematics Newsletter,
Mathematics Education Section
Curriculum Development Institute
Room 403, Kowloon Government Offices
405 Nathan Road
Yau Ma Tei, Kowloon
email: schmathsnewsletter@gmail.com

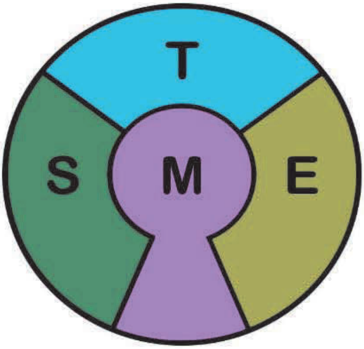
We extend our thanks to all who have contributed to this issue.

Contents

1. What 'Teachers' Can Do to Foster STEM Awareness through the Learning of Mathematics

LAW Huk Yuen
Department of Curriculum and Instruction
The Chinese University of Hong Kong

With the ever-growing pace of technological advancement, STEM (Science + Technology + Engineering + Mathematics) education comes into the focus of attention in our revised elementary (both primary and secondary) Mathematics curriculum. From the perspective of curriculum design, the STEM initiative invites both constraints and challenges (see Herschbach, 2011) as we attempted to incorporate STEM education into our existing curriculum framework. There is as we can see a great variety of how STEM is and would be highlighting (in terms of educational resources) in schools such as through General Studies in the Primary and through Technology Education in the Secondary. And yet, from the epistemological point of view, mathematics as a long-established discipline remains as ever the highly significant core subject in the elementary education. Ontologically speaking, mathematics itself has a unique role to play in our undertaking of meta-ontological theorizing of being through which we may understand better of what is existing in the world we are living in. Thus, I argue that mathematics itself is both the key with which we open and the keyhole through which we vision the world of STEM (see Figure 1 as I used in my writing for SMN, Issue 21). The vision of seeing the world of STEM through the keyhole of mathematics requires teachers to have dedicated mission of reinventing their own teaching (Law, 2013) through researching their practices.



M: Awareness of using mathematics as a language to interpret life experience

E: Awareness of designing model for solving problems

T: Awareness of making tools for solving tasks

S: Awareness of making inquiry for natural phenomena

Figure 1. Making sense of STEM in Maths (HY Law, *School Mathematics Newsletter*, 2017, Issue 21, p. 8)

The three-teacher model

To foster students' STEM awareness through the learning of mathematics, we need to have three 'teachers' – the *learner-as-teacher* (the First Teacher), the *school teacher* (the Second Teacher), and the *environment-as-teacher* (the Third Teacher). Loris Malaguzzi (1994) argues strongly that "We need to define the role of the adult, not as a transmitter but as a creator of relationships — relationships not only between people but also between things, between thoughts, with the environment." With the adoption of such a three-teacher model in the STEM-in-Maths (SIM) classroom (see Figure 2), we need to reinvent the notions of curriculum and assessment – *curriculum as learning* and *assessment as learning*. Such kind of reinvention requires school teachers to see education in general and STEM education in particular as relationship – relating teaching to learning as self-inquiring process (by asking "what do I learn and why I want to learn it?", "how do I know that I have learned?") and in turn to the environment itself as ontological experiment with which we create space for learning.

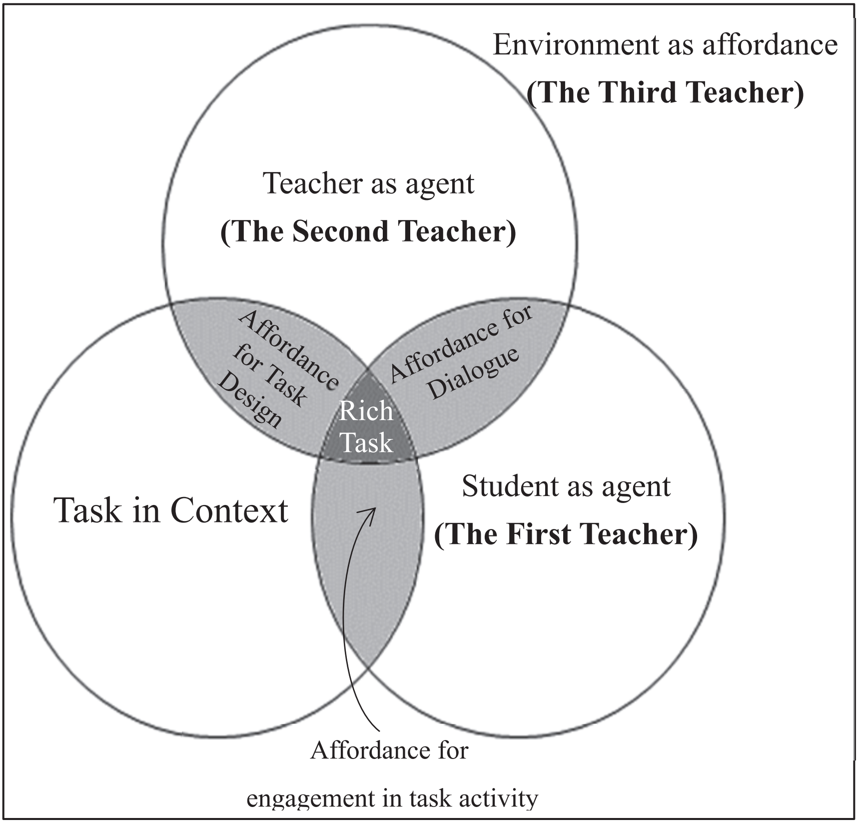


Figure 2. The three-teacher model for fostering STEM-in-Maths awareness

The First Teacher – Student as Agent

All human beings are born into the world as a learner to be. As natural born learners (see Beard, 2018), we are the first teacher who should know best to teach ourselves *the way we learn*. It is the human rather than the technology that teaches us through social interactions. With exploratory actions, we as *self-*

directed learners develop our agency for monitoring and assessing the task as encountered in the environment. Tversky (2019, p.288) argues that "actions in space create abstractions". Yet, making sense of abstractions that we create requires us to engage in voicing through the dialogic space whilst doing the action-oriented task. Hand-on activity is the natural way of developing our spatial thinking (abstract thought of the first kind) that underlies our ability of making sense of STEM world we live in. Engaging in the hand-on activity will develop the kind of emotion or feeling that urges us to assess what kind of learning experience that we have gone through. Such kind of assessment is of vital importance for building up the motivation behind not just the learning of mathematics itself but the fostering of the disposition of self-directed learning (Guglielmino, 2013) for STEM awareness.

The Second Teacher – Teacher as Agent

Learning begins to take place by the moment when the teacher opens the action possibility by creating the task in the classroom and invites the students to engage in it. The teacher as agent for making the difference wants to foster students' response-ability (capability to respond) by creating dialogic space through which she would see the chance of making a collaborative effort at making sense of the classroom discourse. The teacher is well aware that making mathematics meaningful to the students can be possible by having a quality classroom interaction upon the task as engaged rather than by a sheer telling or saying of what to do with the task itself. Knowing how to respond the students' responses enable the teacher to develop the milieu in the classroom for the adoption of *assessment as learning* (Earl, 2003) as she knows well that the kind of feedback to the learners is aimed to develop the thinking about thinking of what kind of learning experience that they would have. As a guide for teachers to design the task for the students, we may make sense of 'rich task' as an equation such as 'Rich Task = Good Problems + Quality Learning Discourse'. In other words, teachers need to adopt various teaching strategies such as questioning and group work to bring in the meaning for the students' learning through quality mathematical communication of the problem task as designed. In the making of the problem task, the teacher needs to have a dialogue with the environment as the third teacher who tells what is going on in our world.

The Third Teacher – Environment as Affordance

Reggio Emilia, the city in southern Italy, is where Loris Malaguzzi's idea of the environment as the third teacher comes about. The Reggio Emilia philosophy values children as central to their own learning via negotiating what to learn with their surroundings (see Strong-Wilson & Ellis, 2007). If we as teachers are to adopt such a philosophy, we would see all kinds of action possibility of designing the task for fostering schoolchildren's STEM awareness. As we see the emergence of what the students want to learn, we come to understand what to-be-designed task constitutes the notion of 'curriculum as learning'. By the very moment we capture the image of our students as a curious learner with inquiring mind, we realise that it is the beginning of teaching something that fits what they are interested to learn (see Malaguzzi, 1994). In other words, environment *affords* learning if we can design the classroom task by adopting the learners' perspectives of how they make sense of the world they are living in – *task in context*. I would argue that the task can be 'rich' only when the classroom teacher can articulate three kinds of affordance – 'affordance for task design', 'affordance for engagement in task activity', and 'affordance for dialogue' (see Figure 2). The concept of 'affordance' that comes from James Gibson (1979) empowers us with the awareness of developing possible actions for undertaking our STEM practices. In the STEM context, I would rather adopt the ideas of 'semiotic affordance' as "responses to a conceivable practical action made possible by habits" (Morgagni, 2012) with which hopefully the teacher can provide her students the kind of semiotic mediation for developing their STEM awareness through the learning of mathematics.

Coda

Fostering STEM awareness through learning mathematics as confined in the classroom context is indeed a challenging mission as it involves a complex interplay among the three entities -- agent, task and environment. And yet, it is worth pursuing if we as teachers do not just want to make sense of

STEM education but to make sense of learning mathematics itself for our schoolchildren. With the design of rich task, it is hoped that the hand-on activity as a way of undertaking the enactive explorations can help learners to make meaning out of what they would have learned in mathematics classrooms.

References

[1] Beard, A. (2018). *Natural born learners: Our incredible capacity to learn and how we can harness it.* London: Weidenfeld & Nicolson.

[2] Earl, L.M. (2003). *Assessment as learning using classroom assessment to maximise student learning*. Thousand Oaks, CA: Corwin Press.

[3] Herschbach, D.R. (2011). The STEM initiative: Constraints and challenges. *Journal of STEM Teacher Education, 48*(1), 96-122.

[4] Gibson, J.J. (1979). *The ecological approach to visual perception.* Boston: Hougton Mifflin.

[5] Guglielmino, L.M. (2013). The case for promoting self-directed learning in formal educational institutions. *SA-eDUC Journal, 10*(2), 1-18.

[6] Law, H.Y. (2013). Reinventing teaching in mathematics

classrooms: Lesson study after a pragmatic perspective. *International Journal for Lesson and Learning Studies, 2*(2), 101-114.

[7] Law, H.Y. (2017). STEM education: The challenge of using mathematics as the initiation for promoting STEM education (in Chinese). *School Mathematics Newsletter* (Education Bureau, Hong Kong), Issue 21, 6-11.

[8] Malaguzzi, L. (1994). Your image of the child: Where teaching begins. *Early Childhood Educational Exchange*, No. 96, 52-61.

[9] Morgagni, S. (2012). Affordances as possible actions: Elements for a semiotic approach. *Proceedings of the 10th World Congress of the International Association for Semiotic Studies.* Universidade da Coruña (Spain), 867-878.

[10] Strong-Wilson, T., & Ellis, J. (2007). Children and place: Reggio Emilia's environment as third teacher. *Theory into Practice, 46*(1), 40-47.

[11] Tversky, B. (2019). *Mind in motion: How action shapes thought*. New York: Basic Books.

2. 數學科差異化教學的理念和設計

張僑平
香港教育大學數學與資訊科技學系

一、引言

香港自 1970 年代推行強迫教育之後，學校教育開始普及，更多不同背景的學生有機會入學。1990 年代推行融合教育，課室裡學生的學習差異加大。這種差異主要表現在學生的不同性向，不同學習能力、學習興趣和學習風格。當然不同的學校文化和課堂文化也是造成學習差異的原因之一。其實不單是香港，在世界範圍來看，如何在學校有效地照顧學生的學習差異也都是一個難題。

跟處理學習差異類似的說法大致有差異化教學(differentiated instruction)、分層教學（tiered instruction）以及適應性教學(adaptive teaching/instruction)。雖然說法不同，他們的內涵其實也都相近。在本文中我們主要使用差異化教學這一提法。目前在學校中的一些常見作法有：抽離式學習、加速學習、按學生能力分組甚至分班，或者借助校外資源進行補救或拔尖的教學。這些方法均或多或少受到特殊教育中的三層級學習支援模式（Response to Intervention Model）（Bender & Crane, 2010; Pierce & Adams, 2004）或資優教育中增潤三分層模式（Enrichment Triad Model）（Renzulli, 1977）的影響。增潤分級支援模式也為目前香港教育局推行資優教育所採用。如果說差異化教學在學校層面主要是上述的分層結構，那麼在課堂層面則主要是教學內容和材料的分層策略，如根據問題的難易程度

劃分淺、中、難或者挑戰題等層級。無庸置疑，最有效和最理想的個別差異處理方式，自然是面對面的個別輔導，或是針對全班不同學生設計出不同教學方式，或是給 8 組學生發展出 8 種不同的學習路徑。但這些做法十分消耗資源，亦難以實施。如何在學校的同一個常規班級（學生能力是混合的）中進行差異化教學，要更為貼近教師的日常。這也是值得我們去深入探討的。

二、差異化教學

差異化教學的專家 Carol Ann Tomlinson（1999）指出，差異化的教學尤其需要教師能從課程的三個要素即學習內容（content）、學習過程（process）及學習成果（product）入手，根據學習個體在學習特徵（learning profile）、學習準備度（readiness）以及學習興趣（interest）等層面的不同，透過多元的教學設計，達到學生的學習參與最大化的目的[2]。簡單來說，跟傳統意義上教師主導或學科內容為本的教學相比，差異化的教學設計更注重不同能力、興趣和風格的學生在課堂中的參與和投入，教師須要儘可能給更多學生提供學習的機會。學習的內容、過程和結果構成了學生在課堂上的完整學習體驗。這些體驗包括作為學習內容的知識和技能，作為過程的思維方式和問題解決的方法，以及通過口頭表達、動手操作、視覺觀察、書寫作答等表現出來的學習結果。

現有的差異化教學理論中，已經總結了不少處理學習差異的策略和方法，比如改變學習內容的深度、調整學習內容的抽象度、改變問題的複雜度、改變舉例的數目（增加或

減少)、改變學習的方式(獨立或合作)等等，Tomlinson 稱為教學設計的調適均衡器（Equalizer Model）(見圖 1)。

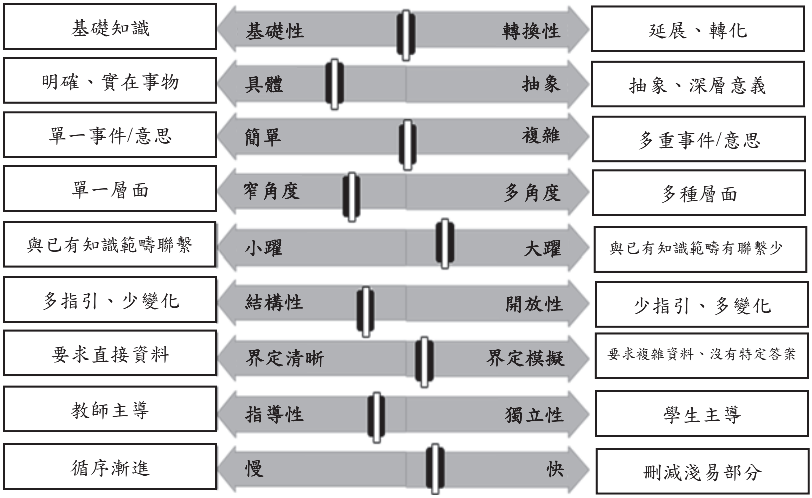


圖 1. 差異化教學設計均衡器

上述列出的均是一般的策略，結合具體學科特點和課程要求的教學策略並不多。差異化教學不是要消除差異。基於課堂的現實，務實而有效的差異化教學策略是能為同一班級絕大多數不同學習需求、學習方式、學習興趣及學習程度的學生，都能提供學習的機會，在同一時間能參與課堂活動，發展自己的思維，達至其最近發展區(*The zone of proximal development, ZPD*) (Vygotsky, 1978)。在數學課堂上，問題解決是學習數學的核心。不同學生的數學學習差異真正就體現在解題思路、解題策略和方法上。如何佈置問題、如何啟動學生的數學思維，達成他們的 ZPD，正是數學科進行差異化課堂教學的重點。

美國 Marina Small 教授基於長期試驗研究和教師專業發展課程的經驗，總結出《基於課程標準的數學差異化教學法》（Great ways to differentiate mathematics instruction in the standard-based classroom）(Small，2017)。她將數學開放題（open questions）結合平行任務題(parallel tasks)（以下簡稱 O-P 策略）融入到日常的數學教學中，結合美國數學課程標準諸多學習領域，包括數和運算、數系、運算和代數思維、運算式和方程、幾何、測量和資料、統計和概率等，在共同的架構下設計差異化的數學教學。當前香港已經推出了新修訂的數學課程文件，儘管和美國的數學課程標準不盡相同,但我們一樣面對處理學生數學學習差異的問題。這種基於課程標準的教學設計，與我們教師日常教學的方式還是比較貼近，主要涉及的是教學任務的設計和內容安排。教師在運用上不會覺得完全陌生。因此，本文對這此差異化數學教學的做法進行一些介紹，希望能給前線教師處理學生數學學習差異提供借鑒，以及帶出一些啟示和思考。

三、《數學差異化教學法》的教學策略和實施模式

學生在數學學習上會有怎樣的差異？在《數學差異化教學法》中作者舉了這樣一個問三年級學生的問題：

在一個櫥櫃中，您有三個架子，每個架子上有五個盒子。這個房間裡有三個櫥櫃。共有多少盒子存放在三個櫥櫃裡？

這個問題涉及的計算關係還是比較複雜的。不同的學生會

有怎樣的回應呢？比如：

- 有的學生可能會畫一幅圖畫，把盒子數畫出來，來描述上面的文字問題；
- 有學生會直接用加法，將所有盒子數相加 5+5+5+5+5+5+5+5+5；
- 有學生可能快一點，先算出一個櫥櫃中的盒子數 5+5+5＝15，然後算出房間裡盒子的總數 15+15+15=45；
- 有的同學可能也熟悉乘法，先計算 3×5=15，然後再加 15+15+15=45；
- 有的學生可能對題意和乘法運算很熟練，直接用乘法 3×5×3=45。

上面學生的這些回答未必全部發生在我們每一位教師的數學課堂，也可能有的課堂中出現更多不同方案，要知道還有一些學生是不出聲的(這不代表他們沒有想法！)。不過，這不正是學生多樣化的數學學習嗎？假如這樣的情境真的發生在我們的課堂，我們該怎麼回應呢？

學生的回答反映出他們不同程度的理解，我們是一個一個去解釋，還是告訴全班同學一個「標準」的理解和計算方法？還是在幫助學生在解決問題的同時繼續挑戰他們的思維呢？在前面的討論中，我們指出，要促進學生的最近發展區（ZPD），透過提供適當的學習任務，讓所有學生都能在老師的指導下或同學的協助下，參與課堂活動，獲得有意義的數學學習經歷。基於此，在《數學差異化教學法》的教學設計中主要凸顯三個重要目標：

- 重視學生預測診斷(Pre-assessment): 預先的診斷評估以確定不同學生的需求，瞭解學生的學習準備度。
- 圍繞課標重要觀點(Big ideas)︰教學的焦點集中於重要觀點且確保都被提出；
- 提供學生選擇機會(Choice)︰無論是在課程內容、實施過程和學習結果都能在某些面向提供學生機會去選擇；

在每一堂課的教學中，教師有時會局限在每一個學習單元或者課題的教學目標，往往忽視了整個學習領域的重要觀點。比如，對數的認識，表示一個數可以有多種表達方式，即便是進行加、減、乘、除這樣的運算，也可以有多種方法。幾何圖形可以有不同的表徵方式。這樣的一些重要想法不只是拘泥在一個課題、一個單元，而是貫穿各個年級的課程。

相信我們的教師都認同，教學須要提供學生參與的機會，但真正實施起來並不容易。不少教師習慣於掌控整個教學的進程和節奏，害怕課堂鬆散和失控（這裡不只是紀律問題，還有學生理解很可能出乎教師的意料以致教師回答不上來）。然而，沒有學生聲音的課堂，如何真正啟動他們的主動思考呢？當我們提供學生表達的機會時，不僅幫助教師瞭解學生，也有助於讓學生建立學習的自信和投入學習。上述三個目標並非是彼此分隔的，如何讓不同能力的學生都有機會參與課堂的學習活動，Small 提出了兩種核心的教學策略—開放問題和平行任務將上述目標有效地結合起來（見圖 2）。

四、開放問題和平行任務

我們經常看到，教師在教學中往往提出一個問題，發現學生不會或少有人回答，於是改為問一個簡單的問題，降低難度或者給學生一些提示。這種策略也能起到一定的效果，但對學生的要求還是比較高，降低到哪個程度也不容易掌握。開放的問題即是在同一個問題中可讓不同的同學依據他們自己的瞭解層次做出回應。比如，如果問學生如何比較$\frac{4}{5}$和$\frac{8}{9}$的大小，對全班學生來說，能回答的學生可能不會太多。如果教師的問題改為問，你認為$\frac{4}{5},\frac{4}{10},\frac{2}{9},\frac{1}{12},\frac{8}{9}$哪兩個分數較易進行比較？這樣問的效果就會不同。

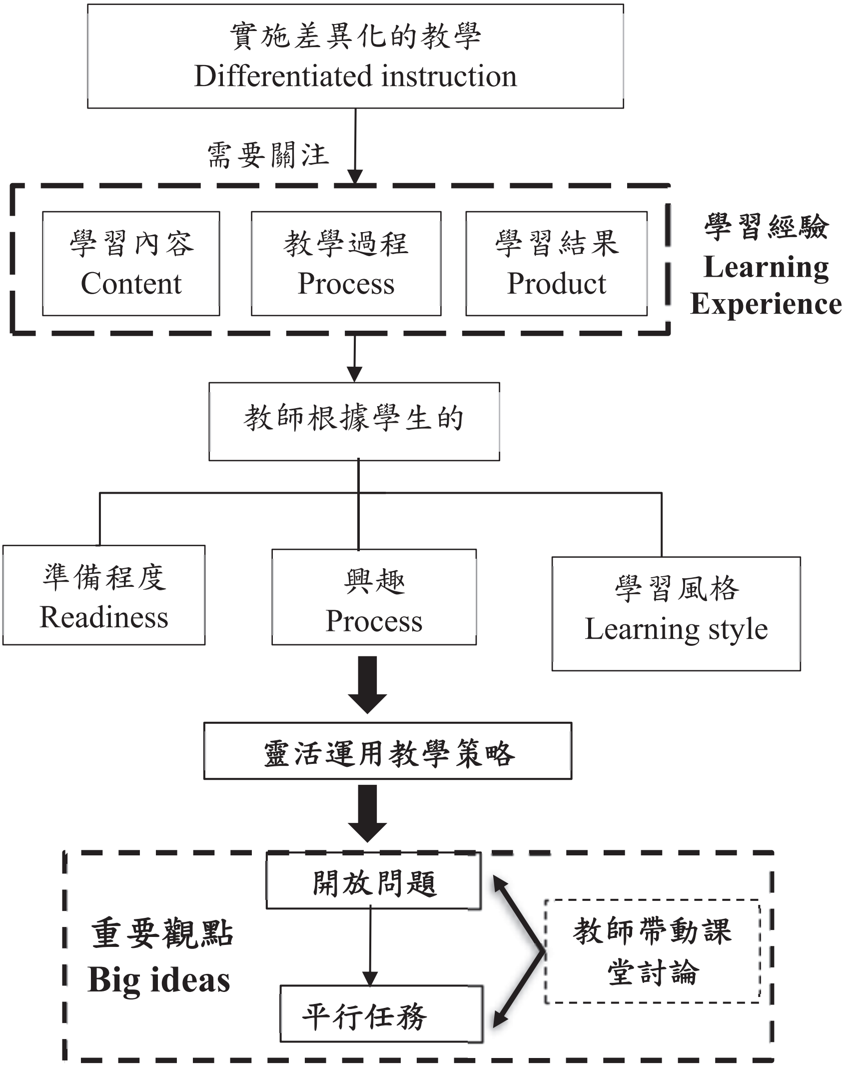


圖 2. 差異化教學的設計模式

將開放的問題引入教學，學生容易獲得學習的信心，而且能開始思考和回答問題。開放性問題提供學習者從不同角度思考的機會，讓學生能學會從多種觀點來思考數學概念，絕大多數的同學均能完全的參與課堂，而且從針對數學問題的課堂討論中獲取知識。Silver（1995）曾簡要地將數學開放性問題化為三大類別：1）給出的問題條件是開放的；2）問題的最終結果是開放的；3）問題的解題過程是開放的。當然，開放題也可以是這三種類型的綜合情況。不少開放題會提供多種不同的解法，有助於學生創造力的培養。教師從學生的回答中，能診斷出學生的問題和學習需求，從而組織下一步的教學。

儘管開放題的設計有很多方式，但不能漫無目的地放寬，需要有數學意義，跟所教授的主題相關（重要觀點 big ideas）。如問學生，看到 2 你會想到甚麼？這當然是一個開放性的問題，但這會導致許多沒有甚麼數學意義的回答。而比如問學生 16 和 18 有甚麼相似的地方？這也是開放問題，有學生可能答都是偶數，有學生可能說都在 10 到 20 之間。這樣的問題就更有數學意義，每一個學生的回答都很有價值，教師跟進討論也能回到授課重點上去。開放題的例子我們前線教師並不陌生，這裏不贅述了。

至於平行任務的問題，其實是有相同的重要觀點而且有比較接近的問題情境的問題。其想法類似於變式教學設計的理念（黃毅英、林智中、孫旭花，2006；黃毅英、林智中、陳美恩、王豔玲，2008），通常是兩個或三個問題作為一組，用來滿足處於不同發展水平的學生需要，讓學生同時練習

和討論。這些任務具有相同的知識要點，並且能在類似情境中討論。透過對問題的討論，能帶出問題情境中重要的數學知識。比如，這樣一組平行任務：

任務 1: 如果你將兩個數相乘，結果是 24000，你會選怎樣的兩個數相乘?

任務 2: 如果你將兩個數相乘，結果是 24.00，你會選怎樣的兩個數相乘?

這兩個任務很相近，都涉及到乘法，也跟十進位有關係。但還是有細微的分別，一個數字相對較大，一個有小數點。有的學生可能選擇做 24000，有的可能選擇做 24.00。學生根據自己的選擇解決任務後，教師需要有跟進的問題。因為是平行任務，教師跟進的問題在兩類學生中都可以作答:

- 你選擇這兩個數位中的任何一個數都可以很小嗎?
- 你選擇這些數中最大的是甚麼數?
- 你知道任務中的數字 24 有甚麼用嗎?

平行任務的佈置可以讓學生基於自己的理解進行自由選擇，也可以分配不同組別。教師結合學生的回答進行綜合講解，學生通過對比不同的情境的問題解決，在變化中掌握其中的不變的主要觀點。下面是一些平行任務的例子。

任務 1: 一個圖形的周長是 30 厘米，它可以是甚麼圖形?

任務 2: 一個圖形的面積是 30 平方厘米，它可以是甚麼圖形?

任務 1: 試舉兩個相同分子的分數，解釋哪一個分數較大。

任務 2: 試舉兩個相同分母的分數，解釋哪一個分數較大。

五、結論與啟示

Tomlinson 在給《數學差異化教學法》寫的序言中，描述了一段自己學習數學不愉快的經歷，小學階段的數學學習已經讓她覺得害怕，以致長大後認為自己就是不擅長數學的人。相信不少人都會有類似的學習經歷，甚至有人在小學已經覺得自己不擅長數學，不會在數學上成功了。這裏我們並不準備展開討論情意因素對個人數學學習的影響。反而，我們想大家思考：是什麼造成學生學習數學的失敗，或者說，怎樣才能幫助不同的學生學好數學。

《數學差異化教學法》儘管給出了一些清晰的、可操作的教學設計模式，但它是否適合每一個教師的課堂，需要實踐檢驗，更需要我們教師專業的判斷。我們想著重指出的是，這套差異化教學設計背後的理念值得我們借鑒。教師給每一個學生充分的學習數學機會，對學生的回應，教師能給予引領和回饋，讓弱的學生獲得成功的體驗，讓能力強的學生接受進一步的挑戰！在整個課堂教學中，將與課題相關的重要觀點貫穿其中，無論是提出開放問題，還是佈置平行任務，亦或跟進的問題和討論，無一不是在構建和促進學生的數學思考。這真正是當前我們不少數學課堂欠缺的。

每個人的學習進度有快有慢，學習方式有不同：有的喜歡符號，有的喜歡圖形，有的喜歡小組活動，有的喜歡個人練習等等。在這些活動中，我們都希望學生還能有數學的思考。2000 年課改以來，各種教學法不斷出現，一時喧鬧非常。無論怎樣，其實都是希望促進學生課堂參與，提升

學習動機，最好還能獲得好的學習結果。差異化的數學教學，正是需要教師搭建起一個促進數學思考的教學環境。開放題的特徵和功能，既重視了過程，也關注了結果。讓學生學會學習，掌握思維的方法。處理數學學習差異，既不是只選用簡單問題將就低程度學生，也不是一味挑選難題給能力高的學生。教師須要先掌握每一個教學單元的核心概念和觀點，為教學目標設定不同層次並提供相應的學習活動。平行任務既能讓普通程度與低程度的學生關注核心知識與技能，有達成學習目標的機會；同時也兼顧能力高的學生促進其高層次思維的發展，在問題變化中發現規律，在不同情境中學會應用。

參考文獻

[1]黃毅英、林智中、孫旭花（2006）。《變式教學課程設計原理：數學課程改革的可能出路》。香港：香港中文大學教育學院香港教育研究所。

[2]黃毅英、林智中、陳美恩、王豔玲（2008）。數學變式課程設計——以小學三個課題爲例。《教育學報》35 卷 2 期，1-28。

[3] Bender, W. N., & Crane, D. N. (2010). RTI in math: Practical guidelines for elementary teachers. Bloomington, IN: Solution Tree Press.

[4] Renzulli, J. S. (1977). The enrichment triad model: A guide for developing defensible programs for the gifted and talented.

Mansfield Center, CT: Creative Learning Press.

[5] Pierce, R. L., & Adams, C. M. (2004). Tiered lessons: One way to differentiate mathematics instruction. Gifted Child Today, 27(2), 58–66.

[6] Small, M. (2017). Good Questions: Great ways to differentiate mathematics instruction in the standards-based classroom (3rd Edition). New York, NY, U.S.A.: Teachers College Press.

[7] Silver, E. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. ZDM-International Reviews on Mathematical Education, 27 (2), 67-72.

[8] Tomlinson, C. A. (1999). The differentiated classroom: Responding to the needs of all learners. Alexandria, VA, U.S.A.: Association for Supervision and Curriculum Development.

[9] Tomlinson, C. A. (2001). How to differentiate instruction in mixed ability classrooms (2nd ed). Alexandria, VA: ASCD.

[10] Vygotsky L. S. (1978). Mind in Society: The development of higher psychological processes. Cambridge, MA, U.S.A.: Harvard University Press.

3. 解方程要驗算嗎？

羅家豪

香港大學數學系

在本地的數學課程中，學生一般在高小開始接觸方程。以下是一個簡單例子：

例一： 解方程 $3x+4=22$。

解：

$$\begin{aligned} 3x+4&=22\\ 3x+4-4&=22-4\\ 3x&=18\\ \frac{3x}{3}&=\frac{18}{3}\\ x&=6 \end{aligned}$$

驗算： 當 $x=6$ 時，左方 $=3(6)+4=22=$ 右方。

因此 $x=6$ 是正確答案。

解方程後可以「驗算」，就是把答案[1]代入原方程看看兩邊是否相等。驗算的定位對於大部分老師和學生來說似乎都是「額外的一步」，也就是說驗算並不是解方程的一部分，只是如果題目要求，或是題目沒有要求但有時間剩下來，我們可以這樣做來檢驗答案是否正確。事實上，連課程文件[2]也是這樣寫的：

---

[1] 何謂方程的「答案」或「解答」？何謂「解」方程？這些問題和本文的主旨環環相扣，暫不在此扯得太遠，稍後會再提及。

[2] 《數學教育學習領域課程指引補充文件》小學數學科學習內容，課程發展議會，2017。

「學生須認識如何在解方程或解應用題後作驗算」

由此可見，驗算是在解方程「後」進行的，它並不被視為解方程的一部分。

現在看看另一例子（這種「牽涉根號且可變換為二次方程的方程」在現行高中課程中已被刪去）：

例二： 解方程 $\sqrt{x-2}=x-4$。

解：

$$\begin{aligned}\sqrt{x-2}&=x-4\\ x-2&=x^2-8x+16\\ x^2-9x+18&=0\\ (x-3)(x-6)&=0\\ x&=3 \text{ 或 } 6\end{aligned}$$

完成了嗎？還沒有！先來驗算一下：

當 $x=6$ 時，左方 $=\sqrt{6-4}=2=6-4=$ 右方。

當 $x=3$ 時，左方 $=\sqrt{3-2}=1$，可是右方 $=3-4=-1$！

因此 $x=3$ 須捨去，從而 $x=6$ 是方程的唯一解。

之前不是說驗算是事後額外的一步嗎？為何在上例中驗算突然變得必要（否則會多出一個錯誤的解）？筆者遇過的老師一般的解釋都是「因為曾經對方程兩邊平方，所以必

須驗算」。這個解釋有點似是而非：我們當然明白「對方程兩邊平方」會出現的問題，可是我們很容易構作出「沒有對方程兩邊平方，但仍必須驗算」的例子，因此這個解釋並沒有觸及問題的核心。

讓我們回到一些現行高中課程會出現的方程。以下例子取自某出版社的高中教科書：

例三：　解方程 $\log(3x-1)=1+\log(x+2)$ 。

解：

$$\begin{aligned}\log(3x-1)&=1+\log(x+2)\\ \log(3x-1)&=\log 10+\log(x+2)\\ \log(3x-1)&=\log 10(x+2)\\ 3x-1&=10(x+2)\\ -7x&=21\\ x&=-3\end{aligned}$$

當 $x=-3$ 時，$3x-1$ 和 $x+2$ 都是負數，即 $\log(3x-1)$ 和 $\log(x+2)$ 都是沒有意義的，所以原方程沒有實數解。

該書在題解後還有一個溫馨提示：

「注意解對數方程後必須進行驗算，以檢查方程中的每一個對數是否有意義。」

可是由此至終還是沒有解釋為何「必須」進行驗算，而跟例二一樣，有關解釋亦只針對特定的「對數方程」，仍是沒有解答在甚麼情況下「驗算」的角色會突然有所不同（我們總不可能把方程分成根式方程、對數方程、三角方程、……，再逐一判斷每種方程中驗算是額外還是必須的吧）。

事實上這樣的例子有很多，除了牽涉根號的方程和對數方程外，另一常見的類別是分式方程，例如：

例四：　解方程 $\frac{4x}{x^2-4}+\frac{1}{x+2}=1+\frac{2}{x-2}$ 。

解方程的步驟從略，反正結果就是其中一個解出來的根須捨去，這次的原因是它會使原方程的最少一個分母變成 0。然而各位讀者不妨思考一下：如果你只有紙筆而要構作一道這樣的試題（即解出來的根中有最少一個必須捨去），有甚麼好的方法？

在說下去之前，先看一個似乎不太相關的例子。

例五：　蛋糕每件售價 5 元，現在特價八折。買 5 打蛋糕，付款 1000 元，應找回多少？

解：　應找回的金額 $=1000-(5\times 80\%)\times(5\times 12)$

$$=1000-4\times 60$$

$$=1000-240$$

$$=760$$

上例與解方程無關，但大家可以注意一下題解的表達方式。在上例中，整個題解基本上只包含了一道等式，我們可以把它在同一行寫出來：

應找回的金額

$=1000-(5\times 80\%)\times(5\times 12)=1000-4\times 60=1000-240=760$ 。

只是習慣上一般會分成幾行，每行一道算式，這樣比較清

晰易讀。再比較一下之前「解方程」的例子，例如例一，不難發現題解的結構是不同的：它每一行都是一道等式(由於等式中含未知數，所以這等式又稱為「方程」)，而我們也無法把整個題解放在同一行表示。

筆者在小學階段曾遇過幾位數學老師教「解方程」，記得其中一位（只有一位）有提及過可以在題解的每行之間加上「⇒」符號，這樣的話例一的題解會變成這樣：

$$
\begin{aligned}
& 3x+4=22 \\
\Rightarrow\quad & 3x+4-4=22-4 \\
\Rightarrow\quad & 3x=18 \\
\Rightarrow\quad & \frac{3x}{3}=\frac{18}{3} \\
\Rightarrow\quad & x=6
\end{aligned}
$$

這樣做的話，「無法把整個題解放在同一行表示」的問題也解決了：

$$3x+4=22 \quad\Rightarrow\quad 3x+4-4=22-4 \quad\Rightarrow\quad 3x=18 \quad\Rightarrow\quad \frac{3x}{3}=\frac{18}{3} \quad\Rightarrow\quad x=6$$

上面提到，例一的題解中「每一行都是一道方程」。特別地，第一行是原來的「題目」，最後一行是「答案」，為甚麼一道方程可以成為另一道方程的「答案」？

所謂「解方程」，基本上就是「求未知數的所有可能值，使等式成立」。在上述「加了 ⇒ 號的例一題解」中，整個邏輯就變成「如果 $3x+4=22$ 成立，那麼 $x=6$ 成立」，也就是說「除了 6 以外，其他 x 值都無法滿足方程 $3x+4=22$ 」。

但這樣並不保證 $x=6$ 滿足方程，因此我們應該要驗算！可是我們也知道，每個「$\Rightarrow$」牽涉的逆命題也成立（這當然須要小心逐一檢查！），於是題解可重寫成

$$3x+4=22 \iff 3x+4-4=22-4 \iff 3x=18 \iff \frac{3x}{3}=\frac{18}{3}$$
$$\iff x=6$$

也就是說「$3x+4=22$ 成立 當且僅當 $x=6$ 成立」，這樣我們就可以確定 $x=6$ 是原方程的唯一解而不需驗算了。

這樣，隨後幾個例子的迷思也就水落石出了。在例二的題解中

$$\sqrt{x-2}=x-4 \Rightarrow x-2=x^2-8x+16$$

的逆命題並非真確（一般來說 $a=b \Rightarrow a^2=b^2$，但 $a^2=b^2 \not\Rightarrow a=b$），因此題解基本上證明了「$\sqrt{x-2}=x-4 \Rightarrow x=3$ 或 6」，也就是說「除了 3 和 6 以外，其他 x 值都無法滿足方程 $\sqrt{x-2}=x-4$」，於是只要（而且須要！）逐一檢驗 3 和 6 是否滿足方程，便能得出方程的所有解。同樣，大家自然也可以立即找到例三和例四「需要驗算」的原因，這裡不再多花篇幅。

以下例子取自 2018 年香港中學文憑試的延伸部分（單元二）：

例六: (a) 若 $\cot A=3\cot B$，證明 $\sin(A+B)=2\sin(B-A)$。

(b) 利用 (a)，解方程 $\cot\left(x+\frac{4\pi}{9}\right)=3\cot\left(x+\frac{5\pi}{18}\right)$，其中

$0 \le x \le \frac{\pi}{2}$ 。

解：(a)　從略

(b)　利用 (a) 部的結果可解方程如下：

$$\cot\left(x+\frac{4\pi}{9}\right)=3\cot\left(x+\frac{5\pi}{18}\right)$$

$$\sin\left[\left(x+\frac{4\pi}{9}\right)+\left(x+\frac{5\pi}{18}\right)\right]=2\sin\left[\left(x+\frac{5\pi}{18}\right)-\left(x+\frac{4\pi}{9}\right)\right]$$

$$\sin\left(2x+\frac{13\pi}{18}\right)=2\sin\left(-\frac{\pi}{6}\right)$$

$$\sin\left(2x+\frac{13\pi}{18}\right)=-1$$

$$2x+\frac{13\pi}{18}=\frac{3\pi}{2}$$

$$x=\frac{7\pi}{18}$$

看到這裡，大家當然立即會想到「需要驗算嗎」的問題。筆者跟一些中學老師討論過，大部分都覺得不需要。在考評局出版的題解和評卷參考[3]中，雖然有驗算的一步，可是不佔分數（也就是以上「沒有驗算」的題解也可得滿分）。題解中的每一步都是等價（即可用「 $\Leftrightarrow$ 」表示）的嗎？的確如此（這也得小心驗證，例如從第四行到第五行牽涉到

3 《香港中學文憑考試 2018 試題專輯：數學（延伸部分）》，香港考試及評核局，2018。

x 的取值範圍）。可是從第一行到第二行我們採用了 (a) 部的結果，而 (a) 部只證明了「若 $\cot A = 3\cot B$ ，則 $\sin(A+B) = 2\sin(B-A)$ 」而沒有證明其逆命題（儘管其逆命題也是真確的，事實上只要懂得證明原命題，則證明逆命題基本上不用多花太多功夫），因此如果使用 (a) 部的結果，從第一行到第二行只有「$\Rightarrow$」而非「$\Leftrightarrow$ 」。也就是說，除非在 (a) 部額外證明了「若 $\sin(A+B) = 2\sin(B-A)$ ，則 $\cot A = 3\cot B$ 」這個逆命題，否則以上的「滿分題解」中存在一個重大的邏輯漏洞！

筆者認為，姑且撇開驗算步驟不佔分數的問題（畢竟評分準則有其局限性和很多考慮因素），但讓學生清楚知道本題中驗算的必要性是很重要的。以上「沒有驗算的題解」即使得到滿分，也只是「幸運」而已（因為在本題最終得出的解毋須被捨去，而評分準則中亦未有把分數分配到驗算的步驟）。

更一般地，我們應培養學生判斷是否需要驗算的能力，而非只記著「解對數方程必須驗算」這類指引。事實上，在很多情況下，這種「每行寫一個命題（命題可以是等式、方程或其他形式）」而不搞清楚命題之間關係的論證方式，很容易會掉進邏輯陷阱。因此解方程時能養成「思考是否需要驗算」的習慣，對學習其他課題也會有所裨益。

以上談到的例子涵蓋根式方程、對數方程、分式方程、三角方程等。大家不妨思考一下一些相關的問題，例如：
解聯立方程要驗算嗎？

承上題，唯一解和無窮多個解的情況是否有分別？
在中學課程中，除解方程外，有沒有類似「需要驗算」的情況？

最後，以一個笑話式的例子作結，藉以說明弄清邏輯關係的重要性。當中所證明的結論固然是不正確的，大家可以細想一下問題的所在。

例七：證明 $2=4$ 。

解：考慮方程 $x^{x^{x^{\cdots}}}=2$ ，其中 $x>0$ 。解方程[4]如下：

$$x^{\left(x^{x^{\cdots}}\right)}=2$$
$$x^2=2$$
$$x=\sqrt{2}$$

再考慮方程 $y^{y^{y^{\cdots}}}=4$ ，其中 $y>0$。解方程如下：

$$y^{\left(y^{y^{\cdots}}\right)}=4$$
$$y^4=4$$
$$y^2=2$$
$$y=\sqrt{2}$$

[4] 當有「多於一層」的指數時，應從「上至下」計算，舉例說，$2^{3^{4^5}}=2^{\left(3^{4^5}\right)}=2^{\left(3^{\left(4^5\right)}\right)}$ 。

由於 $x=\sqrt{2}$ 滿足第一道方程，故此 $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}=2$ 。

由於 $y=\sqrt{2}$ 滿足第二道方程，故此 $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}=4$ 。

結合上述兩條等式，可得 $2=4$ 。

4. 一類分數的小數表示

梁景信
香港教育大學
數學與資訊科技學系

考慮分數 $x=\frac{p}{q}$，其中 $0<p<q$，且 p、q互質，把 x 化成小數，有以下三種情況：

（1） $q=2^m5^n$(m、n是非負整數)：有盡(terminating)小數，如 $\frac{1}{4}=0.25$；

（2） q和10互質：純循環(pure recurring)小數，如 $\frac{16}{33}=0.\overline{48}$（橫綫下的數字不斷重複，即 $0.\overline{48}=0.484848\cdots$）；

（3） $q=2^m5^nv$(m、n是非負整數，v和10互質)：混循環(mixed recurring)小數，如 $\frac{23}{60}=0.38\overline{3}$。

但有盡小數也可以表示成循環小數，如$0.25=0.24\overline{9}$。有關表示分數為循環小數的論述，可參考[1]和[2]。

以下介紹一個表示分數為循環小數的重要定理：

定理 1 假設 $p=a_n10^{n-1}+a_{n-1}10^{n-2}+\cdots+a_210+a_1$(一般寫作$a_na_{n-1}\cdots a_1$，$a_i\in\{0,1,2,\ldots,9\}$) 和 $q=10^n-$

$1 = \underbrace{99\cdots9}_{n個}$，則 $\frac{p}{q} = 0.\overline{a_n a_{n-1} \cdots a_2 a_1}$。

證明： $\frac{p}{q} = \frac{a_n 10^{-1} + a_{n-1} 10^{-2} + \cdots + a_2 10^{-n+1} + a_1 10^{-n}}{1-10^{-n}}$

$= (a_n 10^{-1} + a_{n-1} 10^{-2} + \cdots + a_2 10^{-n+1} + a_1 10^{-n})(1 + 10^{-n} + 10^{-2n} + \cdots)$

$= 0.\overline{a_n a_{n-1} \cdots a_2 a_1}$

應用定理 1，我們可以用簡單筆算，把分母數位是同一數字的分數化為小數。

例 1（分母的數位全是 9）把 $\frac{257}{9999}$化為小數。

解：把257 寫成 0257，應用定理 1 得 $\frac{257}{9999} = 0.\overline{0257}$。

以下例 2 和 3 中的分母可以很容易地倍大成$10^n - 1 = \underbrace{99\cdots9}_{n個}$。

例 2（分母的數位全是 1）把 $\frac{78}{111}$ 化為小數。

解： $\frac{78}{111} = \frac{78\times9}{111\times9} = \frac{702}{999} = 0.\overline{702}$。

例 3（分母的數位全是 3）把 $\frac{29}{33}$ 化為小數。

解： $\frac{29}{33}=\frac{29\times3}{33\times3}=\frac{87}{99}=0.\overline{87}$。

在以下例 4—8 中的分數，雖然分母皆不能倍大成 $10^n-1=\underbrace{99\cdots9}_{n個}$，但可以倍大成 $(10^n-1)\cdot10^m=\underbrace{99\cdots9}_{n個}\underbrace{00\cdots0}_{m個}$，然後把分數表示成 $\frac{x}{10^m}+\frac{y}{(10^n-1)10^m}$，其中 $1\le x<10^m$ 和 $1\le y<10^n-1$，只要找到 x 和 y，便可把原來的分數表示成混循環小數 。

例 4（分母的數位全是 2）把 $\frac{147}{222}$ 化為小數。

解：$\frac{147}{222}=\frac{147\times5\times9}{222\times5\times9}=\frac{6615}{9990}=\frac{x}{10}+\frac{y}{9990}$，其中$1\le x\le9$和$1\le y<999$，解

$999x+y=6615$ 得 $x=6$ 和 $y=621$，所以

$\frac{147}{222}=\frac{6}{10}+\frac{621}{9990}=\frac{6}{10}+\frac{1}{10}\cdot\frac{621}{999}=0.6+0.0\overline{621}=0.6\overline{621}$。

例 5（分母的數位全是 4）把 $\frac{35}{44}$ 化為小數。

解：$\frac{35}{44}=\frac{35\times25\times9}{44\times25\times9}=\frac{7875}{9900}=\frac{x}{100}+\frac{y}{9900}$，其中 $1\le x\le99$和 $1\le y<99$，解 $99x+y=7875$ 得 $x=79$ 和 $y=54$，

所以$\frac{35}{44}=\frac{79}{100}+\frac{1}{100}\cdot\frac{54}{99}=0.79+0.00\overline{54}=0.79\overline{54}$。

例 6（分母的數位全是 8）把 $\frac{573}{888}$ 化為小數。

解：$\frac{573}{888}=\frac{573\times125\times9}{888\times125\times9}=\frac{644625}{999000}=\frac{x}{1000}+\frac{y}{999000}$，其中$1\leq x\leq 999$ 和 $1\leq y<999$，解
$999x+y=644625$ 得 $x=645$，$y=270$ ，所以
$\frac{573}{888}=\frac{645}{1000}+\frac{1}{1000}\cdot\frac{270}{999}=0.645+0.000\overline{270}=0.645\overline{270}$。

例 7（分母的數位全是 5）把 $\frac{21}{55}$ 化為小數。

解：$\frac{21}{55}=\frac{21\times2\times9}{55\times2\times9}=\frac{378}{990}=\frac{x}{10}+\frac{y}{990}$，其中 $1\leq x\leq 9$ 和 $1\leq y<99$，解
$99x+y=378$ 得 $x=3$，$y=81$ ，所以
$\frac{21}{55}=\frac{3}{10}+\frac{1}{10}\cdot\frac{81}{99}=0.3+0.0\overline{81}=0.3\overline{81}$。

例 8（分母的數位全是 6）把 $\frac{127}{666}$ 化為小數。

解：$\frac{127}{666}=\frac{127\times3\times5}{666\times3\times5}=\frac{1905}{9990}=\frac{x}{10}+\frac{y}{9990}$，其中 $1\leq x\leq 9$ 和 $1\leq y<999$，解 $999x+y=1905$ 得 $x=1$，$y=906$ ，所以
$\frac{127}{666}=\frac{1}{10}+\frac{1}{10}\cdot\frac{906}{999}=0.1+0.0\overline{906}=0.1\overline{906}$。

最有趣的情況是分母的數位全是 7，以下例 9－11 都是把分母倍大為999999。

例 9（分母是 7）把 $\frac{2}{7}$ 化為小數。

解：$\frac{2}{7}=\frac{2\times 142857}{7\times 142857}=\frac{285714}{999999}=0.\overline{285714}$。

例 10（分母的數位全是 7）把 $\frac{46}{77}$ 化為小數。

解：$\frac{46}{77}=\frac{46\times 10101}{77\times 10101}=\frac{464646}{777777}=\frac{464646\div 7}{777777\div 7}=\frac{66378}{111111}=\frac{66378\times 9}{111111\times 9}=$

$\frac{597402}{999999}=0.\overline{597402}$。

注意：10101 是 7 的倍數，所以第一次擴分後的分子可以被 7 整除 。

例 11（分母的數位全是 7）把 $\frac{689}{777}$ 化為小數。

解：$\frac{689}{777}=\frac{689\times 1001}{777\times 1001}=\frac{689689}{777777}=\frac{689689\div 7}{777777\div 7}=\frac{98527}{111111}=$

$\frac{98527\times 9}{111111\times 9}=\frac{886743}{999999}=0.\overline{886743}$。

注意：1001 是 7 的倍數，所以第一次擴分後的分子可以被 7 整除 。

習題 1　　　把 $\frac{4321}{7777}$ 化為小數。

提示：把分母倍大為 $\underbrace{99\cdots9}_{12\text{ 個}}$。

習題 2　　　把 $\frac{13579}{77777}$ 化為小數。

提示：把分母倍大為 $\underbrace{99\cdots9}_{n\text{個}}$，$n=?$

習題 3　　　讀者自擬。

提示：不需要。

參考文獻

[1] Hardy, G. H. & Wright, E. M. (1979). *An introduction to the theory of numbers* (5th ed.). Oxford: Clarendon Press.

[2] Niven, I. M. (1961). *Numbers: Rational and irrational.* Providence: American Mathematical Society.

5. The *e* Story

WONG Hang Chi
International Christian Quality Music Secondary and Primary School
CHEUNG Ka Luen
The Education University of Hong Kong

1 Introduction

The Euler's number $e \approx 2.718281828$ plays a very special role in Mathematics, especially in Calculus. In the Mathematics Curriculum in Hong Kong, for instance, students may select the Extended Part Module 1 (Calculus and Statistics) or Module 2 (Algebra and Calculus), in which this concept is introduced. This learning objective is stated clearly in the Mathematics Curriculum and Assessment Guide (Secondary 4–6) provided by the Education Bureau.

For Module 1 students: "recognise the definition of the number e and the exponential series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$" and

For Module 2 students: "recognise the definitions and notations of e and the natural logarithm."

However, the introduction of e requires tactful planning. Some students found it difficult to believe in the existence of the limit of the sequence $\left(1+\frac{1}{n}\right)^n$; some struggled with the sum to infinity of the series $1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$ and wonder why this could possibly be finite; while some just could not imagine how it works. This article aims to discuss a different pedagogical approach to introduce this special number to secondary students, so that students could have a deeper understanding of the definition of e and the exponential function.

In this article, for the sake of simplicity, the sets of natural numbers, rational numbers and real numbers are denoted by $\boldsymbol{N}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ respectively throughout.

2 A Problem Solving Activity

2.1 A Function which Equals its Derivative

The lesson planning we would like to present here reverses the order of the traditional approach, which suggests the existence of e first and then goes on with the differentiation of polynomials, exponential functions, etc. We shall, however, delay the introduction of e after the discussion of the differential calculus of polynomials and the basic differentiation theorems such as Addition Rule, Product Rule, Quotient Rule and Chain Rule.

In this activity, students are first asked to find the derivatives of certain algebraic expressions like polynomials x^2, x^3, x^4,...

and rational functions $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$, ... Then, the teacher may guide students to summarize these results as the Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

where n is a non-zero integer. Or, equivalently stated, we have $x^{n-1} = \frac{d}{dx}\left(\frac{1}{n}x^n\right)$. The interesting part is that this result is not valid if $n = 0$, otherwise the right side of the equality results in the division by zero. Let us also investigate the left side by putting $n = 0$. It equals $x^{0-1} = \frac{1}{x}$. So, the question is: "the derivative of which function gives $\frac{1}{x}$?" From the above discussion, we know that if a differentiable function satisfies $\frac{dy}{dx} = \frac{1}{x}$, it must not be of the form $y = x^n$, where n is any integer. We need to think about other possibilities.

The teacher may then motivate the students to investigate the properties of this sought for function. If such a function $y = g(x)$ exists, then it must satisfy the equation $\frac{dy}{dx} = \frac{1}{x}$. According to the Chain Rule, we then obtain $\left(\frac{dy}{dx}\right)\left(\frac{dx}{dy}\right) = \frac{dx}{dx} = 1$, which implies that $\frac{dx}{dy} = x$. In other words, if x can also be expressed as a function of y, then the derivative of x equals itself! From the observation before, the function $x = f(y)$ cannot be a polynomial of y with any degree n, or otherwise, $\frac{dx}{dy} = f'(y)$ will be another polynomial of degree $n - 1$, which is not equal to $f(y)$. Intuitively, in order that the equality $\frac{dx}{dy} = x$ to hold true, the function $x = f(y)$ should grow more rapidly than any polynomial. At this stage, students might start to be curious to look for something of exponential growth!

2.2 A Limit as the Slope of Tangent

Let us investigate the exponential function to see whether it is indeed the solution to our problem. Let a be a positive constant such that $a \neq 1$. Students had already learned, in the Compulsory Part, that the exponential function $f(y) = a^y$ is well-defined for all $y \in \boldsymbol{Q}$. The teacher might explain to students that $\boldsymbol{Q}$ is dense in $\boldsymbol{R}$, and so this function could, somehow, be extended to be defined for all $y \in \boldsymbol{R}$.

We aim to find the value of a such that $\frac{d}{dy}(a^y) = a^y$, that is, $f'(y) = f(y)$. Because if this "idealistic" value was found, then the function $f(y) = a^y$ would satisfy the property that its derivative equals itself, which in turn solves our original problem.

Putting $y = 0$, in particular, we have $f'(0) = f(0) = a^0 = 1$. By the first principles, let us calculate the limit of

$$\frac{f(h)-f(0)}{h} = \frac{a^h-1}{h} \text{ as } h \to 0.$$

If this limit does exist, then $\lim_{h\to 0}\frac{a^h-1}{h} = 1$.

In general, assume that $y \in \boldsymbol{R}$, we have

$$\frac{f(y+h)-f(y)}{h} = \frac{a^{y+h}-a^y}{h} = a^y(\frac{a^h-1}{h}).$$

Taking $h \to 0$, we obtain

$$f'(y) = a^y \lim_{h\to 0}\frac{a^h-1}{h}$$

provided that the limit exists.

In other words, we may "localize" the study of the derivative at $y = 0$. The solution of the equation $f'(y) = f(y)$ will come down to the understanding of finding the value of a such that

$\lim_{h\to 0}\frac{a^h-1}{h} = 1$.

In the process of problem solving, it is beneficial to us if we can look at the problem in a different perspective. In fact, x and y are merely dummy variables. We shall also write $f'(x) = f(x)$ to represent the same relation. What is the geometric meaning of $f'(0) = f(0) = 1$? We may introduce a rectangular coordinate system to represent the graph.

We know that $f'(0)$ equals the slope of tangent of the curve $y = f(x)$ at $(0, 1)$. Using the slope-intercept form, the equation of this tangent line can be expressed as $y = x + 1$.

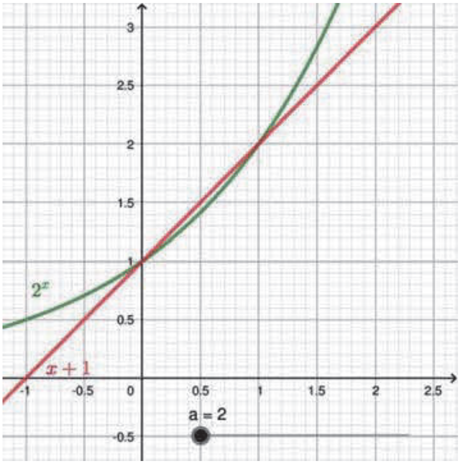
3 Exploration of the Exponential Function

3.1 The Graph of the Exponential Function

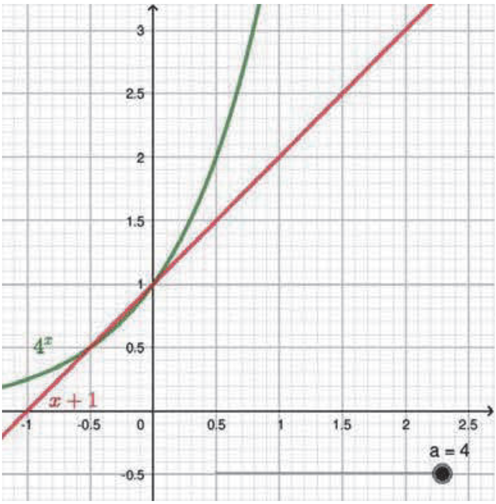
3.1.1 The Relation of $y = a^x$ and $y = x + 1$

One way to illustrate this is to use a Dynamic Geometry Software (DGS), such as GeoGebra, to demonstrate the situation. In the figure below, the slide bar controls the value of a, which is initially set at $a = 2$. As a varies, the software can immediately display the behaviour of the graph of $y = a^x$ so that it can be compared with the fixed straight line $y = x + 1$. This helps students to visualize the shape of the function in the investigation.

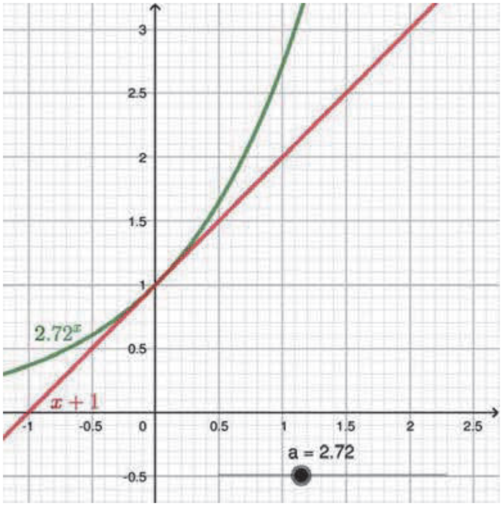
As we can see from the graph, when $a = 2$, the graph of $y = a^x$ intersects the straight line $y = x + 1$ at two distinct points $(0, 1)$ and $(1, 2)$.



On the other hand, if we drag the slide bar to, say, $a = 4$, the curve $y = a^x$ also intersects the straight line $y = x + 1$ at two distinct points $(0, 1)$ and $(-0.5, 0.5)$.



Combining these two observations, it is natural to guess that there must be an "idealistic" position between $a = 2$ and $a = 4$, at which the curve $y = a^x$ and the straight line $y = x + 1$has exactly one point of intersection. By adjusting the slide bar carefully, it could be found that the sought for value occurs at about $a \approx 2.72$, as shown in the following figure. We shall call this idealistic value e.



3.1.2 The Nature of the Points of Intersection

Let us formulate the above results more rigorously. We know that in general, for any constant $a > 1$, the curve $y = a^x$ cuts the straight line $y = x + 1$ at the point $(0, 1)$, and then they intersect again at another point (t, a^t) for some $t \in \boldsymbol{R}$, except when $a = e$, where the curve $y = a^x$ and the straight line $y = x + 1$ touch each other at exactly one point $(0, 1)$.

In the DGS, it could be observed that the point (t, a^t) will move towards $(0, 1)$ when the value of a approaches e. Intuitively, the point $(0, 1)$ serves as the "bending centre" of the curve $y = a^x$, which is concave upward for any $a > 0$ with $a \neq 1$.

The behaviour of the curve, however, changes at $a = e$. Intuitively, when $1 < a < e$, the rate of increase of the curve $y = a^x$ is less than 1 at the point $(0, 1)$, and so the other point of intersection (t, a^t) is at the right side of the y-axis. Therefore, we have $t > 0$. On the contrary, vice versa for the situation of $a > e$, we know that the other point of intersection (t, a^t) is at the left side of the y-axis, which implies that $t < 0$.

These observations can help us establish the relation between e and t (Nelsen, 2015). Thinking backward, we can consider t as an independent variable and express a as a function of t. Our aim is to take t to be really small, so that a can be made very close to e.

If $1 < a < e$, we have $t > 0$. At the point of intersection (t, a^t), we know that $a^t = 1 + t$, and so $a = (1 + t)^{\frac{1}{t}}$. Now, as t approaches 0 indefinitely from above, the value of a can be taken as close to e as we please from the below. The limiting case is that the two points of intersections coincide with one another, and the secant line, with slope 1, becomes the tangent line at the point $(0, 1)$. That is, the **right hand limit** exists.

$$\lim_{t \to 0^+} (1 + t)^{\frac{1}{t}} = e.$$

Similarly, for $a > e$, we have $t < 0$. In this case, we have also $a = (1+t)^{\frac{1}{t}}$. If t approaches 0 indefinitely from below, the value of a can also be taken as close to e as we please from the above. Therefore, the **left hand limit** also exists.

$$\lim_{t \to 0^-} (1+t)^{\frac{1}{t}} = e.$$

Summarizing these two results, we have

$$\lim_{t \to 0} (1+t)^{\frac{1}{t}} = e.$$

3.2 The Study of the Limit

3.2.1 Expressing e as the Limit of a Sequence

In particular, if we substitute $t = \frac{1}{n} > 0$, for $n \in \boldsymbol{N}$, we have $e > a = (1+t)^{\frac{1}{t}} = \left(1+\frac{1}{n}\right)^n$. On the other hand, if we substitute $t = \frac{-1}{n+1} < 0$, for $n \in \boldsymbol{N}$, then $e < a = (1+t)^{\frac{1}{t}} = \left(1-\frac{1}{n+1}\right)^{-(n+1)} = \left(\frac{n+1-1}{n+1}\right)^{-(n+1)} = \left(\frac{n+1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n+1}$. Hence, we have established the inequality

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

for all $n \in \boldsymbol{N}$.

Note that when $t > 0$, the value of $a = (1+t)^{\frac{1}{t}}$ increases towards e when t decreases, as shown in the DGS. Since $\frac{1}{n}$ decreases towards 0 as n increases, we know that the sequence $a_n = \left(1+\frac{1}{n}\right)^n$ is monotonic increasing. By using similar arguments, we can also deduce that the sequence $b_n = \left(1+\frac{1}{n}\right)^{n+1}$ is monotonic decreasing. As a result, we have $2 = a_1 \leq a_n \leq a_{n+1} < e < b_{n+1} \leq b_n \leq b_1 = 4$ for all $n \in \boldsymbol{N}$. Both of the sequences a_n and b_n are monotonic and bounded. By the Monotone Convergence Theorem for sequences, a_n and b_n must also be convergent! Furthermore, since $b_n - a_n = \left(1+\frac{1}{n}\right)a_n - a_n = \frac{1}{n}a_n < \frac{4}{n} \to 0$ when $n \to \infty$, by using the Sandwich Theorem, we know that the limiting values of a_n and b_n ought to coincide at the constant e. This ultimately explains why the definition of the Euler's number can be expressed in this peculiar form of

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

3.2.2 Estimating the Value of e

Once the students are convinced by the definition of e, the teacher may introduce some numerical values by using $a_n = \left(1 + \frac{1}{n}\right)^n$ to get a taste for the students. This sequence, however, converges quite slowly as n increases. Therefore, it would be beneficial to them if a spreadsheet software, such as Excel, is used instead of a scientific calculator. For instance, by putting $n = 10^k$, for $k = 1, 2, 3, \ldots$ and calculating the corresponding value of $\left(1 + \frac{1}{n}\right)^n$ in the software, one could obtain the following approximations, as shown in the table below.

n	$\left(1 + \frac{1}{n}\right)^n$
10	2.59374246
100	2.704813829
1000	2.716923932
10000	2.718145927
100000	2.718268237
1000000	2.718280469
10000000	2.718281694
100000000	2.718281798
1000000000	2.718282052
10000000000	2.718282053

Unfortunately, the drawback of this method to estimate e is that the accumulation error is very large when n exceeds 10^{10}. It is because the precision of the software or calculator is not enough to handle values of such a large scale. The first way to solve this problem is to use a software that support high precision arithmetic. The second way is to apply the exponential series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ at $x = 1$ (Dörrie, 1956). Nevertheless, the technical details are out of the scope of this article.

4 Conclusion

Now, we have found the answer to our original problem. Firstly, as we saw earlier in the previous section, for the exponential function $y = a^x$ to satisfy the equation $\frac{dy}{dx} = y$, the only possible value of a is e (Maor, 1994). Secondly, since the natural logarithm $y = \ln x$ is the inverse function of $y = e^x$, we conclude that $y = \ln x$ is a solution to the equation $\frac{dy}{dx} = \frac{1}{x}$. Moreover, the growth rate of $y = e^x$ is faster than that of x^n, whereas the growth rate of $y = \ln x$ is slower than that of x^n, for any $n \in \boldsymbol{N}$.

As a side product of the above investigations, we also discover the reason why the Euler's number e is chosen as the base of the natural logarithm. Consider $y = a^x$, then $x = \log_a y = \frac{\ln y}{\ln a}$, where $a > 0$ and $a \neq 1$. Consequently, we have $\frac{dx}{dy} = \frac{1}{y \ln a}$, and hence $\frac{dy}{dx} = y \ln a = a^x \ln a$. As a result, we would have a strange constant $\ln a$ multiplied to a^x.

In order to simplify the calculations of these differentiation formulas, the most "natural" choice is to take the base a of the exponential function to be e, so that "trouble maker" $\ln a$ becomes unity. An analogy to this situation is using radian measure in the trigonometric functions. For example, we can see that the formulas $\frac{d}{d\theta}\sin\theta = \cos\theta$ and $\frac{d}{d\theta}\cos\theta = -\sin\theta$ works well with a nice "symmetric" property, if θ is in radian. Nevertheless, if θ is measured in degree, we would obtain more complicated results $\frac{d}{d\theta}\sin\theta° = \frac{\pi}{180}\cos\theta°$ and $\frac{d}{d\theta}\cos\theta° = \frac{-\pi}{180}\sin\theta°$, which are not quite convenient in the computations.

To conclude, the pedagogical approach presented here makes use of DGS to visualize the property of the graph of the exponential function a^x throughout the investigation. It has the advantage that students can observe it more directly when they learn to solve the problem.

Frankly speaking, the rigor of the Mathematics could sometimes be quite terrifying to students. It is the responsibility of the teachers to explain these abstract concepts in a more intuitive way and to allow students to appreciate the beauty of Mathematics. We sincerely hope that the above mentioned teaching activities could benefit students and teachers struggling in the learning and teaching of Calculus.

References

[1] Curriculum Development Council and The Hong Kong Examination and Assessment Authority. (2017). *Mathematics: Curriculum and Assessment Guide (Secondary 4–6)*. Hong Kong: Education Bureau.

[2] Dörrie, Heinrich. (1965). Trans. David Anin. *100 Great Problems of Elementary Mathematics: Their History and Solution*. New York: Dover Publications.

[3] Maor, Eli. (1994). *e*: *The Story of a Number*. New Jersey: Princeton University Press.

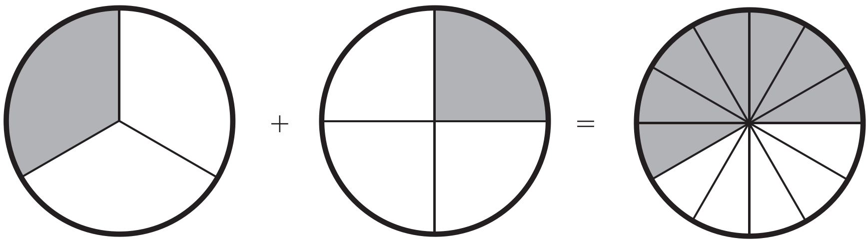
[4] Nelsen, Roger B. (2015). *Cameos for Calculus: Visualization in the First-Year Course.* Washington, DC: The Mathematical Association of America.

6. 正負數的加減[1]

黃毅英

退休數學教育工作者

這可能是耳熟能詳的話題，大家可以用很多圖或隱喻去解釋[2]，又或用數學的運算去推斷，但（起碼在一些學生而言）都搔不著癢處：因為做數（刻意用口語「做數」而不用問題解決，其實是指很具體的操作 manipulate 和運算）時根本不是這麼做！就如分數加減，我們會畫些圖形去解釋，做數時根本不會這麼畫，於是出現做了數才把圖畫回的情況。



$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

先看看以下幾題。

① 38 + 72

② 72 – 38

③ 38 – 72

④ –38 – 72

---

[1] 感謝鄧國俊博士提供寶貴意見。

[2] 例如見榊忠男（2002）。《愛麗絲與孫悟空的數學之旅》。台北：國際村文庫書店。

⑤ 38 + (–72)

⑥ 38 – (72)

⑦ –38 + (–72)

⑧ –38 – (–72)

我們想像一個「成年人」(如各位)如何「做」以上的題目。所謂「成年人」是指受過數學訓練又未忘記數學的人,不需要是數學家這類數學專業人士。所謂「做」當然是不用電腦或計數機去做。

首先,用數線跳前跳後只是一種解釋或輔助的意象,你不會向前跳 72 格吧!

① 是「正常的加法」

當然不用計數機或數線的話,我們會這樣加,這個是大家熟知的了。

$$
\begin{array}{r}
38 \\
+\quad 72 \\
\hline
10 \\
10 \\
\hline
110
\end{array}
$$

這其實用了「結合律」:

$38 + 72 = (30 + 8) + (70 + 2)$

$\qquad\quad = (30 + 70) + (8 + 2)$

= 100 + 10

= 110

② 72 – 38 是大家熟知的了，亦是用了結合律[3]：

72 – 38 = (70 + 2) – (30 + 8)

= (60 + 12) – (30 + 8)

= (60 – 30) + (12 – 8)

= 30 + 4

= 34

如涉及借位便稍複雜點。

③ 38 – 72 不夠減：反向變成 – (72 – 38)。72 – 38 是熟知的了。

④ 抽出負號得 – (38 + 72) 【用分配律「反拆括號」！】變成第①題。

⑤ 拆括號變成 38 – 72 ，即第③題。

⑥ 拆括號變第③題。

⑦ 拆括號後變 –38 – 72，即第④題。

[3] 見黃毅英、張僑平、許世紅、蘇洪雨、陳鎮民、張家麟、黃麗珍、謝明初、蔡勁航（2012）。《數學教師不怕給學生難倒了！——中小學數學教師所需的數學知識》。武漢：華中師範大學出版社。
繁體字版：香港教育局數學教育組（2013）。
https://www.edb.gov.hk/attachment/tc/curriculum-development/kla/ma/res/Cabinet%2014.pdf

⑧ 拆括號後得 –38 + 72，用交換律變成第②題。

一個受過數學薰陶的人很可能傾向於潔淨主義(不敢說「潔癖」:p)，希望能用單一方法處理同類問題。但以正負數加減而言，事與願違。我們很可能要接受不同情境用不同方法。不過是否要搞出八類這麼多呢？也不必。我們（起碼以我而言）的機略[4]（schema）很可能是：

有括號的話
- A：正常加法 (①)
- B：正常減法（即「夠減」）(②)
- C：減法但不夠減 (③)：把「–」號抽出來然後倒轉減
- D：$-a-b$ (a，b 為正數) 形式 (④)：也用上面方法把「–」號抽出來

換言之是 8 類濃縮為 4 類。而其實 C、D 最後會化成 A、B。故此只要

I. 看出不夠減的情況，

II. 懂得把「–」號抽出來（包括 C、D），

以上 4 類數就可化成兩類數。而這兩類數都是「正常」的

[4] 這類強調是做數時我們的機略而不一定是有人教我們必須這樣做。因為問題解決策略可以因人而異。

加減法。

把數題分拆成四類是否太麻煩呢？首先就算麻煩也得接受。此外，我們做到正負數乘除不也是要分成正正得正、正負得負、負正得負、負負得正嗎？

那麼數線解釋真的不會在做數過程中產生作用、變成「白學」嗎？這也未必，人處理問題往往不是單一策略（當然亦因人而異），數線除了給出概念上的完整解釋[5]外，在做數過程中，縱使採用上述機略，數線的意象很可能會產生作用[6]。例如對於 38 + 72（當然不會逐格數）便知道是向前（右）走、再向前，於是數會很大（相加）。72 – 38 會向前走後折返，故此數值會細。38 – 72 也一樣，故此數值（絕對值）不會是 72 及 38 的和那麼大，應該是它們的差。–38 – 72 便是向後走得很遠，應該是「負很多」……

我們只有從「成年人」做數的角度考慮才會既不失概念，亦能協助學生做數！

盲操瞎練、不求甚解當然不是我們想要的，但我們不必以二分法看待概念和運算。處理得宜，可以做到「從執行加深理解」這些都談過了[7]。現補充一點。何以我們好像要拚

[5] 數線亦是對「實數」（real number）最貼切的模型（model）。亦可參考註腳 2 的書。

[6] 亦可參考黃毅英、張僑平（2014）。數學教學的幾個最基本問題：做數、概念與理解。《學校數學通訊》。18 期，1-18。https://www.edb.gov.hk/attachment/tc/curriculum-development/kla/ma/res/smn_18.pdf

[7] 黃毅英（2007）。數學化過程與數學理解。《數學教育》25 期，2-18。

命撇除圖像和實際處境而重視算式呢？首先前者不等於概念。它們也只是概念的一些表像和模型吧了。此外，我們亦要考慮日後的發展。例如對數字很大、很多數字、非整數、甚至高維度......已無圖可畫了。所以我們早晚要脫掉畫圖進入符號世界。正如上面所說，適當的圖示（貼近兒童思維的圖示而不是爲了畫圖而畫圖）不只有助理解亦爲算式做數提供有幫助的意像當然何時「轉軌」就要看老師對學生的了解。〈從分數除法之例看螺旋變式課程〉一文[8]便提出多個連接點的想法。即在一段時間，學生同時運用圖示和算式（以這兩個表像爲例），並平行出現，讓他們知道和熟習它們的「互換」，這對「轉軌」可能順暢點。

[8] 孫旭花、黃毅英、林智中（2009）。從分數除法之例看螺旋變式課程。《基礎教育學報》18 期，1-19。

7. 淺說人工智能之數學方法

陳泳昌

香港大學博士研究生

jsphchan@connect.hku.hk

前言

近年世界各地均關注人工智能之推廣和培訓，香港政府在 2019 財政預算案亦提出「中學 IT 創新實驗室」計劃，向中學提供資助，籌辦有關嶄新資訊科技（包括人工智能、大數據等）的活動，以及購置設備和專業服務，學界一般做法是把這項目交予資訊及通訊科技科（ICT）教師負責[1]。然而人工智能是一門跨科的學問，除了電腦程式編寫外，還涉及廣泛的數學知識。今年筆者為中學教師多次主講人工智能的基礎培訓班，考慮到參與教師可能任教不同科目，筆者設計課程時少談數學，然而美中不足的是到關鍵之處，只能概括帶過。

自 2019/2020 學年開始，本港一些大學開辦人工智能學士學位課程，這些課程的必修科目包括線性代數（Linear Algebra）、多元微積分（Multivariate Calculus）、概率、統計等，可見學習人工智能，須有紮實的數學基礎。誠然，

[1] 香港政府資訊科技總監辦公室曾在 2019 年 5 月向立法會財務委員會提交有關「中學 IT 創新實驗室」之撥款申請，同年 6 至 7 月亦曾舉辦三場計劃簡介暨分享會，向學界介紹計劃的背景資料及申請辦法。然至截稿之日，立法會財務委員會議程有所調動，是項撥款申請暫時押後。

在中學推行人工智能教育，必須顧及學生能力，不能揠苗助長；然而數學教師亦未嘗不可輔助配合，替學生打好根基，為香港培育創新科技人才。本文試舉數項人工智能所運用之數學方法，有些方法稍為超出中學數學教育的範疇，教師閱畢本文，或可視作持續進修，文中提及一些有關人工智能的實際應用，亦可作為談助，提高學生學習數學之興趣。

線性代數

在中學數學教育，矩陣往往是用來求解線性方程組，然而事實上，很多實用問題均能用矩陣表示。且舉一例：表一是六間物業的資料（包括面積 x_1、房數 x_2、層數 x_3、樓齡 x_4、樓價 y），我們希望建立一線性模型以預測樓價，其中 θ_1、θ_2、θ_3、θ_4 是模型參數，表示 x_1、x_2、x_3、x_4 的權重（Weight）。用 θ_0 表示線性模型的截距（Intercept，又稱偏置項 Bias，即不包含任何自變量的常數項），又設 x_0 恆等於 1，如此整個線性模型可統一格式，寫成 $y = \theta_0 x_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \theta_4 x_4$。

	x_0	面積 x_1	房數 x_2	層數 x_3	樓齡 x_4	樓價 y
物業甲	1	2104	5	1	45	460
物業乙	1	1416	3	2	40	232
物業丙	1	1534	3	2	30	315
物業丁	1	852	2	1	36	178
物業戊	1	564	2	1	15	250
物業己	1	745	3	2	20	220

表一：物業資料及價格數據

上述線性模型可用矩陣表示，設 X 是物業資料矩陣，θ是模型參數矩陣，Y 是樓價矩陣：

$$\mathrm{X} = \begin{bmatrix} 1 & 2104 & 5 & 1 & 45 \\ 1 & 1416 & 3 & 2 & 40 \\ 1 & 1534 & 3 & 2 & 30 \\ 1 & 852 & 2 & 1 & 36 \\ 1 & 564 & 2 & 1 & 15 \\ 1 & 745 & 3 & 2 & 20 \end{bmatrix}, \ \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}, \ \mathrm{Y} = \begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \\ 250 \\ 220 \end{bmatrix}$$

建立樓價預測模型，即是找出模型參數矩陣θ，以使 Xθ儘量接近 Y（Xθ是矩陣乘法）。用正規方程（Normal Equation）可求出θ，公式是 $\theta=(X^TX)^{-1}X^TY$（篇幅所限，容筆者省去公式證明），其中 X^T 是 X 的轉置矩陣（Transpose），X^{-1} 是 X 的逆矩陣（Inverse）。如上例，可得：

$$\theta = \begin{bmatrix} 234.2603 \\ 0.1978 \\ 29.5394 \\ -64.0579 \\ -6.0921 \end{bmatrix}$$

即樓價預測模型是 $y = 234.2603 + 0.1978x_1 + 29.5394x_2 - 64.0579x_3 - 6.0921x_4$。上例旨在說明矩陣有多方面用途，不單是為了令算式看起來更簡潔而已。一些計算軟件（例如 MATLAB，以及 Python 的 NumPy 套件）能專門處理矩陣運算，若電腦程式使用這些套件，運算速度可大幅提升。

上文所論的樓價預測模型，其實是線性回歸（Linear Regression）問題。一些人工智能教材以此為引子（例如見周志華，2016，53 頁），這是因為處理線性回歸的數學手

法，堪可用於人工智能方面。首先，人工智能須要進行大量矩陣運算。例如人工神經網絡（Artificial Neural Network）雖早於 1950 年代提出，但此後數十年間，一般研究大多限於 3 至 4 層的淺層網絡，學者嘗試研究深層網絡時遇到不少困難，其中之一是當時電腦硬件尚未足以應付計算需求。後來發現人工神經網絡和電腦圖像處理之算法有相通之處，二者皆須處理大量的矩陣運算。2012 年，以 Alex Krizhevsky 為首的多倫多大學團隊借用圖形處理器（Graphical Processing Unit，GPU，是專門處理圖像的晶片）以加速訓練深層網絡，參加 ImageNet 大規模視覺識別競賽（ImageNet Large Scale Visual Recognition Competition，ILSVRC）並獲得冠軍（Krizhevsky et al., 2012），自此運用 GPU 晶片以加速矩陣運算，已成為現代人工智能系統不可或缺的技術。

多元微積分

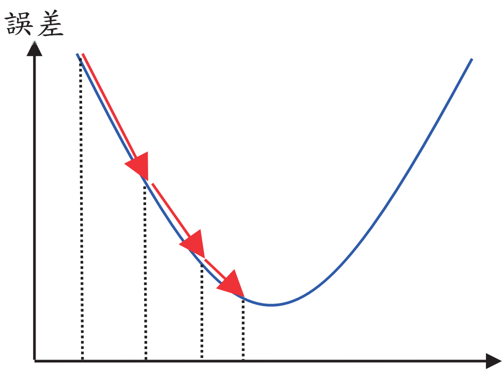
上文談到可用正規方程求出線性模型的參數，然而此方法不一定實際可行。這是因為正規方程須要計算逆矩陣，某一矩陣 M 的逆矩陣，等於 M 的伴隨矩陣（Adjoint Matrix）除以 M 的行列式（Determinant）。如果 M 的行列式十分接近 0，逆矩陣算法須要除以一個接近 0 的數，計算可能出現困難。另外，逆矩陣算法十分繁複，如果矩陣十分龐大（例如有數十萬間物業資料），即使用電腦計算，亦會耗費不少時間。因此要找出線性模型的參數，我們可用梯度下降法（Gradient Descent），具體算法是：

1. 隨機選取數字（任選皆可），作為模型參數的估計值（用 $\hat{\theta}$ 表

示）；

2. 定義誤差函數，例如可用均方誤差（Mean Square Error）；
3. 計算此模型的預測樓價（用 $\hat{Y}$ 表示），即 $\hat{Y} = X\hat{\theta}$；
4. 計算預測樓價 $\hat{Y}$ 與實際數據 Y 的誤差；
5. 根據誤差，調整模型參數估計值 $\hat{\theta}$；
6. 重覆步驟 3 至 5，使總誤差值不斷減小，達致最優化，即預測樓價 $\hat{Y}$ 逐漸逼近實際數據 Y，$\hat{\theta}$ 逐漸逼近用正規方程求出的模型參數 θ。

換言之，我們希望用迭代法（Iterative Method），找出一模型參數組合，令誤差函數達致最小值。步驟 5 是這迭代法的要點，要找到誤差函數的極小值，我們調整模型參數估計值 $\hat{\theta}$ 的方向，須與誤差函數當前點的梯度（Gradient）方向相反。見圖一，我們任意選取的一點，誤差較大，誤差函數的梯度為負數，所以模型參數 $\hat{\theta}$ 的調整方向為正數。由於要計算誤差函數的梯度，這就須要使用微分法，本文只略示大概，詳細算法不贅。



圖一：梯度下降法

中學數學教育只考慮一元函數的微分，然而上文談及的線性模型參數共有 θ_0、θ_1、...、θ_4，誤差函數亦是由 θ_0、θ_1、...、θ_4 組成，因此我們便要運用偏微分（Partial Differentiation）

以求多元函數的梯度。偏微分其實不難理解，例如有一函數 $z = x^3y^2 + 4xy + x$，其中 x 和 y 皆為自變量，若求 z 關於 x 的偏微分(用 $\frac{\partial z}{\partial x}$ 表示)，則視 y 為常數，因此 $\frac{\partial z}{\partial x} = 3x^2y^2 + 4y + 1$；與之相若，求 z 關於 y 的偏微分，則視 x 為常數，即 $\frac{\partial z}{\partial y} = 2x^3y + 4x$。

梯度下降是一重要算法，在不同場合均有實際應用。於人工智能方面，訓練人工神經網絡，本質同樣是運用梯度下降，不斷調整網絡的參數，以令誤差函數達致最小值，即預測值與實際數據最密合。這套反向傳播算法（Backpropagation）是 Geoffrey Hinton 在 1986 年提倡（Rumelhart *et al.*, 1986），至今仍是人工智能的核心算法之一。由於有 GPU 加速矩陣運算，加上其他種種算法改進，人工神經網絡可以不斷擴張，現今已出現數百層甚至一千餘層的網絡。2012 年 Alex Krizhevsky 團隊勝出 ILSVRC 競賽後，深層神經網絡成為主流技術，此後歷屆 ILSVRC 競賽優勝團隊所使用的人工神經網絡，其參數數量有幾千萬甚至逾一億之多（見表二）。換言之，訓練人工神經網絡，便是運用多元微積分及梯度下降，以解決一個牽涉幾千萬至一億餘個變量的最優化問題。

ILSVRC 競賽年份	人工神經網絡名稱	參數
2012	AlexNet	62,378,344
2014	GoogLeNet	6,998,552
2014	VGG16	138,357,544

ILSVRC 競賽年份	人工神經網絡名稱	參數
2014	VGG19	143,667,240
2015	ResNet50	25,610,216
2015	ResNet101	44,654,504
2015	ResNet152	60,344,232

表二：著名人工神經網絡的參數數量

概率

機器學習（Machine Learning）是人工智能的一個重要分支，希望電腦分析數據後，自行掌握規律並能作出準確預測，上文談及的人工神經網絡，便是屬於機器學習的範疇。要作出事件預測，便須處理概率。

機器學習的其中一項重要應用，是為資料作分類（Classification），入門教材一般會用 MNIST 手寫數字資料集（LeCun *et al.*, 1998）作示範。這個資料集（見圖二）提供 60000 張手寫數字圖像作模型訓練，另有 10000 張圖像作模型測試。我們讓電腦自行分析訓練集內 60000 張圖像後建立模型，希望這模型能準確推知測試集內 10000 張圖像是對應何種數字。模型訓練完成後，我們

圖二：MNIST 手寫數字資料集

（圖片出處：維基百科）

輸入一張手寫數字圖像，電腦便會給出 10 個數字 p_0、p_1、…、

p_9，分別代表這圖像對應數字 0 至 9 的概率。例如一張圖像似 1 又似 7，那麼電腦分析後便會給出 10 個概率，p_1 和 p_7 的數值應該較高，其他 p 值則較低。由於已知 MNIST 手寫數字資料集只有數字 0 至 9 的圖像，因此 $p_0 + p_1 + \ldots + p_9 = 1$。這類問題稱為多類別分類（Multi-class Classification），即是有眾多互斥類別（Mutually Exclusive Class），我們希望求出資料屬於各種類別的可能性。

除此之外，又有所謂多標籤分類（Multi-label Classification），這些標籤是各自獨立，並不互斥。且舉一例，我們希望人工智能系統把報刊新聞分類，標籤有三：科技新聞、財經新聞和體育新聞。有些新聞固然可以明確分類為科技、財經或體育其中一類，但亦有新聞可能科技、財經二者相關（例如科技公司的股價表現）；財經、體育二者相關（例如某一公開上市球會的財務狀況）；科技、體育二者相關（例如運動科學）；三者皆相關（例如運用高科技舉辦盛大運動會，帶來可觀的財政回報）又或三者皆無關（例如時事新聞）。人工智能系統分析報刊新聞後，便會給出 3 個數字 p_t、p_f、p_s，分別代表科技、財經、體育標籤的概率，例如有關科技公司股價的報刊新聞，p_t 和 p_f 的數值應該較高，p_s 的數值則較低。然而與多類別分類不同，多標籤分類的各項概率皆各自獨立，總和不一定是 1。

由此可見概率計算對人工智能系統之重要性。上例所談只是十分基本的概率應用，中學數學教育亦談及條件概率（Conditional Probability）和貝葉斯定理（Bayes' theorem），這些概率知識在不少人工智能研究皆派上用場。例如有關

寫作或翻譯的人工智能系統，每寫一個字，皆須考慮上下文脈絡，而不是隨機寫作。人工智能當中一些有關自然語言處理（Natural Language Processing，NLP）的研究課題，是探討如何創立語言模型（Language Model），即是設定短語或句子的概率分佈。假如給出某一句子的前半部（用事件 A 表示），這些概率分佈可以幫助我們預測其下一個字或下半部句子（用事件 B 表示）。此類人工智能研究，就涉及條件概率之應用，即已知事件 A 發生的情況下，計算事件 B 發生的概率。

<u>統計</u>

機器學習本質上是由已知求未知，即是讓電腦先分析現有數據，建立模型後，希望用這模型處理新的數據，亦有準確的預測。要評估這些模型，可以使用統計學的置信區間（Confidence Interval）。例如一些人工智能的研究項目，提出某一模型的預測準確度若干，亦會給出預測準確度的95%置信區間，如此我們便有更全面的評估。

另一方面，近年人工智能系統的重大成就，是建基於大量的數據分析。但假如我們首次接觸某個數據集，便把數據直接輸入電腦，進行機器學習，效果不一定理想，這是因為我們對這些數據完全陌生。此時首要工作，是運用各種統計工具（例如最大值、最小值、平均數、中位數等等），結合各種統計學圖表，把數據呈現眼前，使我們對數據有一基本理解，知道應該運用何種手法進行機器學習，才有理想效果。這步驟稱為探索性數據分析（Exploratory Data Analysis，EDA）。

結語

上文談及的數學知識，大致相關於香港現時高中數學科的延伸部分，包括單元一（微積分與統計）及單元二（代數與微積分）。現今人工智能的發展方興未艾，筆者希望本文引起教育工作者以至社會各界人士對高中數學科延伸部分之重視，以期積極培育人才，推動本港創新科技之發展。

參考文獻

[1] 周志華（2016）。《機器學習》。北京：清華大學出版社。

[2] Krizhevsky, A., Sutskever, I., and Hinton, G. E. (2012). ImageNet classification with deep convolutional neural networks. In *Advances in neural information processing systems*, 25, 1097-1105.

[3] LeCun, Y., Bottou, L., Bengio, Y., and Haffner, P. (1998). Gradient-based learning applied to document recognition. *Proceedings of the IEEE*, 86(11), 2278-2324.

[4] Rumelhart, D., Hinton, G., and Williams, R. (1986). Learning representations by back-propagating errors. *Nature*, 323, 533-536.

8. 在數學科推行 STEM 教育

何保光老師、龐家樑老師、陳穗園老師
香港道教聯合會雲泉學校

STEM 的起源

隨著現今社會科技急速的進化，單一的學科已不能配合社會發展需要，科技學習已無法再依循過去的「試誤學習」模式，或是「手工訓練」模式，而是必須嘗試統整與設計製作產品的相關數學或科學原理，以作為其改良或創新的依據(游光昭、林坤誼，2007)。因此，美國早於 2003 年已開始提倡發展 STEM 教育(Stem Education)，美國總統奧巴馬更於 2014 年投放 31 億美元於 STEM 教育，可見 STEM 教育早已成為近年教育方向的新指標。香港教育局亦於 2015 年初政策文件中表示會強化發展 Science(科學)、Technology(科技)、Engineering(工程)及 Maths(數學)四個主要範疇，STEM 便是由這四個範疇組合而成，目的是裝備學生面對社會在經濟、科學與科技一日千里的發展。有部分學校更會加入 Art（視藝）的元素，稱為 STEAM。

本校推行 STEM 教育計劃

STEM 教育可分為兩種模式：

模式一：建基於一個學習領域課題的學習活動，讓學生綜合其他學習領域相關的學習元素

模式二：透過專題研習讓學生綜合不同學習領域的相關學習元素。

有見及此，本校於 2015 年成立 STEM 發展組，開始規劃推

行 STEM 教育的安排，如安排教師培訓、統籌各科組的協調等。並於 2016 年開始正式推行以問題為本方式的跨學科活動，名為「STEAM Project」。計劃從四至六年級甄選了八十位學生分成十六組進行為期三個月的專題研習，每組由一位數學科或常識科老師指導學生進行科探和如何設計製作產品，因為教學的最終目標是要使學生能自主地解決各種問題(施良方，1996)，其他科組老師會各司其職去配合活動，如中文和英文科老師會教授演說技巧；視藝科老師會提供設計展示攤位的意見；電腦科亦會教授學生製作簡報和三維空間的立體圖形等。在活動的完結時，學校會舉行 STEAM 學習成果分享會。

過往三年的 STEAM Project 詳情如下：

2016-2017 年度：

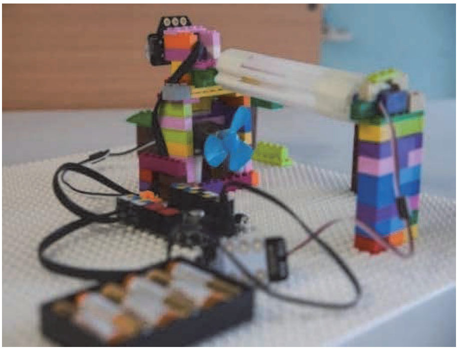
主題：飲食實驗室				
題目：	1)天然清潔方法	2)健康汽水	3)零碳煮食	4)種植法研究
				

2017-2018 年度：

主題：空氣與水	
1. **Return of the clean air**	2. Our beloved ones deserve cleaner air
3. **The dirty air, the cleaning home**	4. Elimination of dirty air
5. **Clean drinking water in the Park**	6. Test heavy metal inside our drinking water
7. **Turning river water to clean drinking water**	8. Find the clean drinking water in the wild
9. **Saving poor chef**	10. Find the perfect mask
11. **Saving bloody nose**	12. Helping "Hot Dog"
13. **Less is more**	14. Water tank won't break the bank
15. **Who stole my sticky**	16. Time reserve

2018-2019 年度：

主題：Better Living from a small step to a BIG SMILE	
1. **B for Bench, P for People**	2. Elderly Trainer
3. **Happy Bird**	4. IDD Bus Stop
5. **PC Swing**	6. Mosquito Terminator
7. **"Power Ranger" Exercise, Charge, Share**	8. Reality Map
9. **Sharing Electronic Wheelchair**	10. Shiny Sun Shelter
11. **Smart Lampposts**	12. Smart Zebra Crossing
13. **The "Food Spreads Caring" App**	14. The Smart Bin
15. **The "Wun Tsuen Go" Exercise App**	16. Wheelchair Partner – Anywhere Board



如何在數學科推行 STEAM 教育

本校已經累積了幾年舉辦 STEAM 教育的經驗，但是以問題為本的 STEAM 教育都是以常識科和科學科作為主導，而數學科只是擔當數據收集、分析和量度的支援角色。為了讓學生進一步體驗數學與 STEM 息息相關的關係，本校自 2017 年開始除了進行 STEAM Project 外，還參與由教育局舉辦「探討及發展於小學數學推展 STEM 教育的有效策略種籽計劃」，以一個學習領域課題的學習活動，體驗 STEM 教學的樂趣。

計劃會於每學年挑選兩個學習領域的課題而進行，為了減輕推行 STEM 時對科組同事的影響，計劃會在不同年級進行。在正式教授課堂前，老師會與教育局的人員合作設計課堂的內容，除了搜集資料外，還會進行實驗的測試，讓教師更了解和體驗課堂的難點，從而令老師更有效地運作課堂。

推行 STEM 數學教育，我們的理念是希望透過生活情境上遇上的問題作為引入，使學生明白學習課題的需要，並刺激學生設計出產品或利用圖像分析資料。課堂的設計包含以下的模式來進行：

投入(Engage): 引起學生的好奇、興趣、和投入。

探索(Explore): 提供學生建構學習經驗的機會。

解釋(Explain): 學生解釋所學到的東西，並加以改良。

工程(Engineer): 學生將所學的知識，應用到日常生活，將概念和技術應用到主要問題，以獲得更深的理解。

評鑑(Evaluate): 讓師生瞭解學習的效果。

於 2017-2018 年度進行的 STEM 課題和班級：

班級	課題
四年級	密鋪
六年級	圓形圖

於 2018-2019 年度進行的 STEM 課題和班級：

班級	課題
二年級	光與影
二年級	重量

以四年級 STEM 課堂為例，選擇「密鋪」為課題的原因是與學校生活情景有關（曾經有學校職員於較滑的地磚上滑倒而導致骨折），所以老師建議重鋪學校內較滑的地磚，以免同類事情再次發生。老師須要引導學生學習「密鋪」的需要，讓學生明白「密鋪」的特性，然後以科學教學的「預測、觀察、解釋」模式，讓學生估計哪些圖形可以「密鋪」，之後分組動手嘗試及觀察結果，再解釋「密鋪」的原理。在明白「密鋪」的原理後，配合其他 STEM 元素，透過觀察不同種類的地磚及防滑物料（科技）去了解「摩擦力」（科學），再運用網上資源協助自行設計一個地磚的樣板及嘗試進行「密鋪」（工程）。

推行 STEM 教育的總結及展望

推展 STEM 教育提供了一個很好的契機讓老師重新審視教學的設計，將課堂內容更貼近生活，更能配合其他 STEM 知識一起運用，使課堂的學習更加豐富，令學生明白數學科與生活息息相關。

要全面在數學科推展 STEM 教育，除了透過科組同事之間的通力合作，鍥而不捨的探索學科知識和對 STEM 的熱衷；與有 STEM 教育經驗的專家進行交流，亦可以讓老師汲取更多寶貴的經驗，從而啟發更多可行的方案去設計課堂，令教學更有效地去豐富學生的學習效能，這才是 STEM 教育成功與否的關鍵。在現今科技一日千里的社會，STEM 教育確實能有效提升學生的學習興趣及養成良好的思考能力，令學生能建立自信迎接未來的新挑戰！

參考文獻

[1]游光昭，林坤誼(2007)。數學、科學、科技統整課程對不同學習風格學習者在學習成效上之影響。教育研究學報。

[2]施良方(1996)。學習理論。高雄：麗文文化。

9. Effective Teaching of Extended Modules --- Reorganization of Compulsory Part

IP Ka Fai Gavin
Kwun Tong Government Secondary School
gavinip@edb.gov.hk

As a senior form Mathematics teacher since the implementation of new course structure in 2009, I have been keeping my close eyes on the trend of development of our advanced level Mathematics – Extended Module 1 and 2, and feel regret on their relatively low intakes for the time being.

In my observation, I would summarise the following reasons briefly for an unsatisfactory number of candidates in taking Extended Modules in Hong Kong.

1. Non–elective Status / University Entrance Requirements
2. Restrictions on intake Requirements for S.4 Selection
3. Insufficient School Support on Time Allocation of Lessons
4. Quality / Varieties of Textbooks of the Extended Modules
5. Suitability of Taster Programme to arouse Students' Interest
6. Parents' and Alumni's Influence on Students' Preference

Some factors are out of our control like the school management's decision on intake pre–requisites and number of lessons assigned, influence of parents and alumni on selection, etc. Nevertheless, as professional teachers we can always do something to change the worsening situation and to make our teaching and learning more effective from the departmental point of view. My major concern in this article is about the possible reorganisation of Mathematics (Senior Form Compulsory Part) which sounds effective to stimulate the number of intakes for the time being.

We should be aware that in our Extended Modules Curriculum, there are certain topics which require the pre–requisite knowledge covered in the Compulsory Part.

Extended Module 1

- Logarithmic Transformation --- Logarithmic Definition / Properties
- Equation of Tangent (Differentiation) --- Equation of Straight Line
- Whole Statistical Module --- Advanced Probability and Statistics

Extended Module 2

- Mathematical Induction --- Factor Theorem
- Trigonometry --- More About Trigonometric Identities
- Equation of Tangent (Differentiation) --- Equation of Straight Line

What is the current scenario, however? This may be most frustrated in reality. We can find that the majority (if not all) of the pre–requisite knowledge of two Extended Modules have to be taught in their lessons, because the teaching sequence of our Mathematics curriculum set by the publishers does not match with the curriculum of Module 1 or 2.

Our teachers thus need to arrange large quantity of after–school and long–holiday supplementary lessons on the Extended Modules to cope with this challenge. Nevertheless, it is not rare that the students' participation in extra–curricular activities or private tutorials will obviously affect their rate of attendance, while their lack of motivation or tiredness would also have negative impacts on the overall learning efficiency.

By considering the inadequately allocated lessons, such time wastage should be completely avoidable and it is very necessary to identify some measures to optimise our efficiency in lesson conduction. My candid suggestion is collaboration and determination of our panel members in Reorganization of Teaching Sequence of our Mathematics (Compulsory Part) with the main objectives as follows:

- ✓ The pre–requisite knowledge of Extended Modules need not be taught
- ✓ All pre–requisite knowledge are taught earlier under the Compulsory Part of Mathematics Curriculum
- ✓ Relieve the long–existing insufficient time of the

Extended Modules

- ✓ Optimize the time efficiency teaching Extended Modules in lessons
- ✓ Minimize the number of supplementary lessons of the Extended Modules
- ✓ Arouse more the students' interest in learning the Extended Modules
- ✓ Allocate more time for both Modules in drilling higher–order questions
- ✓ Minimise impact on topic transition of Mathematics (Compulsory Part)
- ✓ More collaboration among the colleagues of Mathematics Department

We could move forward to analyze how the curriculum of Mathematics may be reorganized for our goal achievement. After extensive analysis, I have designed the following teaching sequence which could fulfill all the set targets as mentioned above:

Suggested Teaching Sequence in Secondary 4 Term 1

1. **Quadratic Equations in One Unknown (I)** --- Foundation in problem–solving about Binomial Theorem of M1 / M2
2. **Exponential and Logarithmic Functions** --- Foundation in teaching Logarithmic Transformations of M1
3. **Equation of Straight Line** --- Useful to find Equations of

Tangents in Differentiation part
--- Can collaborate with colleagues in Physics Department to teach it in S.3

4. **Polynomials** --- Factor Theorem is useful in Mathematical Induction of M2
5. **Trigonometry (I)** --- Useful in Trigonometric Module of M2
6. **Permutation and Combination** --- Foundation for the whole Statistical Part of M1

Suggested Teaching Sequence in Secondary 4 Term 2

1. **Quadratic Equations in One Unknown (II) ---** Discriminant, Sum / Product of Roots and Complex Numbers
2. **Functions and Graphs** --- Quadratic Functions, Axis of Symmetry, Optimisation
3. **Probability** --- Foundation for Statistical Part of M1 (Addition and Multiplication Laws, Complementary Events, Conditional Probability, etc.)
4. **Statistics** --- Foundation for Statistical Part of M1 (like Standard Deviation and Variance, concepts of Normal Distribution, etc.)
5. **Linear and Quadratic Inequalities**

Suggested Teaching Sequence in Secondary 5 Term 1

1. **More about Equations** --- Higher Order, Fractional, Algebraic or Radical Equations
2. **Graphs of Function in Transformation** --- Useful for Graph Sketching (Differentiation) of M2
3. **Trigonometry (II)** --- Sine and Cosine Formulas, Area of Triangles, Herons Formula, 3–Dimensional Solids
4. **Basic Properties of Circles** --- All Angular and Chord Theorems

Suggested Teaching Sequence in Secondary 5 Term 2

1. **Variation**
2. **Arithmetic and Geometric Sequences**
3. **Locus**
4. **Equation of Circles**
5. **Linear Programming**

As a frontline teacher, I understand that any reform would encounter hesitation and worries among our teaching colleagues as they have to be responsible to different stakeholders. I have thus outlined several points drawing your attention and see whether they are useful to clarify your concerns:

- ✓ The pre–requisite knowledge of both Extended Modules will be taught as early as possible to facilitate their smooth teaching. The curriculum of S.4 Mathematics is reconstructed the most to cater for such changes, but this new teaching sequence is workable and logically sensible after considering the pre–requisite knowledge of different topics of Compulsory Part as well.

- ✓ The workload of each school term above is similar and reasonable. Under such arrangement, it is expected that no supplementary lesson has to be arranged and all Secondary 6 lessons could be allocated for comprehensive review and practices. If the teacher colleagues prefer to teach slower, some Secondary 5 topics could be shifted to Secondary 6 Scheme of Work flexibly. It is because they are not the pre–requisite knowledge of Extended Modules that they can be taught at any reasonable time in Secondary 5 or 6 to cater for school individual needs.

- ✓ An innovative design above is breaking the chain order of Quadratic units in Secondary 4. Some colleagues may argue whether it is unfeasible in doing so. In my opinion, however, putting them together does not mean the students would learn better. My scheme can in turn provide buffering time for the lower achievers to grasp the concepts gradually. The Mid–Yearly Examination would compel student’s revision on techniques in solving quadratic equations and possession of clearer concepts to learn Nature of Roots and Graphical Properties in Term 2.

- ✓ One crucial factor in implementation is whether your school has determination to make such a dramatic change. This needs the consent of Panel members and the successful convince against our school management. I would ask the fellow colleagues to think seriously if it is really necessary to stick to the teaching sequence of textbook publishers and whether the curriculum reorganisation as suggested can really benefit our students of Extended Modules in long term.

Why don't we propose initiatively to the authority for a reform? The potential adaptation of my designed teaching sequence with school allocation of more lessons can result in termination of endless supplementary lessons, which will likely arouse the students' interest and help to increase the number of candidates taking two Extended Modules. Is that what we desire for more students taking advanced level Mathematics courses?

In the meanwhile, we can extensively analyse the relationship between our Extended Modules and the STEM Education to upgrade their level of significance and the students' interest in study. Some possible examples include Statistical Models of Business Decision Making, Modeling for Surveillance of Infectious Disease, Environmental Protection on Wastage of Natural Resources, etc. It is highly appreciated if our colleagues are willing to devote time in its promotion and to share the valuable resources for more brainstorming.

New Senior Secondary Academic Structure has been implementing in Hong Kong over 10 years. It is delighted to see that our local universities have started to weight the two Extended Modules more regarding their status of JUPAS applications, and their level of attractiveness is envisaged to be increased with more high calibers to choose for the time being. If we could optimise the lesson time of our Extended Modules in teaching and learning, I am pretty sure more senior form students would like to take their courses for maintaining our competitiveness in the territory.

10. 由零開始的電子學習

孔令仁

元朗公立中學校友會鄧兆棠中學

當我初作為一位數學老師的幾年，我花了非常大的力氣去設計每一個課堂、教學工作紙、小測、練習，當時我告訴我自己花下的這些時間建立起來的材料，在我餘下的教學生涯中也可以用到，所以花多少時間也是十分值得的。

大家可以想像到，要在學校推動電子學習困難重重。現今改變速度比以前快很多，大家想想當 2008 年索尼的藍光碟擊敗東芝的 DVD，不消數年 DVD 已被 USB 所取代，光碟機已不再是一台電腦的必需品了，及後雲端技術的普及，和別人分享只需一個雲端硬碟便可輕而易舉地辦到，短短十年間已有如斯變化。問一個出賣年齡的問題，大家還記得那張 3.5 吋 1.44MB 的 Floppy Disk 嗎?由 1981 年推出直至 2003 年開始式微，2009 年正式停產，經歷達 28 年之多，改變速度之不同大家可見一斑。

面對改變，有些人會否認世界及環境的變化，並認為不須對這些變化作出反應。不知大家有沒有聽過以下的說話:
“我現在的教學方法行之有效，學生公開試成績理想，我想不到為甚麼要作出改變。”
“電子學習只是一場大龍鳳，只是做給人看，沒實際用途的。”
“我一聽到電子學習就覺得很煩。”

但我個人認為面對變化，唯一方法是以變化應對變化，我們教導學生“學如逆水行舟，不進則退”。現今世代中，教學也何嘗不是呢? 但我不認為電子教學是要來一場文化大革命除舊迎新。數學教學的發展應該建基於過往的經驗上，配合資訊科技提高課堂的深度和廣度，突破傳統教學的盲點以達致更佳的教學效果。

電子教學紮根在良好的土壤上才能健康地成長開花，而這土壤便是教師，良好的土壤為電子教學提供養分，帶來新氣象。因此如何帶動教師推行電子教學，是一項極為艱難同時十分重要的一環。一所機構內一定不會所有人也是樂於成為改變先驅者，這類型的人大約只有 30%左右，其餘當中有不少是抱觀望態度的。

作為帶領電子學習的牽頭人，一年前我對電子學習是一無所知，例如：甚麼叫學習管理系統 LMS，工具型 App 與內容型 App 的分別，甚麼是 BYOD? 所以須把握每一個學習的機會，參與教育局舉辦的研討會、觀看不同網頁介紹各種工具、與朋友提問討論。只有增值自己豐富自己才能為同事提供支援，電子學習才能得以推行。另外，一個人能力有限，透過招募不同學習領域的先導教師建立開荒團隊，共同研究，提供不同視野，電子學習才能顧及不同的層面。

在嘗試的過程中，我曾經迷失，我亦看到很多同事經歷著我曾犯的錯誤。開始時，我學習不同的資訊科技教學工具，然後不斷在想如何在課堂中使用這些工具，慢慢地我變得為使用這些工具而設計我的課堂，每一個堂也是為了“展

示”這些資訊科技工具，變成了為用而用。我很記得那天凌晨時分，我在為下星期的開放給同事觀課的電子學習課堂籌劃著，我突然想起我的教學法，如：教學目標、課前預習、同儕協作教學、教師提問、點撥……。為甚麼不是我設計課堂時最核心的部份，我知道我迷失了。突然間想起陶淵明<<歸去來兮>>中的一節: 悟已往之不諫，知來者之可追。實迷途其未遠，覺今是而昨非。跟著我思考一個課堂教學法和資訊科技工具的關係，以下是我個人的一點看法：

資訊科技的角色

Role of Information Technology

一個課堂中最重要的是這個課堂的教學目標，無論教學法還是資訊科技工具最終也是為了成就教學目標，永不能離開教學目標這個核心。教學法與資訊科技工具是相輔相乘的，有些時候資訊科技工具幫助教學法實行得更有果效，同時資訊科技工具也刺激了新教學法的創新及發展，最後達致教學目標更有效地達到，讓學生學得更好。就如我之前所說，電子學習不是除舊迎新，它和自主學習、協作學習是良好的合作夥伴。

對我來說，另一個最令我苦惱的問題是網絡的負荷與穩定

性問題。其中一個令教師對電子學習有戒心的就是網絡穩定性問題，這方面的擔憂不難理解，當你設計了一堂電子學習課堂，在進行時突然發生硬件、軟件、網絡等技術問題時，對於任教老師來說可以用“災難”來形容。我自己也經歷了多次技術性的故障，在學生的課堂、教職員專業發展活動、工作坊等，我只能說那種徬徨簡直是非筆墨所能形容。對網絡認識的迫切性是我在學期中後段參與教育局舉辦關於 BYOD 的研討會及工作坊時才開始明白的，但作為一個當年非主修電腦/資訊科技的大學生，網絡設置當中所牽涉到的知識對我來說極度艱深，我甚至連和 ITSS 溝通的語言也沒有。結果我付出了很多時間去認識相關知識，讓自己有一定的基礎去了解基本情況及和相關同事溝通。若要推行電子學習，我建議非相關主修的同工與校內資訊科技主任保持良好的關系與溝通，他會成為你推動電子學習的助力。反之没有他的支援，電子學習會舉步為艱，很難成功。

電子學習的發展是一條長遠的路，很難短期內達到成果，計劃須以乎合校情的步伐推行，亦要兼顧同事相異的進度，欲速則不達。我服務的學校剛完成三年發展週期的第一年，電子學習是一棵剛發芽的幼苗，一不小心便會毀掉。下一年是週期的第二年，目標是深化與普及化第一年建立的內容，不少問題開始表面化，每一步就像摸著石頭過河，本人不才實在不知最後是否能成功推行，令莘莘學子學得更好，同事能享受電子學習帶來的優點。以上僅為本人膚淺的感受，還望各位包涵，更望同工多多提點讓我知所不足而有所進。

11. 電子教學於數學科的推行與反思

林澔基老師、關子雋老師

隨著第四個資訊科技策略於數年前開展，各學校在無線網絡的基建及流動裝置的購置大致完成。老師們心裡不禁要問這一波的電子學習浪潮，跟以往的有何不同呢？

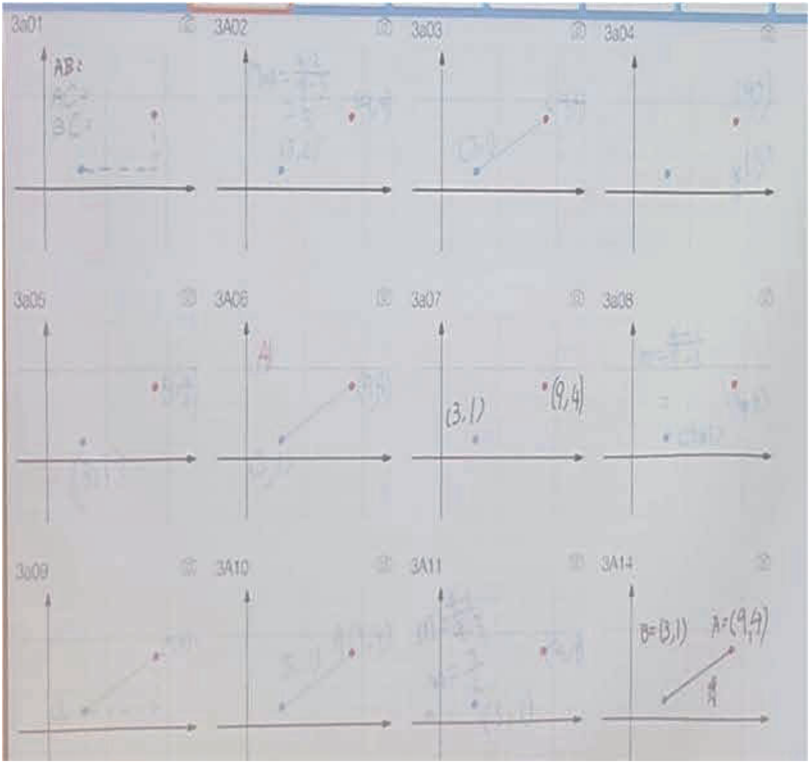
筆者近年跟不同學校的資訊科技統籌、數學科老師有不少接觸，發覺大家對電子學習的想法的差異很大。而這些多樣性，究竟會令電子學習百花齊放，還是會迷失方向呢？

為了探討這個問題，筆者在一次教師發展課程中，問了在座老師兩條問題：「甚麼是電子學習？」和「你期望電子學習做到甚麼？」。過程中，筆者用了標籤雲工具來收集老師的意見。而以下是他們的意見截圖：

對於電子學習的定義，老師們的想法較多樣，有無紙化、教材用多媒體及電子化、網上學習、用電子產品學習等......。但當問及電子學習的期望時，意見則較集中於提升學與教的互動、提高學習深度和效度以及如何更有效地善用課時。這眾多的想法，若老師對以上的想法沒有理念的一一實行，恐怕老師會陷入一次又一次的事倍功半的困境，隨時還會招來同事譏諷為「搞大龍鳳」。

每當筆者遇到熱心於推動電子教學的老師遭遇以上的困境時，筆者心裏不禁想起過往十多年來走過的冤枉路，亦會提醒同工推行電子學習的 SAMR Model。該 Model 由 Dr. Ruben Puentedura 提倡，他定義了資訊科技在學與教的四個階梯，依次為最底層的 Substitution (S，取代)、Augment (A，擴增)、Modification (M，改良)以及 Redefinition (R，重新定義)。Dr. Ruben 建議在課堂中若要運用資訊科技的話，儘量應在較高 M 或 R 階梯才使用，以獲取較高的課堂效能。若以廿年前香港第一波的電子學習浪潮的主角－PowerPoint 來衡量的話，它屬於哪個階梯呢？一般來說，PowerPoint 投影片是書本內容的重點節錄。就這種投影片來說，它屬於取代黑板或用高影片的用具，是 S 的階梯。那麼在投影片中加入動畫，影片或音效呢？這會使該投影片移上一個階梯，到 A 的階梯。但是，Dr. Ruben 並不鼓勵在這兩個階梯使用資訊科技，因為這兩個階梯的使用效能不高，對教學無顯著的幫助。這亦印證了當年不少老前輩認為運用 eBook 或 PowerPoint 等工具輔助教學是「搞大龍鳳」的想法是不無道理的。

同工們該怎樣做才能善用電子教學呢？回到本文開初時的問題，就是同工該如何看待電子教學。同工們會傾向以現有教學工具來理解電子教學，這是人之常情，但若以 SAMR model 來審視的話，這難免會陷入流於取代的圈套，令電子學習的優勢未能發揮。究竟電子學習的優勢在哪裡呢? 筆者認為，要回答這問題，首先要回到事物的本質，電子教學的背後是電腦科技，無論是平板、手機或手提電腦，都是電腦的一種，是不同規格(Form Factor)的電腦的展現而已。多年來電腦化在我們日常生活中的應用有（一）自動化，即將重覆的事物有效率的處理，如批改多項選擇題，複製文件等。（二）數據化，將紙本的檔案變為電子檔，儲存教學材料時既節省地方亦更方便分析。（三）微型化，把以往多部機器，如電腦，相機，錄音筆等都化成一體了。當認識到電腦的優點後，在處理電子教學時亦應按以上長處出發。以下筆者會介紹一下電子工具的種類和特點。一般來說，電子教學工具分為兩大類：學習管理平台(Learning Management System， LMS) 以及一般應用程式(Applications，坊間稱之為 App)。兩者的區別是 LMS 能夠管理學生的電子書/作業或習題並將學習過程記錄下來，而後者則多為單一用途的程式。這兩種軟件宜分工合作，並互相整合以發揮電子教學的優勢。例如老師可在 LMS 中連結不同 App 所發出的家課，並將學生在不同 App 的家課得分在 LMS 中整合，以助評估學生表現。在以上的例子當中，如何展現筆者所提及電腦的三大優勢呢？在 LMS 中，當老師分發電子堂課時，會將堂課複製給每一位學生並將學生的堂課連結到老師帳戶，如圖(一)所示：

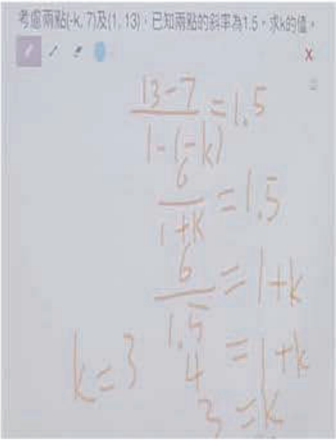


圖(一)

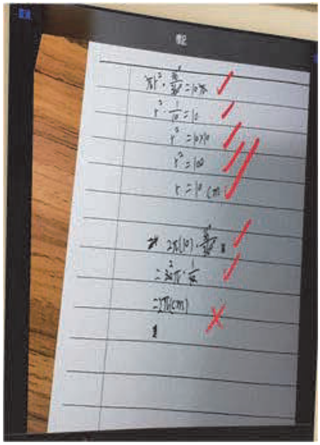
運用適切的 LMS，老師能對學生的堂課表現一目了然，哪批學生較快完成，哪些毫無頭緒，一清二楚。這就是要發揮老師專業的時候了：照顧學習差異和提供及時適切的回饋。例如，教師可透過系統分發附加作業予提早完成堂課的同學；而同時亦可對無頭緒的學生作出及時的提示。另外，教師在展現學生課業時不再受匯報人數以及時間所限制，讓老師有更多的時間與學生討論學習成果並點出同學們的常犯錯誤。在課後，學生們亦可登入系統重溫上課所作的堂課和老師的回饋，這樣便能加深學生的記憶。在這

示例中，老師發揮了電腦自動化和數據化的優點，而且達到 SAMR Model 中的改良等級。因為電子工具的應用不單取代(Substitute)紙本堂課，亦能藉著即時展現學生的學習情況而收擴增(Augment)之效，而老師能藉此節省課堂時間並作出更多的適切的回饋以達課堂改良(Modification)之效。

究竟上述的課堂有沒有局限？而老師該怎樣做才能讓電子課堂達到(Redefinition，重新定義)的境界呢？第一條問題較易回答。因為現實中沒有甚麼是沒有局限的，但限制在哪裡呢？ 以圖(二)為例，學生只寫了數行便沒空位寫下去，要寫在左旁才可。為甚麼平板的尺寸並不少，但為何學生寫不了幾行就沒位了？值得留意是這位同學並沒有用平板筆來寫，而是用了手指來寫。用指頭實再難以把算式寫得字小而整齊，亦寫得較慢。怎辦？解決方法有二，(一)、使用專用平板筆，但缺點是較少平板支援專用平板筆，而該類筆亦不一定隨平板附送，或要另外多付數百元，加重使用成本，而且有些平板筆需要充電或更換電池。這些缺點令不少學校卻步。另一種解決方法則是拍攝堂課簿/工作紙，如圖(三)。



圖(二)



圖(三)

方法（二）的好處不少，可以保留學生的書寫習慣的同時，亦利用到大多平板均配備的相機把習作電子化，沒有明顯成本。這能實現實有賴電子產品微型化所賜。

那麼同工應怎樣開展電子教學呢？我認為，小步子和耐性是需要的。近年，關愛基金為子女於推行 BYOD 的學校就讀的低收入家長提供經濟資助，而不少公司亦為相關學校提供解決方案。筆者聽聞有部分未有太多電子教學經驗的學校直接借助有關的方案推行 BYOD，步伐之快實在令人擔心。或許讀者會問，筆者作為電子教學的推動者，反而不建議儘快推行是何道理？其實一個課堂要做好，是極考驗老師的教學法的。老師們要熟習手上的工具是要時間練習的，所以，小步子運用電子教學有其需要的，例如，老師可選一些應用程序(App)如 Nearpod、Kahoot 等小試一兩個課堂活動。由於此類應用程序功能大多較單一，亦沒有

需要設定登入名稱及密碼。這類的程式需要較低的技術門檻便能操作，較易上手，實在適合初用電子學習的老師嘗試。

隨著老師在課堂運用 Apps 輔助教學的技巧日漸純熟，而學生亦熟習運用電子工具上課時，教師們便可以考慮使用學習管理系統(Learning Management System，LMS) 來管理不同平台的電子課業並記錄學生學習進程，以進一步發揮電子工具的威力。當大多數科目都如此推行時，這樣才值得考慮推行 Bring Your Own Device, BYOD 的政策。這樣既能確保電子器材的充分利用，亦有理據要求學生家長為子女添置平板電腦作為學習裝置。有了完備的學習管理系統，方便家長查看學生的電子習作進度，這亦能使家長更放心學校推行電子教學。

總括而言，要使電子教學成功，老師的努力，學生的熟練，家長以及學校的支持是缺一不可的，希望筆者多年來在推行電子教學碰的釘，撞的板，能幫助有心推動電子教學的同工們能省一點時間，多一點成功感。

12. When Probability meets STEM

WONG See Yan
Yuen Long Merchants Association Secondary School

Our school joined the captioned "Seed" project since 2016. Three years have passed and we have tried out seven STEM activities. In this article, the writer would like to share one of the recent STEM activities conducted at secondary five classes in this school year.

Genetics and DNA (More about Probability)

At the beginning of the lesson, the writer told the class a story about the inheritance. To arouse students' interests, students first were attracted by a legacy of $1 000 000 000 000 000 000 000 that will be inherited to a rich man's children only. They were then given a checklist on the man's characteristics, like "Curly tongue" and "Thumb bending". They were asked if they fitted all the characteristics and thus they might be the child of this rich man. Students were eager to check the characteristics and compared which one got the most items.

The teacher introduced the meaning of dominant gene and recessive gene. The teacher then told the students a news related to a couple of Fujianese carriers of the thalassemia gene. They gave birth to his son four years ago. His son unfortunately inherited his severe illness of thalassemia and suffered from long-term blood transfusion. In order to get the cord blood transplant to save the first child, even if only a quarter of a healthy baby, they still decided to risk a second child. They were then asked how they could get the probability of $\frac{1}{4}$. Using "A" stood for dominant gene and "a" for recessive gene, most of them could get the value by either using the tree diagram and tabulation method they learnt in junior secondary, or by using the probability rules they learned in the senior secondary curriculum.

<u>Tabulation method</u>

F \ M	A	a
A	AA	Aa
a	Aa	aa

<u>Method of using the multiplication rule in the senior secondary</u>

$$P(\text{healthy baby}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Students were also assigned to complete the 2×2 table for other combinations.

There should be 6 combinations.

F \ M	*A*	*A*
A	*AA*	*AA*
A	*AA*	*AA*

F \ M	*A*	*A*
A	*AA*	*AA*
a	*Aa*	*Aa*

F \ M	*A*	*A*
a	*Aa*	*Aa*
a	*Aa*	*Aa*

F \ M	*A*	*a*
A	*AA*	*Aa*
a	*Aa*	*aa*

F \ M	*A*	*a*
a	*Aa*	*aa*
a	*Aa*	*aa*

F \ M	*a*	*a*
a	*aa*	*aa*
a	*aa*	*aa*

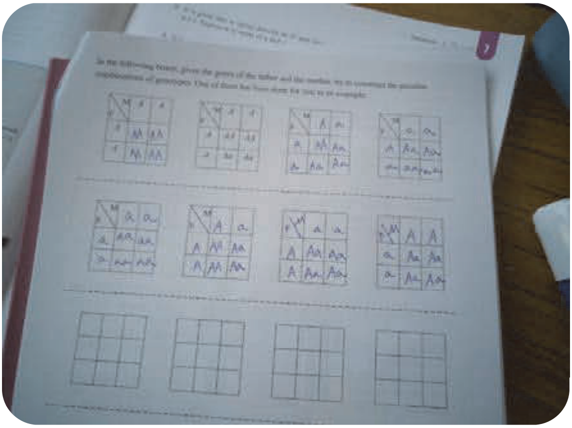
However, some students claimed that they found 9 combinations. After checking, they found out that three of them were redundant.

F \ M	*A*	*a*
A	*AA*	*Aa*
A	*AA*	*Aa*

F \ M	*a*	*a*
A	*Aa*	*Aa*
A	*Aa*	*Aa*

F \ M	*a*	*a*
A	*Aa*	*Aa*
a	*aa*	*aa*

<u>Student's work</u>

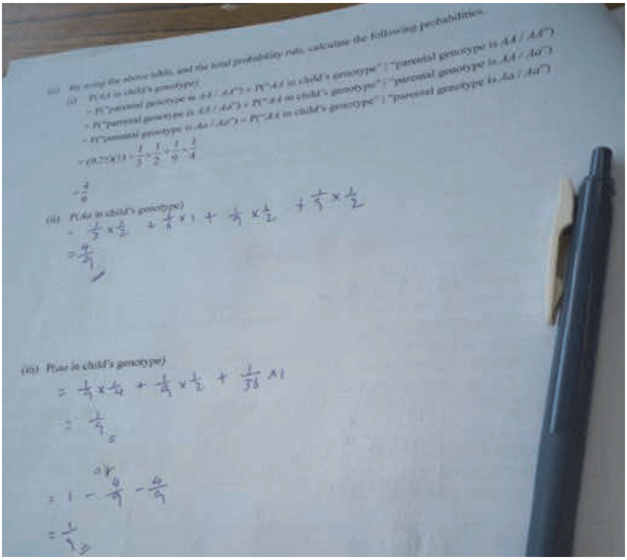


Would the population with characteristics represented by the dominant gene increased after several generations, and the population with characteristics represented by the recessive gene became zero after hundred years? Hardy, a British Mathematician in year 1908, showed that this phenomenon would NOT occur. Hardy with Weinberg, who was a biologist, published this result in a magazine. This was called the "Hardy–Weinberg principle".

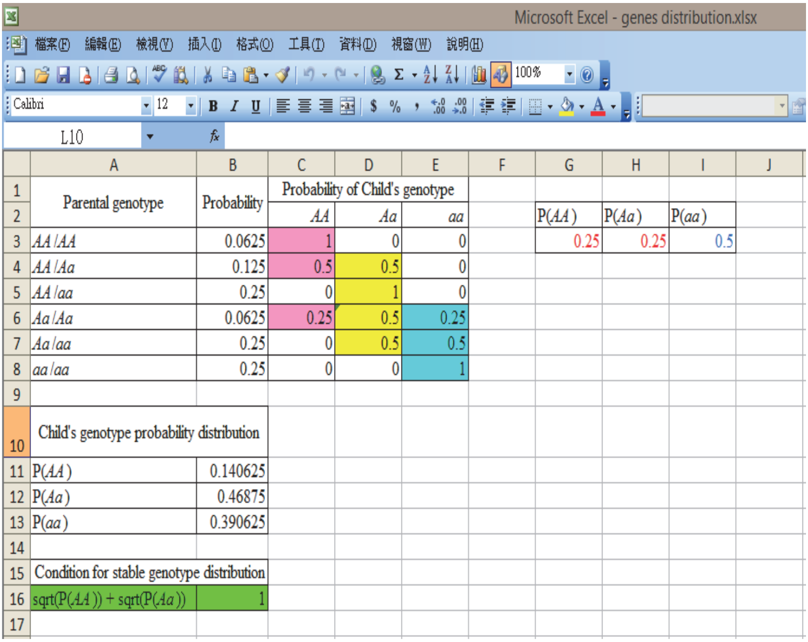
The climax was when students were going to verify the "Hardy–Weinberg principle" by using the multiplication rule in probability. The activity was divided by two parts: one by using numerical value, and one by using algebra.

For the numerical value part, we first assumed that in parental genotypes, $P(AA) = \frac{1}{2}$ and $P(Aa) = \frac{1}{3}$. Then students found out the probabilities of *AA* in child's genotype, *Aa* in child's genotype and *aa* in child's genotype.

<u>Student's script</u>



With the help of the spreadsheet program, students could easily got the results for different values of P(*AA*) and P(*Aa*).



	A	B	C	D	E	F	G	H	I	J
1	Parental genotype	Probability	Probability of Child's genotype							
2			*AA*	*Aa*	*aa*		P(*AA*)	P(*Aa*)	P(*aa*)	
3	*AA* / *AA*	0.0625	1	0	0		0.25	0.25	0.5	
4	*AA* / *Aa*	0.125	0.5	0.5	0					
5	*AA* / *aa*	0.25	0	1	0					
6	*Aa* / *Aa*	0.0625	0.25	0.5	0.25					
7	*Aa* / *aa*	0.25	0	0.5	0.5					
8	*aa* / *aa*	0.25	0	0	1					
9										
10	Child's genotype probability distribution									
11	P(*AA*)	0.140625								
12	P(*Aa*)	0.46875								
13	P(*aa*)	0.390625								
14										
15	Condition for stable genotype distribution									
16	sqrt(P(*AA*)) + sqrt(P(*Aa*))	1								
17										

Most students were able to verify, by numerical values, that

$$\sqrt{P(AA \text{ in child's genotype})} + \sqrt{P(aa \text{ in child's genotype})} = 1.$$

However, was the result true for ALL possible values of P(AA) and P(Aa)? Algebra would be the solution to this part.

Assume that in parental genotypes, P(AA) = p_1, and P(Aa) = p_2. Then, P(aa) = p_3, where $p_3 = 1 - p_1 - p_2$.
Therefore, we only needed to know the values of p_1 and p_2 in order to perform our calculations.

Parental genotype	Probability of such combination of parental genotypes	Probability of child's genotype for such combination		
		AA	*Aa*	*aa*
AA / *AA*	$(p_1)^2$	1	0	0
AA / *Aa*	$p_1p_2 + p_2p_1 = 2p_1p_2$	$\frac{1}{2}$	$\frac{1}{2}$	0
AA / *aa*	$p_1p_3 + p_3p_1 = 2p_1p_3$	0	1	0
Aa / *Aa*	$(p_2)^2$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
Aa / *aa*	$p_2p_3 + p_3p_2 = 2p_2p_3$	0	$\frac{1}{2}$	$\frac{1}{2}$
aa / *aa*	$(p_3)^2$	0	0	1

According to the above table, the probability distribution of the child's genotypes would be as follows.

P(AA in child's genotype) $= p_1^2 + \frac{1}{2}(2p_1p_2) + \frac{1}{4}p_2^2$

$$= p_1^2 + p_1p_2 + \frac{1}{4}p_2^2$$

$$= \left(p_1 + \frac{1}{2}p_2\right)^2$$

$$= \left(\frac{2p_1 + p_2}{2}\right)^2$$

P(*Aa* in child's genotype)

$$= \frac{1}{2}(2p_1p_2) + 2p_1p_3 + \frac{1}{2}p_2^2 + \frac{1}{2}(2p_2p_3)$$

$$= p_1p_2 + 2p_1p_3 + \frac{1}{2}p_2^2 + p_2p_3$$

$$= \frac{p_2^2 + 4p_1p_3 + 2p_1p_2 + 2p_2p_3}{2}$$

$$= \frac{p_2(p_2 + 2p_1) + 2p_3(p_2 + 2p_1)}{2}$$

$$= \frac{(p_2 + 2p_3)(p_2 + 2p_1)}{2}$$

P(*aa* in child's genotype) $= \frac{1}{4}p_2^2 + \frac{1}{2}(2p_3p_2) + p_3^2$

$$= \frac{1}{4}p_2^2 + p_3p_2 + p_3^2$$

$$= \left(p_3 + \frac{1}{2}p_2\right)^2$$

$$= \left(\frac{2p_3 + p_2}{2}\right)^2$$

In what condition would give rise to a stable genotype distribution?

A stable genotype distribution meant that: the ratio of the population with characteristics represented by the dominant gene, to the population with characteristics represented by the recessive gene, would become constant, i.e. the ratio of the child's genotype would be the same as the parental genotype.

Since P(*AA* in child's genotype) and P(*aa* in child's genotype) were $\left(\frac{2p_1+p_2}{2}\right)^2$ and $\left(\frac{2p_3+p_2}{2}\right)^2$ respectively, so,

$\sqrt{P(AA \text{ in child's genotype})} + \sqrt{P(aa \text{ in child's genotype})}$ would be equal to

$$\sqrt{P(AA)}+\sqrt{P(aa)}=\frac{2p_1+p_2}{2}+\frac{2p_3+p_2}{2}$$

$$=\frac{2p_1+2p_2+2p_3}{2}$$

$$=p_1+p_2+p_3$$

$$=1$$

Therefore,

$$\boxed{\sqrt{P(AA \text{ in child's genotype})} + \sqrt{P(aa \text{ in child's genotype})} = 1}.$$

A very interesting result was that:

Conversely, if $\sqrt{p_1}+\sqrt{p_3}=1$ was true, then by squaring both sides of the equation,

$$p_1+2\sqrt{p_1p_3}+p_3=1$$

$$2\sqrt{p_1p_3}=1-p_1-p_3$$

$$2\sqrt{p_1p_3}=p_2$$

$$4p_1p_3=p_2^2$$

$$p_1(1-p_1-p_2)=\frac{p_2^{\,2}}{4}$$

$$p_1-p_1^{\,2}-p_1p_2=\frac{p_2^{\,2}}{4}$$

$$p_1=p_1^{\,2}+p_1p_2+\frac{p_2^{\,2}}{4}$$

$$p_1 = p_1^2 + 2p_1\left(\frac{p_2}{2}\right) + \left(\frac{p_2}{2}\right)^2$$

$$p_1 = \left(p_1 + \frac{p_2}{2}\right)^2$$, by completing square

$$\therefore \quad p_1 = \left(\frac{2p_1 + p_2}{2}\right)^2$$ and the case for another formula

$$p_3 = \left(\frac{2p_3 + p_2}{2}\right)^2$$ was similar.

This condition stated that whatever the value of p_1 and p_3 at the beginning, the second generation of the child's genotype became a stable genotype distribution.

This activity was conducted in two Secondary 5 classes with a diversity in their mathematical abilities. Students could fill in the table to construct the possible combinations of genotypes. They could also (although take time) calculate the probabilities when a concrete numerical value was given. However, some of them were scared of algebra. They were unable to complete the squares and factorising by grouping terms.

A questionnaire was conducted after the tryout lesson. 81% of students agreed that this STEM activity was challenging. Thus, the writer suggested that teachers who wanted to adopt this teaching activity might consider trimming some parts of the activity.

Teacher spirit in the promotion of STEM was more important than the project itself

A Ming Pao news on 13th March 2019 reported an investigation on the stress level of the current HKDSE candidates. 8.3% of students who joined STEM activities claimed that they would seek communication with their family members when dealing with the parental expectations, compared with 7.2% who did not join any STEM activities.

During this seed project, my students did not join any STEM competition, nor were they pulling-out for the elite class activities. Instead, only in-class activities were designed for both average-ability students and relatively lower achievers. The writer believed that STEM for all was the best way to promote STEM education in school.

Vote of Thanks

I would like to express my sincere gratitude to the Curriculum Development Officers Mr CHAN Sau-tang and Mr Gary LEE Kin-sum who had offered guidance and supports by giving me lots of invaluable suggestions and informative resources. I would also like to express my deep thanks to the two seconded teachers, Mr Gavin IP Ka-fai and Ms Kitty CHENG Po-chun, who were also my university classmates, for their passions in helping me to design and prepare the STEM tasks. Special thanks would be also given to my two Principals, Ms IP Wai-ching and Mr YAU Chi-leung, for providing resources for me to test different tasks before actual implementation. Last but not the least, I would like to thank my dear students, S.5M 2016-2017, S.3C 2017-2018, S.4C / S.5C 2017-2019, and S.5A 2018-2019, for playing and enjoying the lessons with me!

Reference

[1] 蕭文強、林建（2010）。《概率萬花筒》。香港：政府物流務署印。

[2] 中文口試開考 討論鼓勵捐血 調查：考生備戰 DSE 時間減 - 20190313 - 教育 - 每日明報 - 明報新聞網

Author’s e-mail: snoopyjoy@gmail.com

13. 小一 STEM 教育初探

聖公會聖十架小學：陳寶儀主任、潘慧妍老師

源起

我校自 2016 年遷進啟德新校舍，便配合社區環境，每年進行以航空為主題的三天專題研習活動。

一年級的主題在這三年換了三次，不是因為學習內容太深，便是活動稍欠跨學科學習元素。今年我們累積了以往的經驗，設計了以「航空遊樂場」為主題的專題研習活動。一年級的學生以自己居住的社區啟德作為研習的背景，了解社區由舊機場發展至現在的歷史。透過遊公園的活動認識區內的設施，例如啟晴公園是以航空為主題設計，也是全港唯一的航空遊樂場。我們安排小一學生化身成為啟晴公園遊樂場設計師，為啟晴公園規劃以 Bee-bot 帶領的巡遊路線，為遊人作導航。

STEM 教育元素

「航空遊樂場規劃」包含了小一數學科及常識科的核心學習元素，配合電腦課應用編程，發展學生的計算思維，配合中文科，提升學生的匯報及說話能力，配合英文科提升學生聆聽及跟從指令的能力。

有關的學習元素如下：

學科	核心學習元素
數學	■ 順數、倒數 ■ 長度和距離 ■ 方向和位置：「左」、「右」、「前」、「後」
科學 (常識)	■ 公園 ■ 物料的測試
工程 (常識)	■ 設計：「航空遊樂場」模型 ■ 測試及改良：是否能正確依既定的路線巡遊
科技 (電腦)	■ 編程：計算思維 ■ 計算所需動作的次數，來決定應執行移動命令的次數 ■ 能在平面圖上讓 Bee-Bot 走到預定的目標/方向 ■ 增加或減少移動命令模塊來改變序列的步驟 ■ 更正錯誤來完成程式
語文	■ 導賞 (中文說話) ■ 聆聽指令(英文方向) ■ 詞彙 (中、英公園設施)
視藝	■ 設計 ■ 繪畫

<u>發展共通能力</u>

透過「航空遊樂場規劃」，提升學生共通能力：

1. 協作能力：

   了解自己的角色，主動思考、分享意見、與組員協作構思設計並製作成品。

   在實驗過程中，分工協作完成測試及紀錄。

2. 溝通能力：

   運用適當的語言及態度與組員進行討論、量度數據和分析結果。

3. 創造力：

   透過應用不插電編程的基本概念，發展計算思維能力以作新的跳飛機遊戲。

4. 明辨性思考：

   學習分析及判別優劣，從而找出合適路線的設計。

5. 解決問題能力：

   透過反覆測試，驗證預測成效。

6. 自我管理能力：

   自我評估，訂定目標。

7. 研習能力：

   根據思考、討論、分享完成研習日誌。

8. 運用資訊科技的能力：

   發展計算思維。

9. 運算能力

<u>過程</u>

小一學生遊啟晴公園後，在設計 Bee-Bot 巡遊路線前，須有以下的輸入：

目標	教學活動	學生任務
1. 認識方向(左右)	**活動一** – 齊齊做個小士兵 1. 向學生介紹左右方。 2. 根據老師指令，請學生舉起左手、右手、左腳或右腳。 3. 全班學生完成一個指令後，須復原站姿。 4. 重複步驟 2 和 3 進行比賽。 5. 如有時間，可每組派出表現最佳的學生，在黑板前進行全班最佳小士兵（淘汰賽）。	聽取指令來完成任務。
2. 認識 Bee-Bot 指令碼	**活動二** – 我是 Bee-Bot 1. 介紹課室的四個方位（分別在四個方位貼上 1-4 的數字，黑板(1)、窗(2)、壁佈(3)及門(4)） 2. 學生先面向黑板(1) 3. 提示學生每次轉向只能轉向一個新數字，不能跳過其他數字（如 1 只能轉向 2 或 3）。	根據指令左轉或右轉一個直角*，向前進或後退。 *小一學生不須使用「直角」一詞

目標	教學活動	學生任務
	4. 老師發出指令左或右轉向另一個新方位。 5. 重複步驟 3 和 4 ，使學生熟練直角左轉及右轉。 6. 重複指令 老師可加入向前行一步及向後退一步的指令。 *（老師發出指令時，須留意學生的活動空間，注意安全）*	
3. 學生能發出口述 Bee-Bot 指令	**活動三** – 領取軍糧 每組派出兩名組員及帶上同色的色帶（一位擔任指揮官，另一人擔任人型 Bee-Bot） 1. 指揮官及人型 Bee-Bot 必須先完成跳飛機任務，進入秘密基地，才能開始領取軍糧的任務。 2. 老師在指定地方放置軍糧（小欖球）。 3. 指揮官發出口述指令（左轉、右轉、向前，向後），指示 Bee-Bot 到達糧倉，領取糧食（小欖球）。 *提示 Bee-Bot 只能在軌道（階磚隙）上行走	發出指令，指示同學到達目的地，完成任務。

目標	教學活動	學生任務
	4. 交換角色，重複步驟 3 。	
4. 學生能輸入 Bee-Bot 指令	**活動四** – Bee-Bot 取花蜜 1. 約三人一組。 2. 老師派發 iPad，學生使用 Bee-Bot 軟件 3. 輪流透過輸入 Bee-Bot 指令，完成不同的關卡。	輸入正確的指令碼，指示 Bee-Bot，完成任務。

<u>學習成果</u>

學生應用常識及數學科所學，配合編程設計智能導航員，透過手腦並用的探究活動，引起他們學習的興趣及好奇心。

一年級學生設計的航空公園，充滿創意：

檢討

在數學課堂中，讓學生學會從觀測者的角度以「上」、「下」、「左」、「右」、「前」、「後」和「之間」描述物件的相互位置，進行專題研習活動時，學生便能正確地應用方向編碼指示 Bee-bot 向特定方向移動，在所設計的情境中鞏固他們對方向及數數的概念。當然，還有小部分空間感較弱的學生仍未能掌握，須個別跟進指導。

在期考中，整體學生在方向的作答表現較往年有所提升，可見引導學生從「做中學習」，進行探究性活動，這種互動的學習模式，不但能提高學生學習數學的興趣，也能加強學習的效能。

延伸及期望

透過進行一連三日的專題研習日，讓學生能夠接觸多元化的跨學科主題學習活動，當中包括應用 Bee-bot 編程深化學生已有的知識，讓初小學生能夠運用資訊科技進行學習，逐步發展學生對科技探索的興趣及好奇心。從初小開始培養學生發展計算思維的能力，與時並進，增加學生對處理資訊的能力及運用科技的觸覺，日後能夠運用數碼科技技術改善生活質素，發展智慧城市。

學生的學習成果反映這種情境化的學習，能讓學生綜合應用學科所學，讓學生掌握共通能力，並培養勇於嘗試的學習態度。因此，在課程規劃上，將持續推動跨學科學習的學與教模式，綜合運用數學科所學的數學語言及概念，透過電腦科技工具應用於常識科中解決日常生活中的問題，期望學生能繼續從多元化的學習活動中「多動腦筋」、「多

動手做」，以深化學生的學習體驗，幫助學生自己建構知識。

總結

計劃的推行須經過不斷的檢討，持續優化改善，過程中最重要的是團隊的合作精神，上下一心朝著同一的方向，為培育下一代共同努力，透過 STEM 教育，以助學生掌握二十一世紀所需要的重要技能和素質。

(註：陳寶儀老師為行政長官卓越教學獎(2018/2019)獲嘉許狀老師)

14. Exploration and Development of Effective Strategies for Promoting and Implementing STEM Education in Secondary Mathematics

CHENG Po Chun, Kitty
Tak Nga Secondary School

Introduction

In the 2015 and 2016 Policy Addresses, the Government pledged to promote STEM education with a view to nurturing students' learning interest, enhancing their creativity, collaboration and problem solving skills as well as developing their innovativeness (Education Bureau 2017).

A one-off grant of HK$100,000 was disbursed to each primary school in the 2015/16 school year, and HK$200,000 to each secondary school in the 2016/17 school year by the Education Bureau (EDB) to support school-based STEM education and activities. In this regard, school heads and teachers began to think of the strength and needs of the school to promote STEM education in their curriculum development. To align with the global trend of implementation of STEM education, my school joined the "Seed" project which aimed at exploring and developing effective strategies for promoting and implementing STEM education in Secondary Mathematics in the 2017/18 and 2018/19 school years. I was offered this unique opportunity to be seconded in the Mathematics Education

Section of Curriculum Development Institute (CDI) of the Education Bureau in these two school years.

Experiences of 2-year Secondment

During these two years of secondment, problems arose when devising the examples of STEM learning and teaching activities from central mathematics curriculum that supported the development of school-based learning tasks. To solve the problems encountered, CDI officers and I worked cohesively with "Seed" schools teachers. I sought advice from peer Science and Technology teachers and referenced other resources from books, e-resources and good practices of STEM activities implemented in other countries as well. Through the collaboration with the participating schools, numerous examples of STEM activities were tried out by the experienced frontline mathematics teachers and followed by evaluations of the learning and teaching effectiveness with both the teachers and the students. In this article, I would like to share 2 STEM examples for S2, of which one of them was implemented by my school.

STEM Example 1 - Decode a QR Code Symbol

School
Tak Nga Secondary School

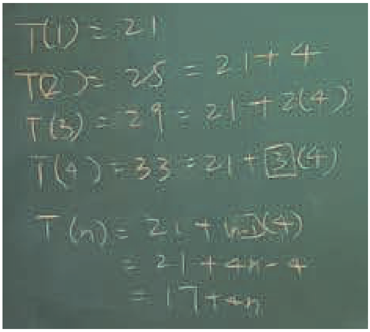
Objective
Apply the knowledge of the binary number system and the denary number system to decode the messages in a QR code symbol by hand.

Lead-In
The differences between a QR code and a barcode were compared. The advantages of a QR code (abbreviated from Quick Response code) over a barcode were being discussed.

Development
A. <u>Structure of a QR code symbol</u>
1. There were forty versions in the QR code. The larger the version number, the larger the amount of data could be stored in the QR code symbol. The smallest version was version 1, 21 modules by 21 modules in size. Each module was stored as a binary number (1 and 0). A dark module was nominally a binary 1 and a light module was nominally a binary 0.

2. Students were guided to deduce the general term T(n) of the nth version from the table given below.

Version	Modules
1	21 × 21
2	25 × 25
3	29 × 29
4	33 × 33
⋮	⋮
10	57 × 57
⋮	⋮
25	117 × 117
⋮	⋮
40	177 × 177



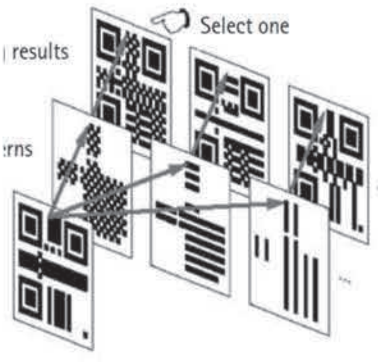
B. XOR Operation

1. The original QR code symbol is laid over each of the masking pattern. The modules will reverse from light to dark or vice versa which correspond to dark modules of the masked pattern through the XOR operation.

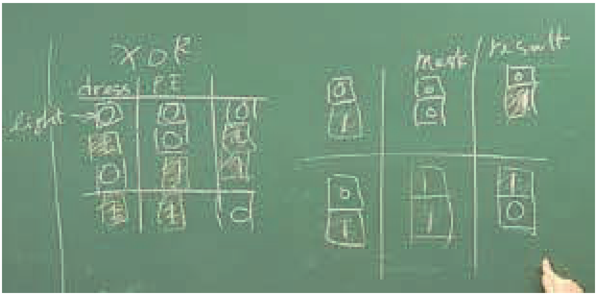
Masked Symbol

Masking Pattern

Original Symbol



2. An example of explaining the XOR operation was cited before teaching data unmasking. The student wearing the school uniform or the PE uniform to school was "1"(the dark module). In other words, the student not wearing the school uniform or the PE uniform to school was "0"(the light module). The student allowed to enter the school was "1" and the student not allowed to enter the school was "0"

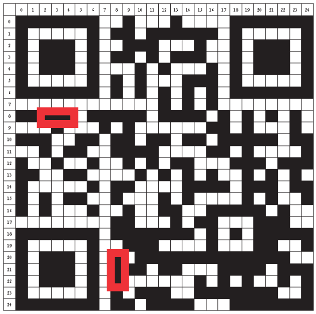


3. The masked module could be worked out by the students as below.

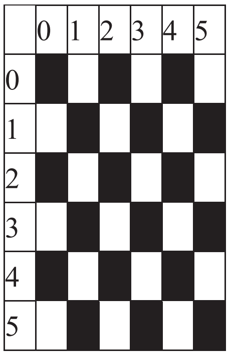
Input (Original Code)	Input (Masking Pattern)	Output (Masked Module)
1	1	0
0	0	0
0	1	1
1	0	1

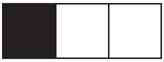
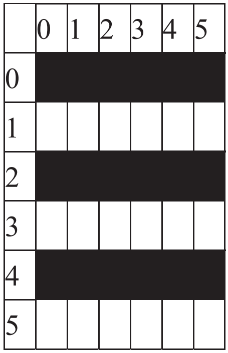
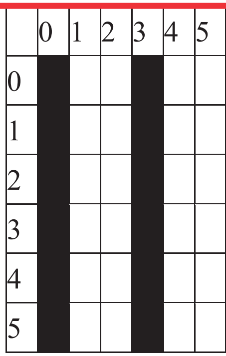
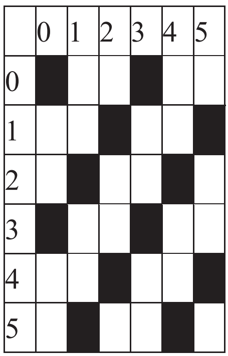
C. Data Mask Pattern

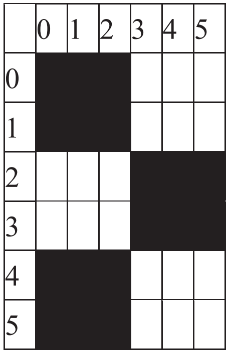
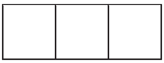
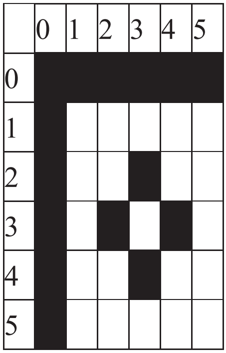
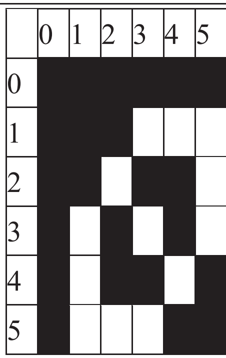
1. A QR code symbol after data masking was shown below. The mask pattern could be found at (8, 2), (8, 3), (8, 4) and (20, 8), (21, 8) and (22, 8) in the QR code.

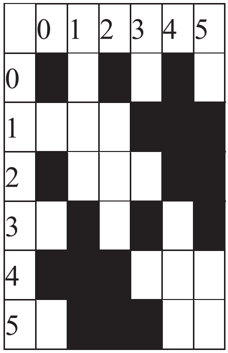


2. The data masked pattern was determined from the data mask pattern reference modules.

	Data mask pattern reference modules	Data mask pattern generation condition	Data mask pattern
(a)		$(i + j) \bmod 2 = 0$	

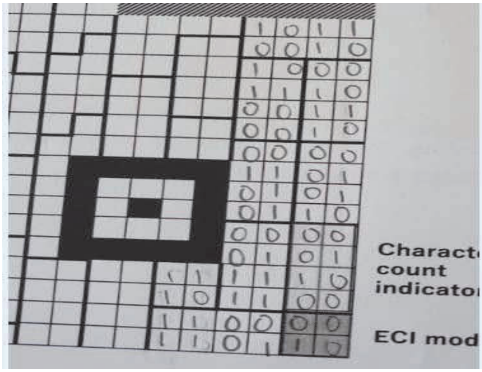
	Data mask pattern reference modules	Data mask pattern generation condition	Data mask pattern
(b)		$i \bmod 2 = 0$	
(c)		$j \bmod 3 = 0$	
(d)		$(i + j) \bmod 3 = 0$	

	Data mask pattern reference modules	Data mask pattern generation condition	Data mask pattern
(e)		$((i \text{ div } 2) + (j \text{ div } 3)) \text{ mod } 2 = 0$	
(f)		$(ij) \text{ mod } 2 + (ij) \text{ mod } 3 = 0$	
(g)		$((ij) \text{ mod } 2 + (ij) \text{ mod } 3) \text{ mod } 2 = 0$	

	Data mask pattern reference modules	Data mask pattern generation condition	Data mask pattern
(h)		$((i + j) \bmod 2 + (ij) \bmod 3) \bmod 2 = 0$	

D. <u>Data Unmasking</u>

1. Using the exclusive OR operation, the teacher demonstated how to unmask the Extended Channel Interpretation (ECI) mode and the character count indicator in the QR code provided. The original module would change from “1” to ‘0” or vice versa if the mask was a dark module.
2. 4-bit and 8-bit binary numbers were found according to the order of placing the data bits in the modules.
3. 8-bit binary number was then converted to a decimal number and the length of the message in the QR code could be worked out.



Character count indicator tells the length of a message. The number of characters in the QR code can be found from it.

(i) Write down the 8-bit binary number from the **Character count indicator**.

00011000_2

(ii) Change the 8-bit binary number into a decimal number.

$00011000_2 = 16 + 8$

$= 24$

E. Decoding the Data Area

1. The students worked in groups to deduce the message in the QR code symbol from a byte mode table.

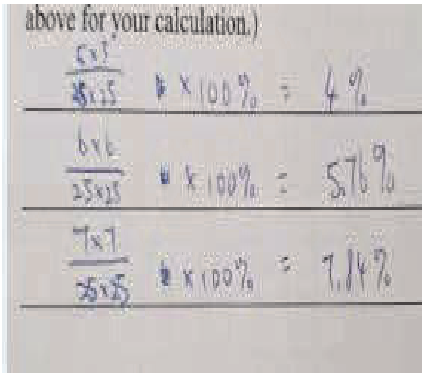
Block	8-bit Binary Number	Decimal number	Character
1	**01101000**	**104**	**h**
2	**01110100**	**116**	**t**
3	**01110100**	**116**	**t**
4	**01110000**	**112**	**p**
5	**00111010**	**58**	**:**
6	**00101111**	**47**	**/**
7	**00101111**	**47**	**/**
8	**01110111**	**119**	**w**
9	**01110111**	**119**	**w**
10	**01110111**	**119**	**w**
11	**00101110**	**46**	**.**
12	**01110100**	**116**	**t**
13	**01100001**	**97**	**a**
14	**01101011**	**107**	**k**
15	**01101110**	**110**	**n**
16	**01100111**	**103**	**g**
17	**01100001**	**97**	**a**
18	**00101110**	**46**	**.**
19	**01100101**	**101**	**e**
20	**01100100**	**100**	**d**
21	**01110101**	**117**	**u**
22	**00101110**	**46**	**.**
23	**01101000**	**104**	**h**
24	**01101011**	**107**	**k**
Terminator	0000	0	NUL

2. They scanned the QR code with their iPad provided and checked if the decoding message was correct.

Application

A. Error Correction

1. An error correction capability to restore data of the QR code was introduced if it was damaged or unreadable.
2. 3 school badge cutouts of (5 × 5), (6 × 6) and (7 × 7) modules in size and the QR code symbol of (25 × 25) modules were given to each group. From the approximate recovery capacity % of the QR code, the students calculated the maximum length of the square school badge in module.



3. The students chose the size of the school badge embedded in the QR code. Feasible locations of the badge would be tested by the iPad and the brief explanation of the structure of the QR code would be introduced.

STEM Example 2 – A Self-made Instrument

School

CMA Secondary School

Objective

Make an instrument out of drinking straws and a cardboard through exploring the length ratios between each note of the C Major scale where the black keys would not be taken into consideration.

Lead-In

The teacher showed a self-made instrument and played the song of ‘Twinkle, Twinkle, Little Star’. The students were asked to make similar instruments, play their songs and record them with an iPad. The file would be sent to the teacher through the software “AirDrop”.

Development

A. An Octave

1. The background information of a mathematician Pythagoras was introduced. The teacher further explained why he was also considered to be the “father of harmony.”
2. With the help of a keyboard diagram, the teacher explained what an octave was. The note C5 is one octave higher than the note C4, the note C5 is the 8th note counting from the note C4 to the right, including C4. The notes C4 and C5

sound “harmonies”.

3. Assume that the length of the string of the note C4 is a. The students were asked to find the lengths of the strings of the notes C3, C5, C6 and C7 from C4 where the length of the string of the note C4 was 2 times longer than that of the note C5.

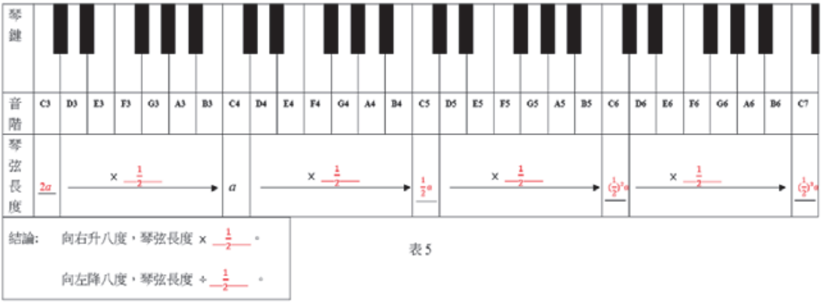
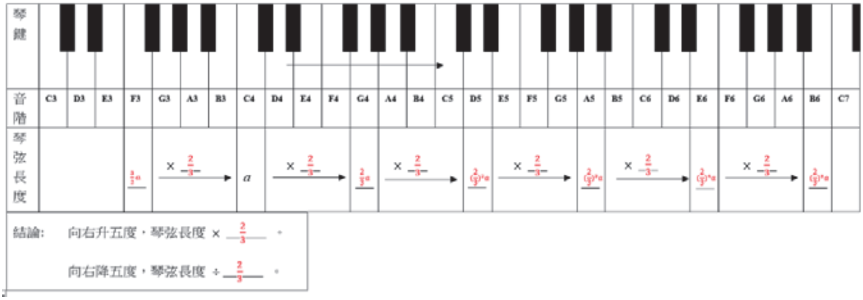


表 5

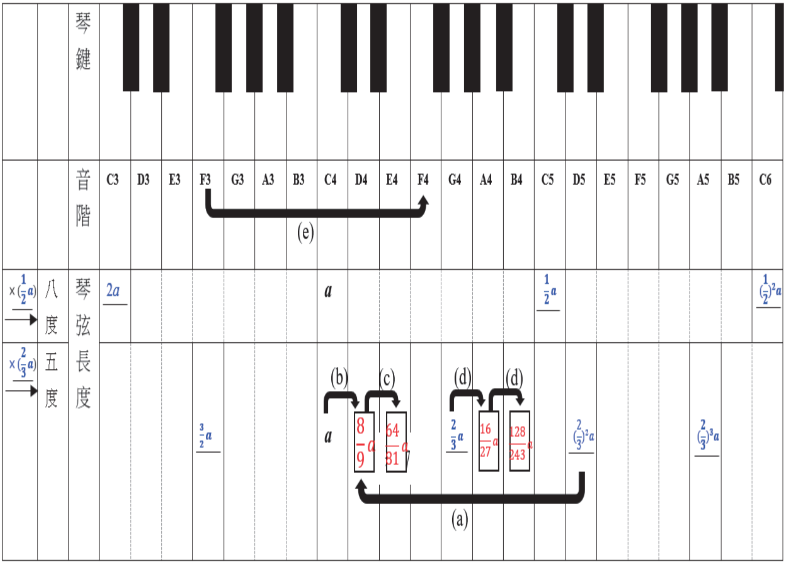
B. <u>A Perfect Fifth</u>

1. The notes C4 and G4 sound “harmonies” too. The note G4 is one perfect fifth higher than the note C4, the note G4 is the 5th note counting from the note C4 to the right, including C4.
2. The length ratio of the strings of the notes C4 and G4 is 3:2. Assume that the length of the string of the note C4 is a. The students were asked to deduce the lengths of the strings of the notes F3, G4, D5, A5, E6 and B6.



C. <u>Lengths of the Strings of the Notes D4, E4, A4 and B4</u>

1. Students were asked to copy the answers from the 2 tables in the above.
2. The length of the string of the note D4 was deduced from the length of the string of the note D5. Making use of the length ratio 8:9, the lengths of the strings of the notes E4, A4 and B4 could be calculated.

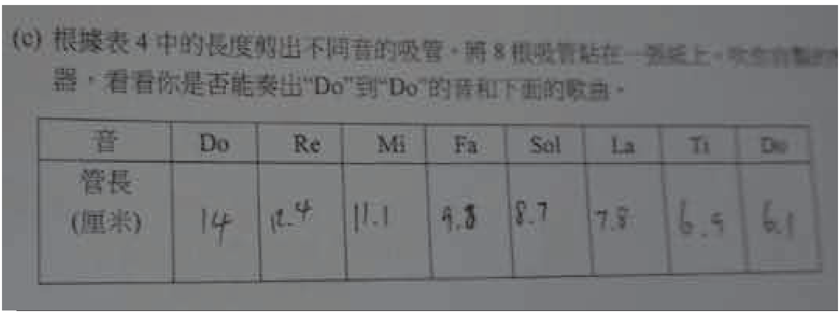


3. The length of the string of the note F4 could be easily deduced from the length of string of the note F3.

Application

A. <u>Making an Instrument</u>

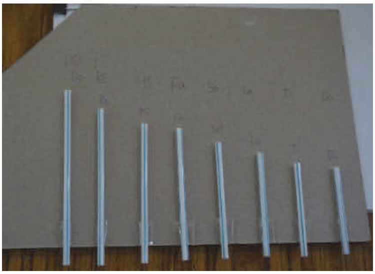
1. 8 drinking straws, a hard cardboard, a pair of scissors and a sellotape were given to each group. The length of the

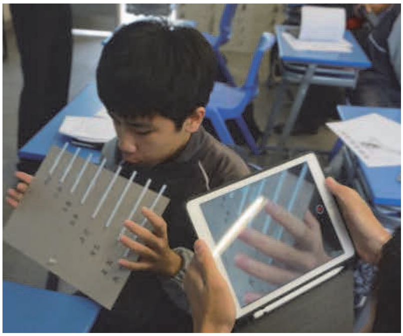

(c) 根據表 4 中的長度剪出不同音的吸管，將 8 根吸管貼在一張紙上，……器，看看你是否能奏出"Do"到"Do"的音和下面的歌曲。

音	Do	Re	Mi	Fa	Sol	La	Ti	Do
管長 (厘米)	14	12.4	11.1	9.8	8.7	7.8	6.5	6.1

drinking straw of the note C4 was 14 cm. The lengths of the drinking straws of notes from D4 to C5 were figured out in respect of the ratios deduced before.

2. The students cut the drinking straws and made the self-made instruments. Some students tried to play the instruments during the lesson.





Vote of Thanks

I would like to express my sincere gratitude to my supervisor Mr CHAN Sau-tang, the Curriculum Development officers, Dr NG Yui-kin, Mr LEE Kin-sum, Ms HO Yee-hung and other officers, who have offered guidance and supports by giving me lots of invaluable suggestions and informative resources. I

would also like to express my deep thanks to my school Principal who has provided full support for my secondment. Furthermore, I would like to thank the colleagues from the participating schools during these two years. All these have enriched my subject and pedagogical knowledge in mathematics education and have opened my eyes to educational software for instructional and learning tools as well. Last but not least, I will keep close communication with EDB officers and other school teachers to stay abreast of all innovative ideas and developments in education.

Reference

[1] 蔡聰明。音樂與數學:從弦內之音到弦外之音。〈數學傳播〉 十八卷一期 民 83 年 3 月。

[2] Education Bureau (2017). *Provision of One-off Grant to Secondary Schools for the Promotion of STEM Education.*

[3] ISO/IEC 18004:2015(E). *Information technology – Automatic Identification and Data Capture Techniques – QR Code Bar Code Symbology Specification*