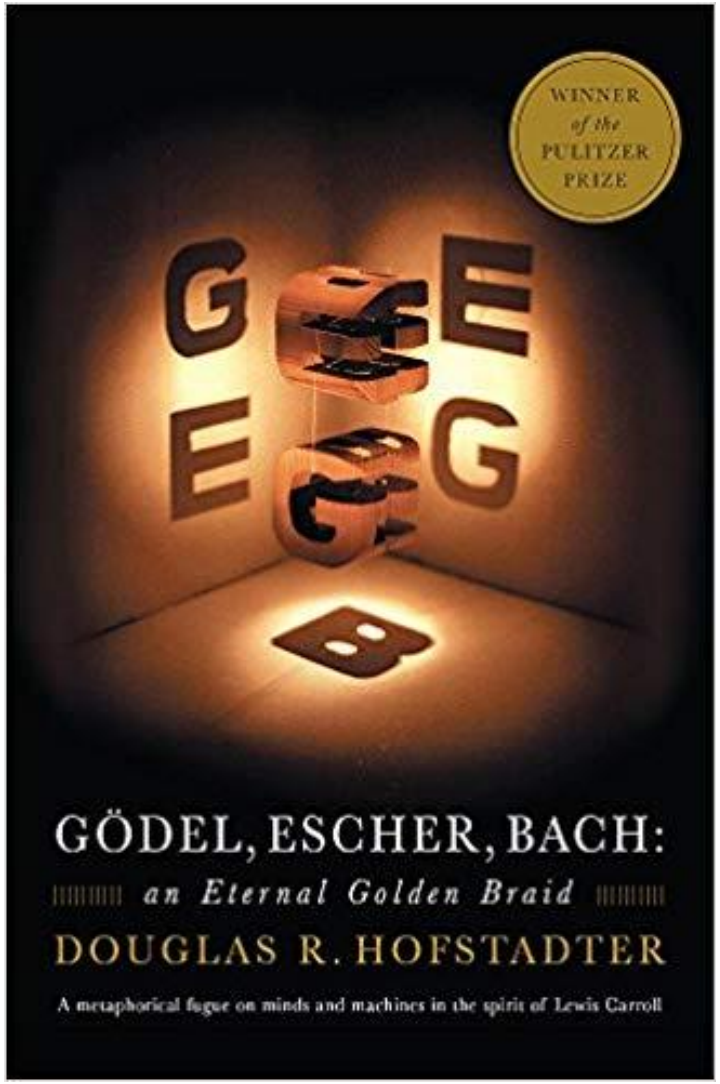
Paradoxes

NG Yui-kin

2019.07.08&10



Douglas. R. Hofstadter, 1979

Paradox

A scenario that involves an **argument** that

- begins with premises that seem to be true
- proceeds with reasoning that seems to be valid
- arrives at a conclusion that is unacceptable (such as a falsehood, a contradiction, or an absurdity)

See, for example, Cook (2013)

Once upon a time …

in a time machine …

Paradoxes

Fallacies

Mistakes

Mistakes

$1 + 1 = 3$

Fallacies (sophisms)

2 = a number
$3 = \text{a number}$
Therefore, $2 = 3$

Paradox

$$\frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}$$

Paradox

$$\frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}$$

$$\frac{x+5-5(x-7)}{x-7} = \frac{4x-40}{13-x}$$

$$\frac{-(4x-40)}{x-7} = \frac{4x-40}{13-x}$$

$$\frac{-1}{x-7} = \frac{1}{13-x}$$

$$7-x = 13-x$$

$$7 = 13$$

Pinocchio paradox

Unspecified premises

Barber paradox

There is a village in which there is a barber named "Bertrand". Bertrand shaves all the men in the village who do not shave themselves, And Bertrand shaves none of the men in the village who do shave themselves.

Question, does Bertrand shave Bertrand, or not?

Liar paradox

This sentence is false

Liar paradox

This sentence is false

Case 1 The sentence is true
Case 2 The sentence is false

Liar paradox

This sentence is false

The sentence is neither true nor false

Liar paradox

This sentence is false

Case 1 The sentence is true
Case 2 The sentence is false
Case 3 The sentence is neither true nor false

Strengthened Liar paradox

This sentence is not true

See, for example, Rieger (2001)

Liar paradox

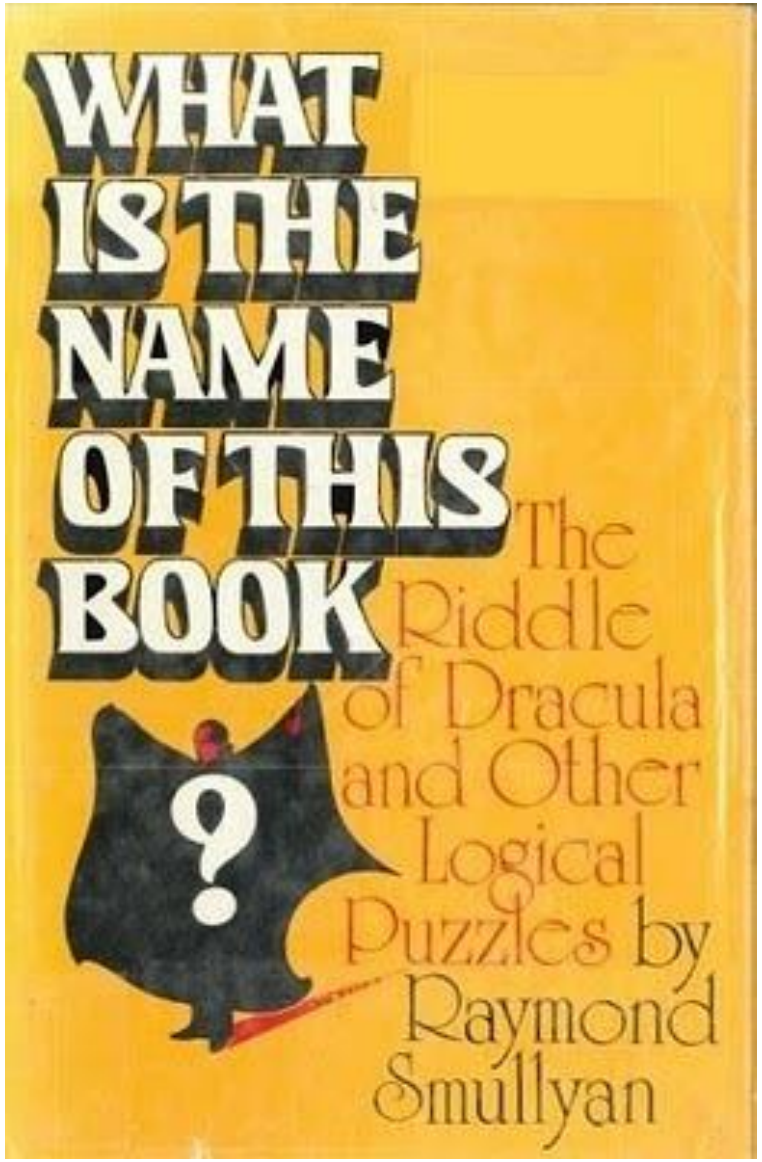
This sentence is false

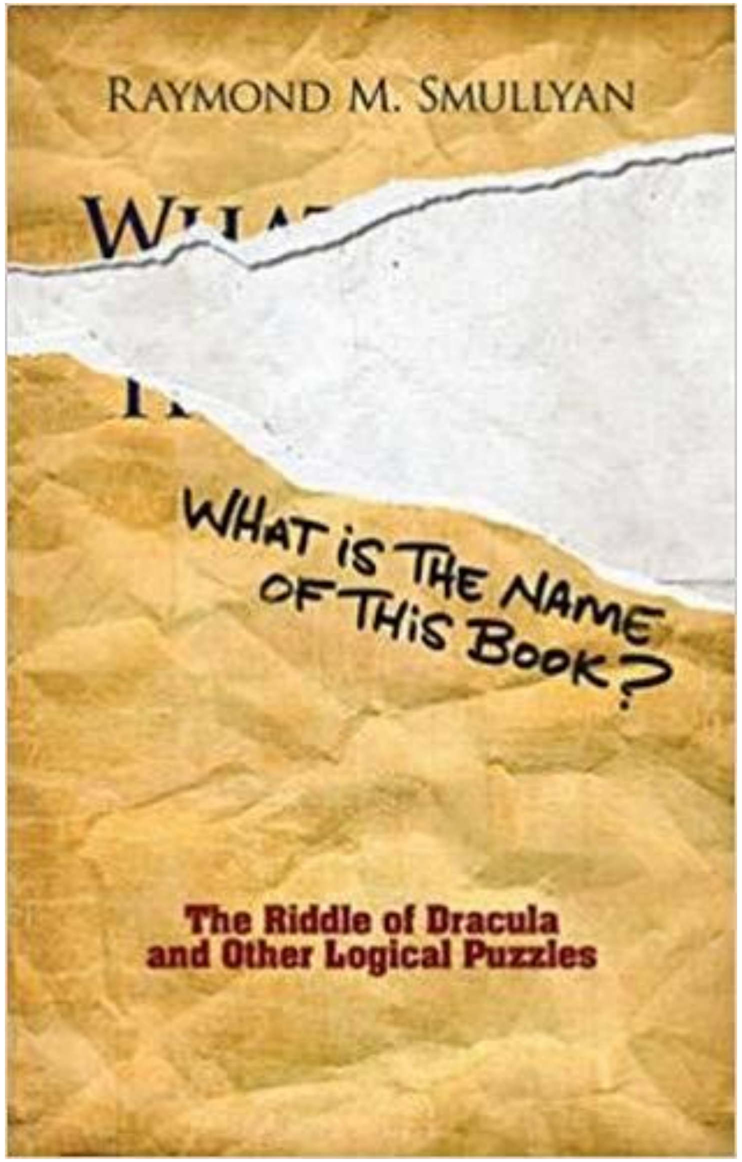
Self-reference should be prohibited

Liar paradox

This sentence is *written in English*.

Self-reference should be prohibited


WHAT IS THE NAME OF THIS BOOK
?
The Riddle of Dracula and Other Logical Puzzles by Raymond Smullyan


RAYMOND M. SMULLYAN
WHAT IS THE NAME OF THIS BOOK?
The Riddle of Dracula
and Other Logical Puzzles

REVISED AND EXPANDED (AGAIN)

There Are Two Errors in the the Title of This Book*

A SOURCEBOOK *of* PHILOSOPHICAL PUZZLES, PROBLEMS, *and* PARADOXES

ROBERT M. MARTIN

* See pages xx and yy about this title; see page zz about the butterfly and city

Multi-sentence Liar paradox

(A) : Sentence (B) is true

(B) : Sentence (A) is false

No Self-reference

Multi-sentence Liar paradox

(A) : Sentence (B) is true
(B) : Sentence (A) is false

Circularity should be prohibited

Yablo paradox

(A_1): For all $m > 1$, (A_m) is false

(A_2): For all $m > 2$, (A_m) is false

. . .

. . .

(A_n): For all $m > n$, (A_m) is false

. . .

. . .

No circularity

See, for example, Yablo (1985); Cook (2014)

Assume that there is a r such that (A_r) is true

For all $m > r$, (A_m) is false

(A_{r+1}) is false

For all $m > r + 1$, (A_m) is false

(A_{r+1}) is true

Contradiction! That implies that the assumption is false

For all r, (A_r) is false

(A_1) is false

For all $m > 1$, (A_m) is false

(A_1) is true

Contradiction!

“Applications”

繡球在這匣子裡

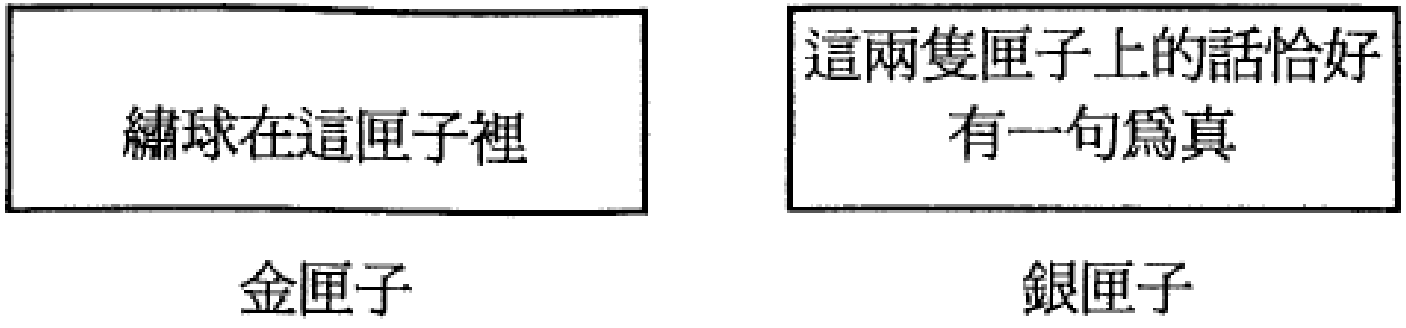
金匣子

這兩隻匣子恰好有一隻
由皇上題字

銀匣子

吳銳堅 (1998)

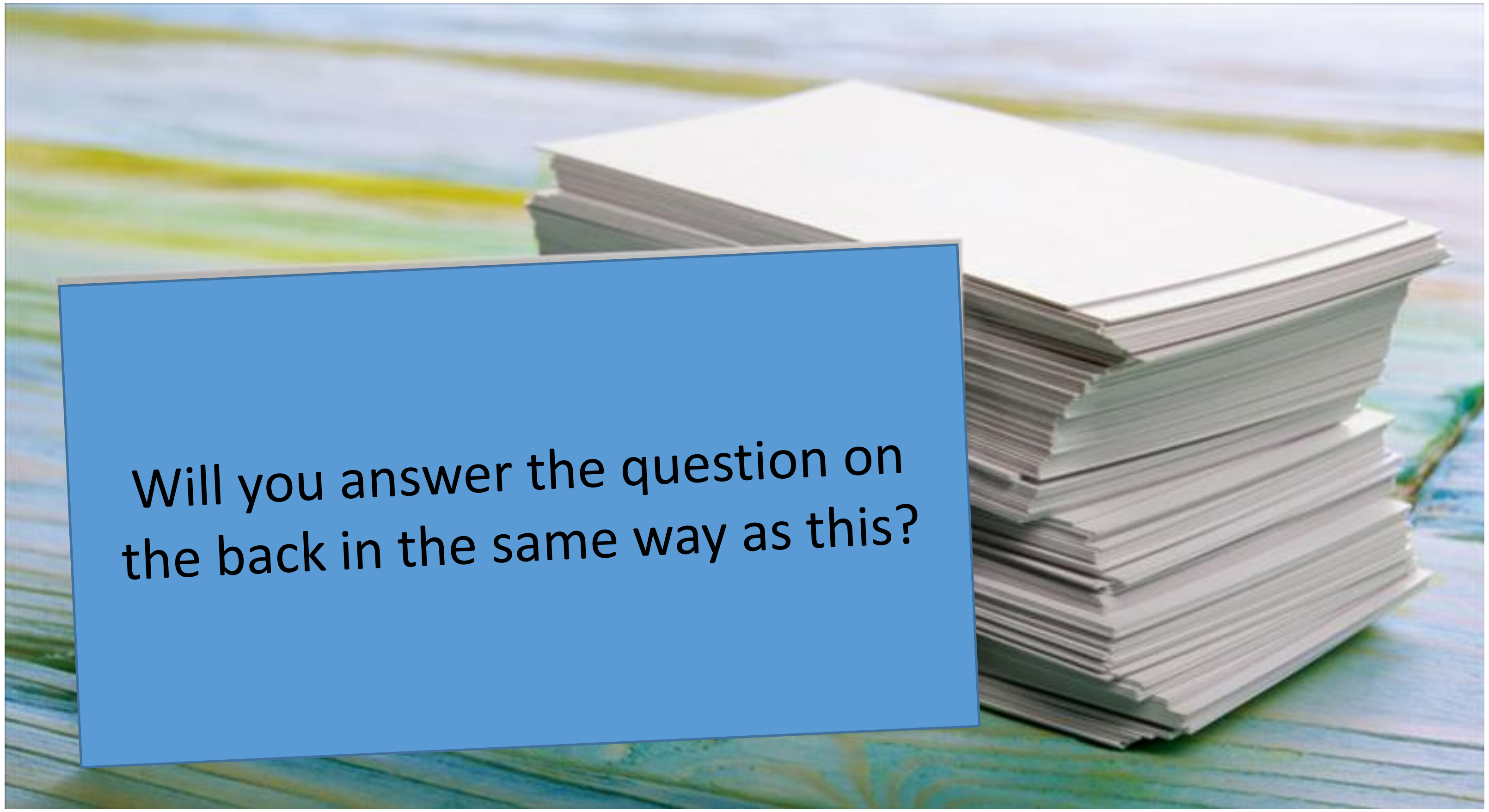
“Applications”



吳銳堅 (1998)

"Applications"

Two yes-no questions



R.M.Smullyan

Implications

Surprises & challenges:
Motivation

Revisit the concepts
and reasoning

Paradox

A common algebraic proof of 1 = 2

Suppose $x = 1$

$$x^2 = x$$

$$x^2 - 1 = x - 1$$

$$(x + 1)(x - 1) = x - 1$$

$$(x + 1) = 1$$

$$2 = 1$$

Paradox

Another algebraic proof of 1 = 2

$$16 - 36 = 25 - 45$$

$$16 - 36 + \frac{81}{4} = 25 - 45 + \frac{81}{4}$$

$$(4 - \frac{9}{2})^2 = (5 - \frac{9}{2})^2$$

$$4 - \frac{9}{2} = 5 - \frac{9}{2}$$

$$4 = 5$$

$$0 = 1$$

$$\therefore \quad 1 = 2$$

Paradox

The 3rd algebraic proof of 1 = 2

$$\frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}$$

$$\frac{x+5-5(x-7)}{x-7} = \frac{4x-40}{13-x}$$

$$\frac{-(4x-40)}{x-7} = \frac{4x-40}{13-x}$$

$$\frac{-1}{x-7} = \frac{1}{13-x}$$

$$7-x = 13-x$$

$$7 = 13$$

$$0 = 6$$

$$0 = 1$$

$$1 = 2$$

Paradox

A calculus proof of 1=2

$$2+2=2^2$$

$$3+3+3=3^2$$

$$4+4+4+4=4^2$$

$$\cdots\cdots$$

$x+x+\ldots+x=x^2$, where x is a positive integer

$$1+1+\ldots+1=2x$$

$$x=2x \qquad \therefore \quad 1=2$$

Paradox

Another calculus proof of 1 = 2

$$\int \frac{1}{x} dx = x \cdot \frac{1}{x} - \int x d(\frac{1}{x})$$

$$\int \frac{1}{x} dx = 1 - \int x \cdot \frac{-1}{x^2} dx$$

$$\int \frac{1}{x} dx = 1 + \int \frac{1}{x} dx$$

$$0 = 1$$

$$\therefore \quad 1 = 2$$

Paradox

Number of roots

The number of roots of a quadratic equation in one unknown is at most 2?

Number of roots

The number of roots of a quadratic equation in one unknown is at most 2?

Suppose a, b, c are three different numbers. The following equation

$$\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-a)(x-c)}{(b-a)(b-c)}=1$$

has three different roots a, b, c.

Number of roots

Coefficient of x^2 =

$$\frac{1}{(c-a)(c-b)}+\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}=0$$

Paradox

Number of roots

Coefficient of x =

$$\frac{-(a+b)}{(c-a)(c-b)}+\frac{-(b+c)}{(a-b)(a-c)}+\frac{-(c+a)}{(b-a)(b-c)}$$

$$=\frac{(a+b)(a-b)+(b+c)(b-c)+(c+a)(c-a)}{(a-b)(b-c)(c-a)}$$

$$=0$$

Paradox

Number of roots

The constant term =

$$\frac{ab}{(c-a)(c-b)} + \frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-a)(b-c)}$$

$$= \frac{-ab(a-b)-bc(b-c)-ac(c-a)}{(a-b)(b-c)(c-a)}$$

$$= 1$$

Paradox

Number of roots

Hence, the equation is:

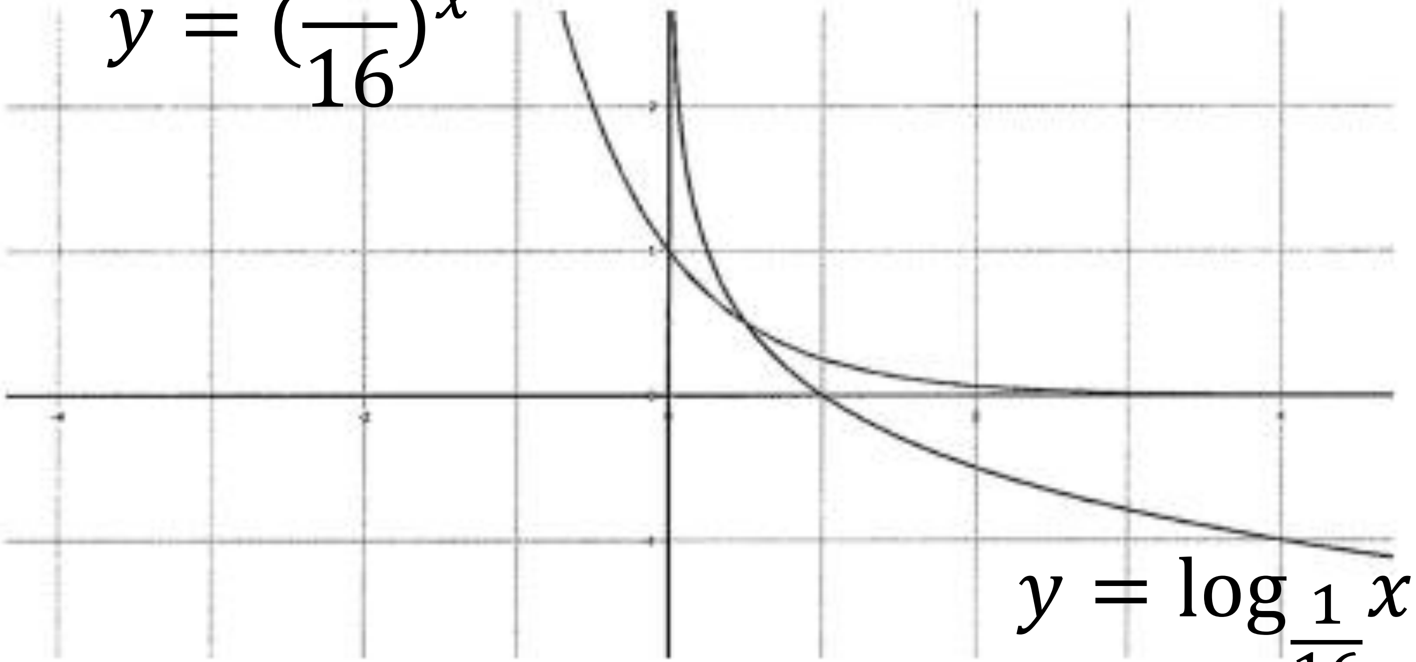
$$0x^2 + 0x + 1 = 1$$

Paradox

Intersection of graphs

$$\log_{\frac{1}{16}} x = \left(\frac{1}{16}\right)^x$$

$$y = \left(\frac{1}{16}\right)^x$$



$$y = \log_{\frac{1}{16}} x$$

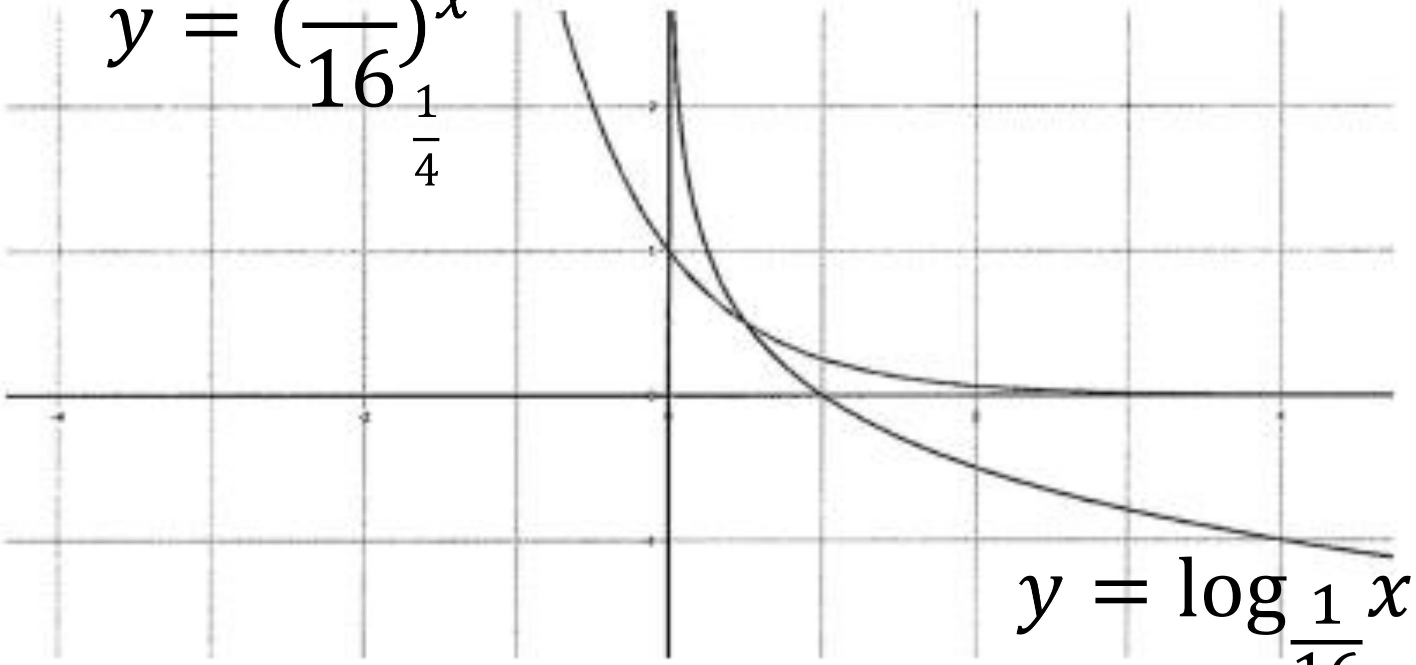
Paradox

Intersection of graphs

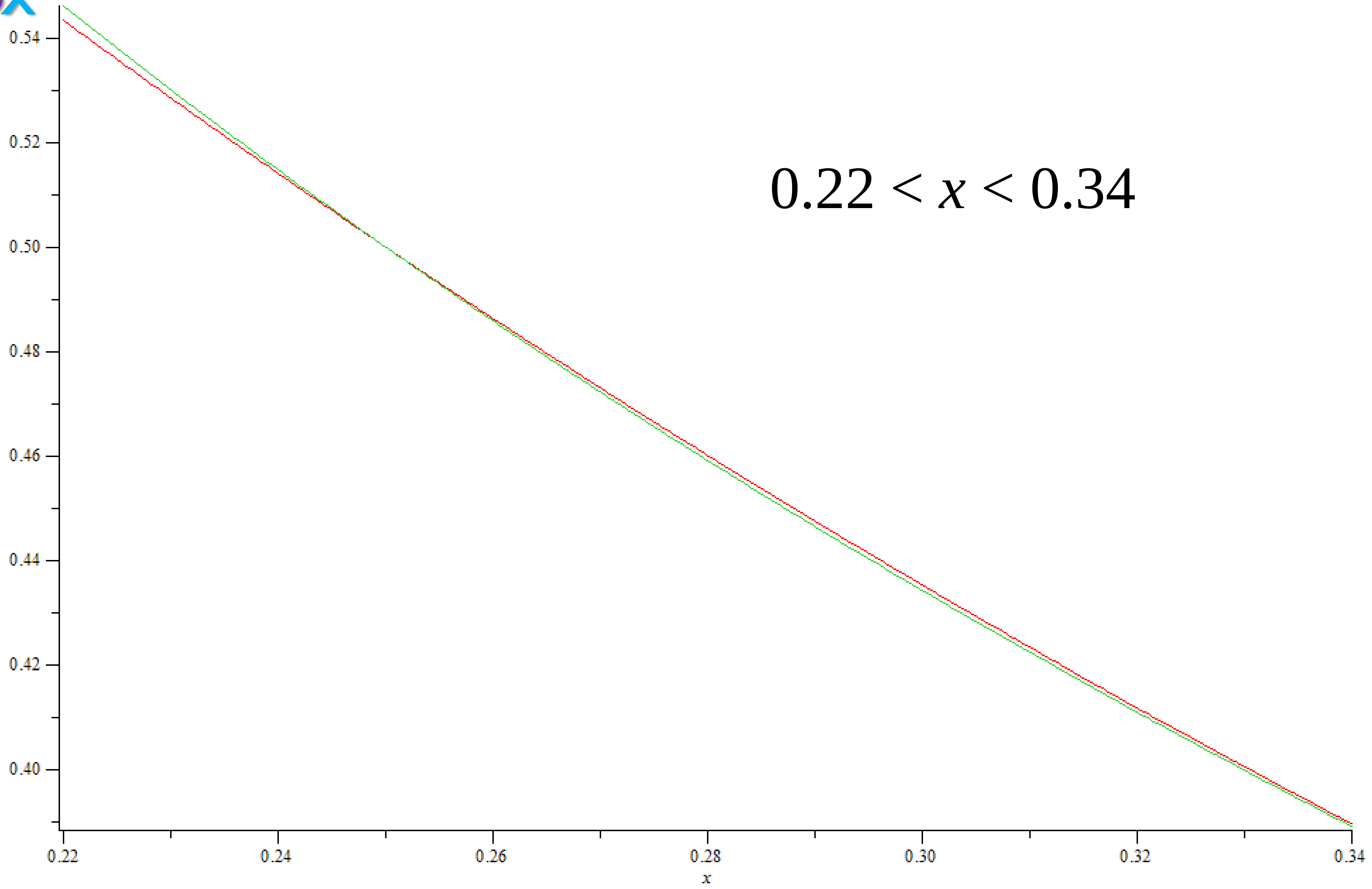
$$\log_{\frac{1}{16}} x = \left(\frac{1}{16}\right)^x$$

But there are at least two roots:

$$\frac{1}{2}, \frac{1}{4}$$

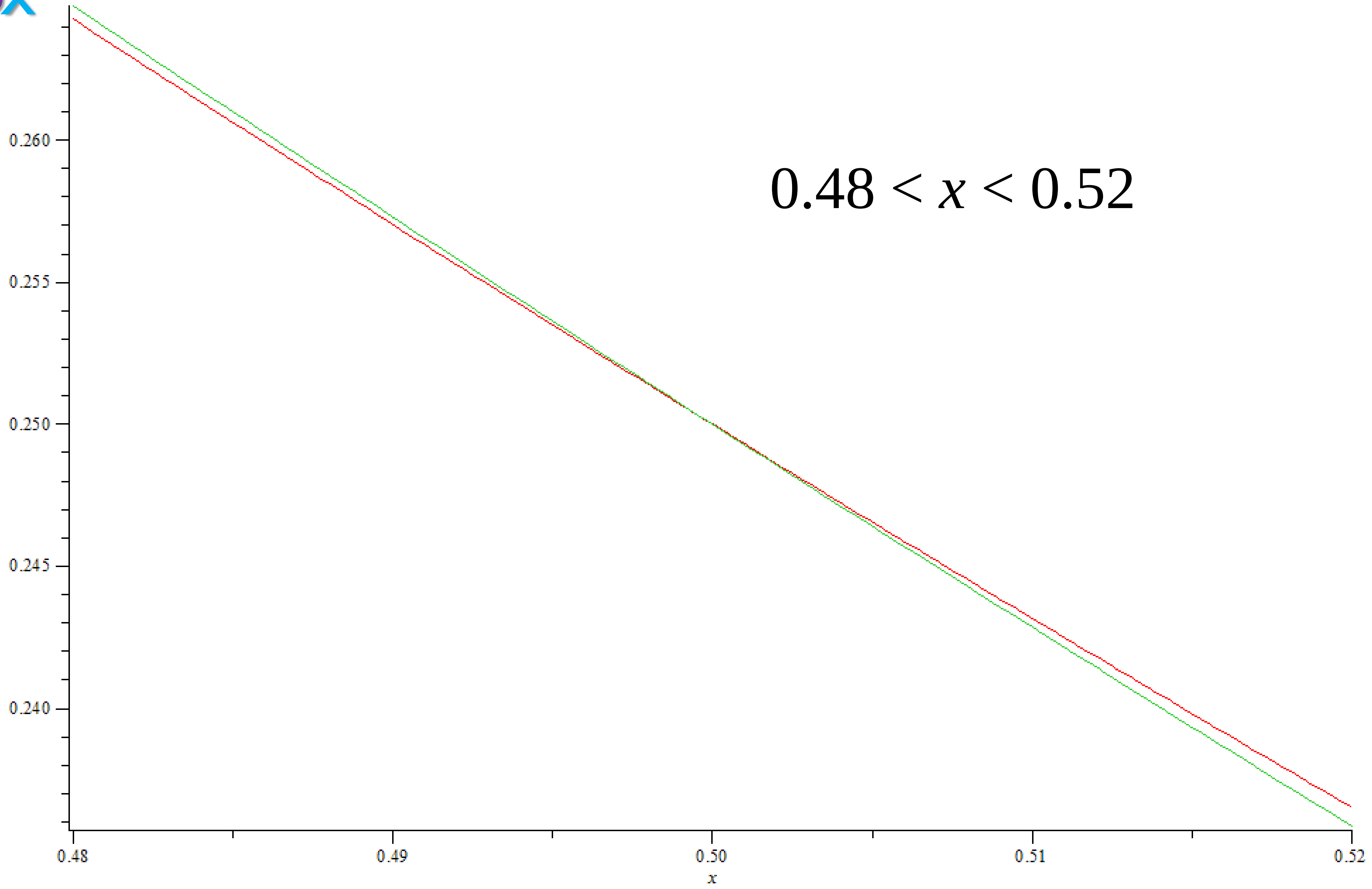


Paradox



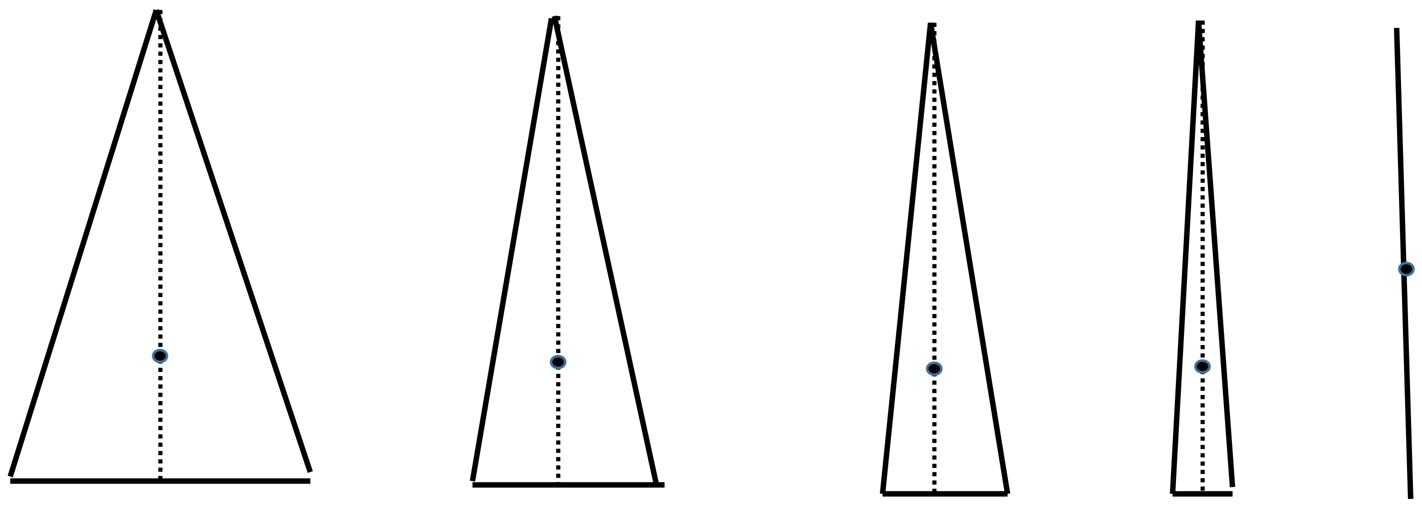
$$0.22 < x < 0.34$$

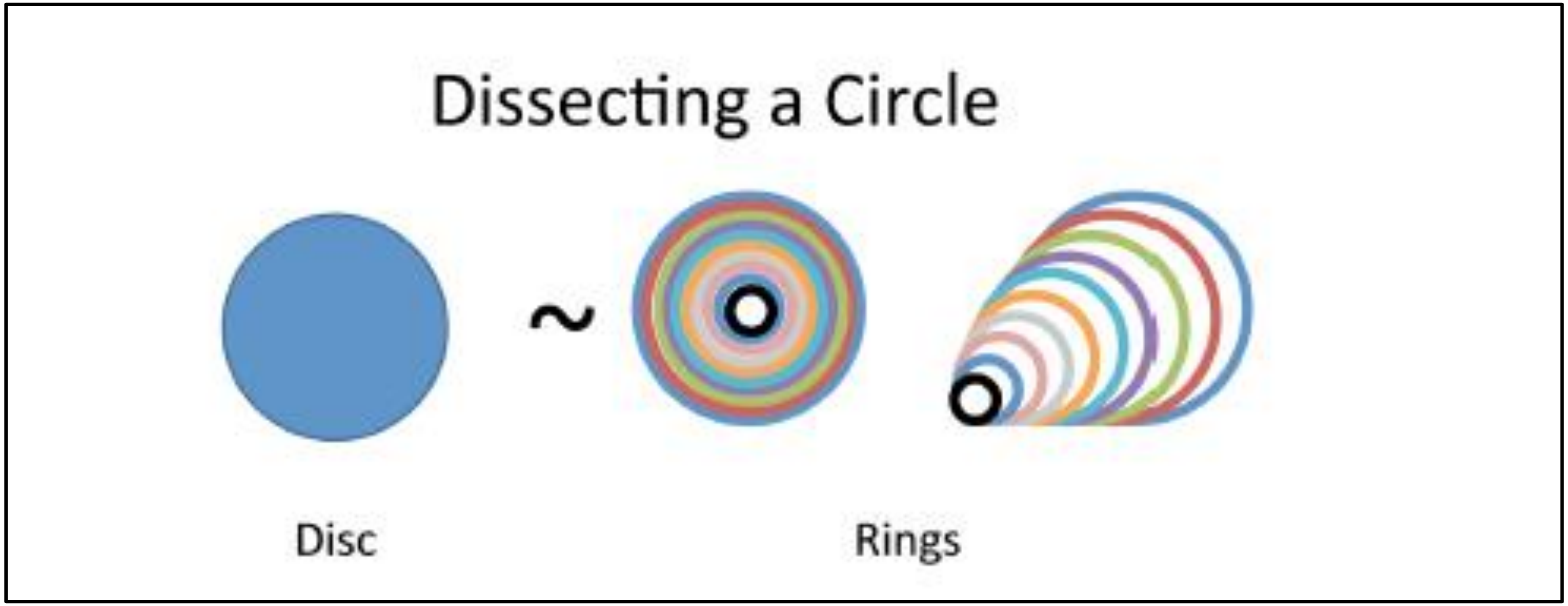
Paradox



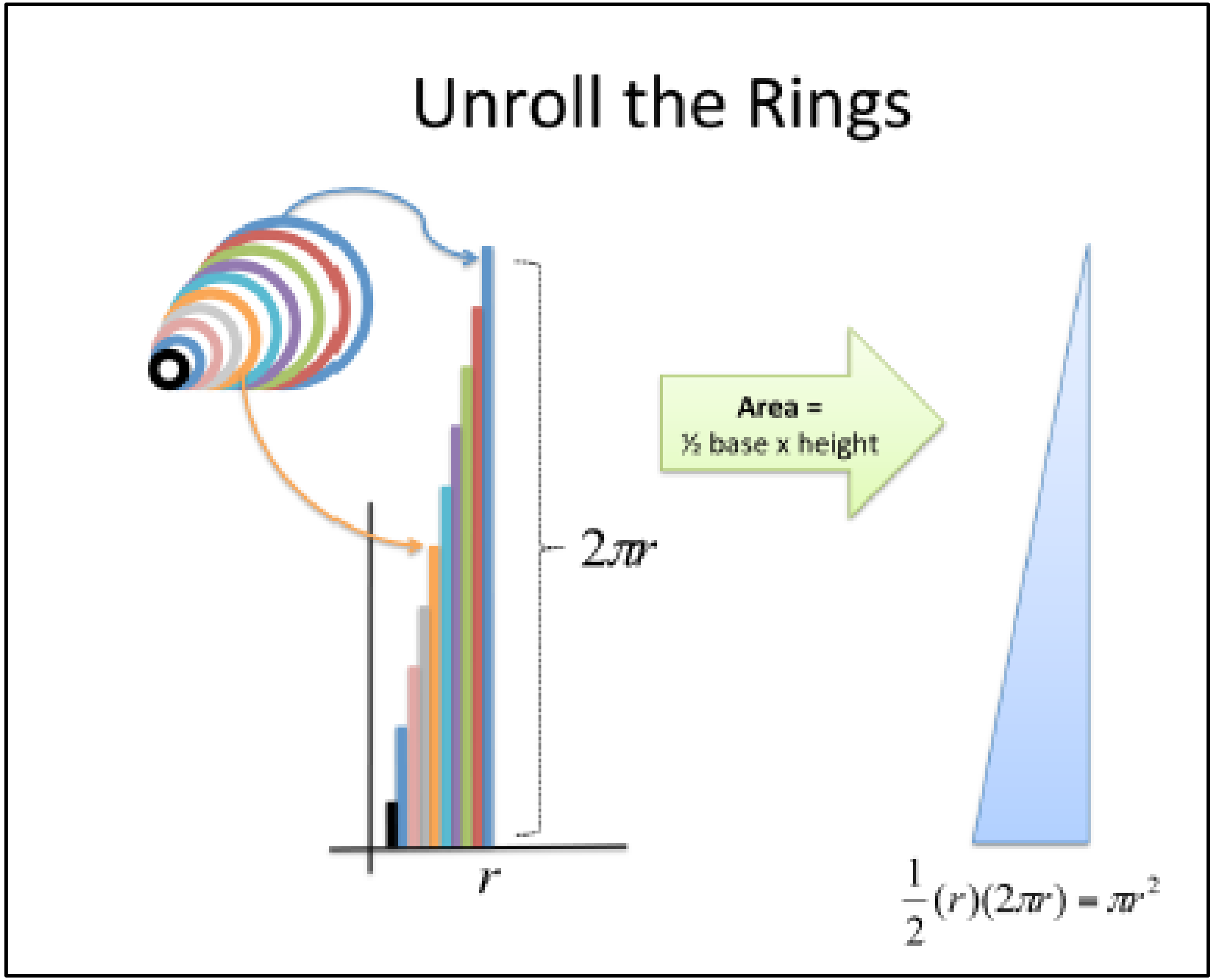
$0.48 < x < 0.52$

Centre of gravity paradox

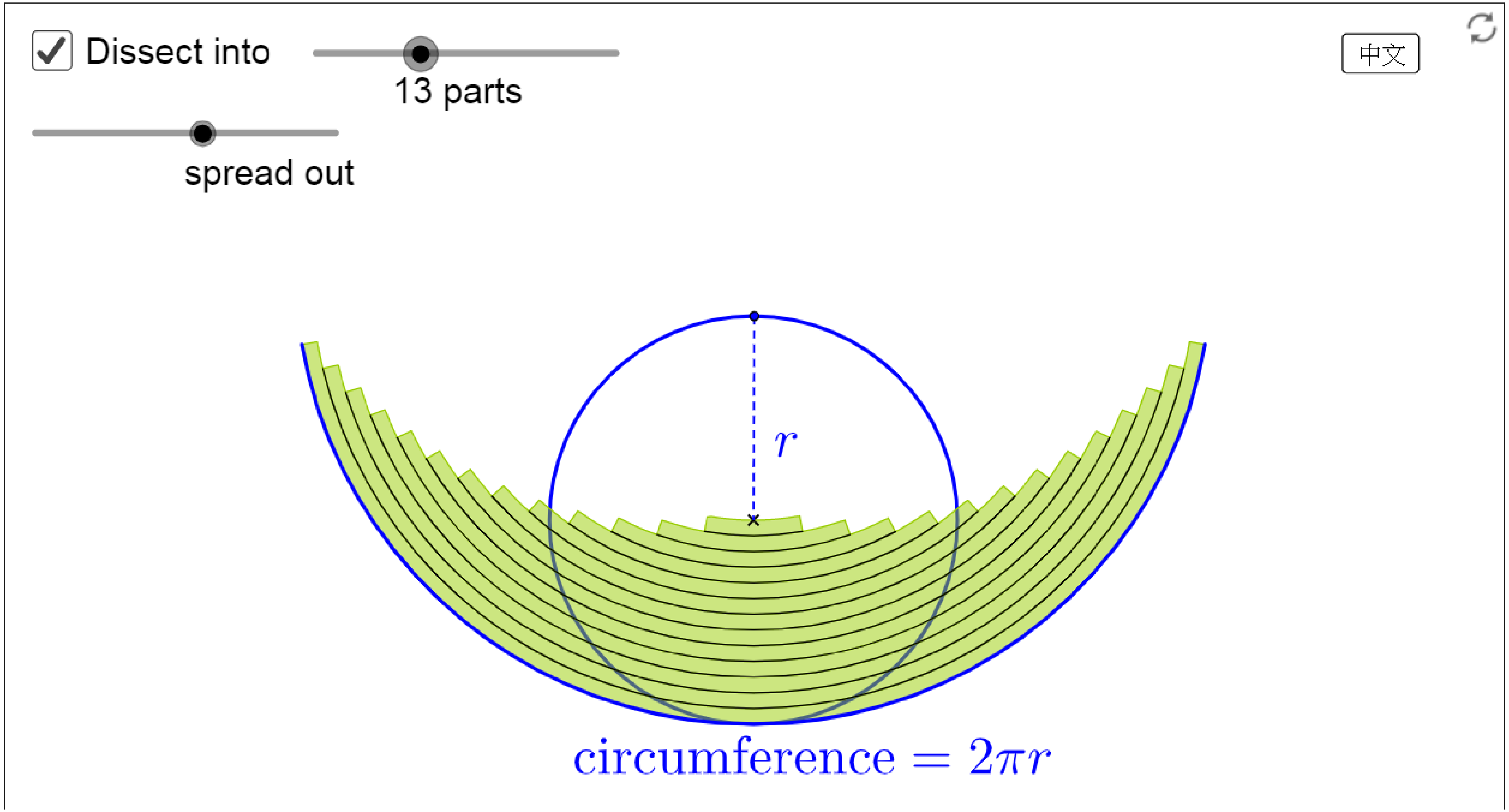




(Azad, 2013; 2015)

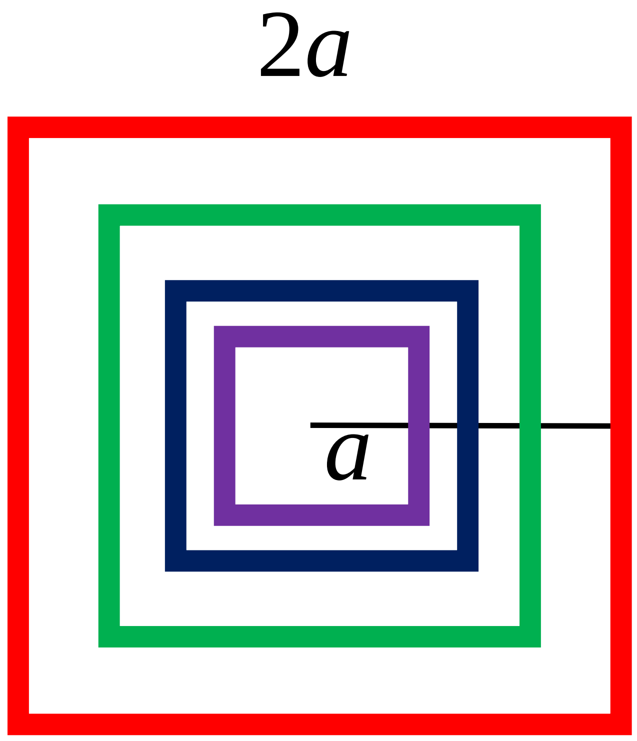


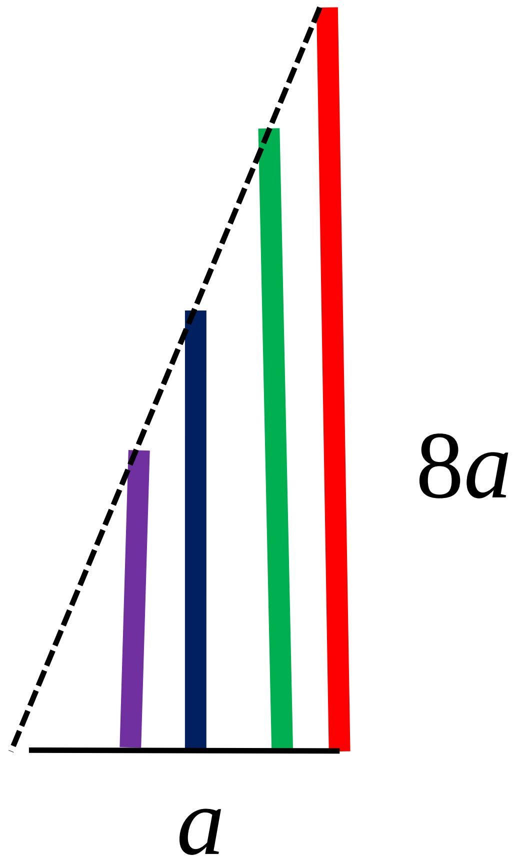
(Azad, 2013; 2015)

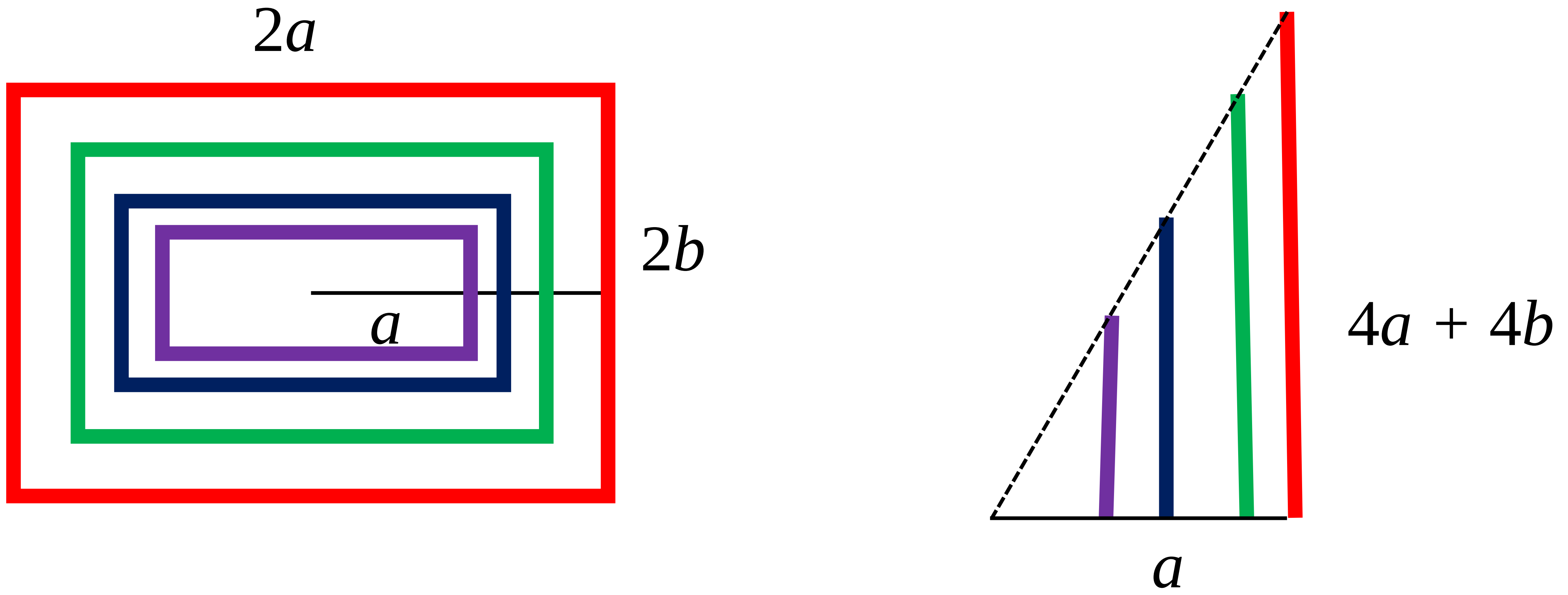


https://www.geogebra.org/m/r5VBs842

$$\text{Area} = (a)(8a)/2 = 4a^2$$

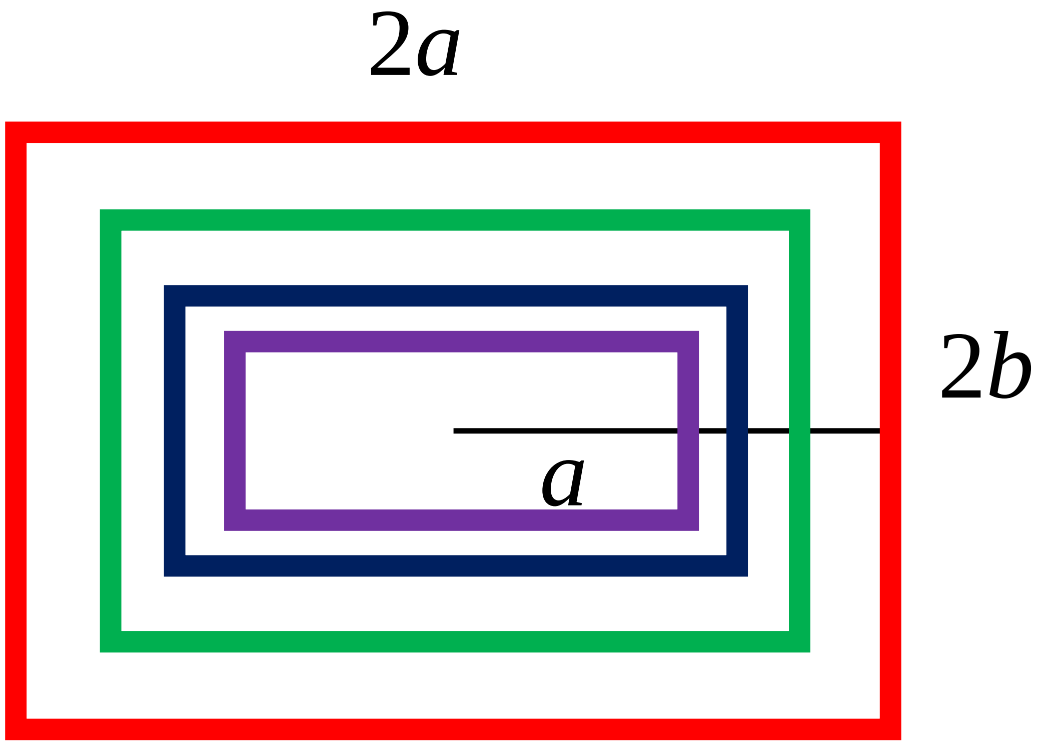


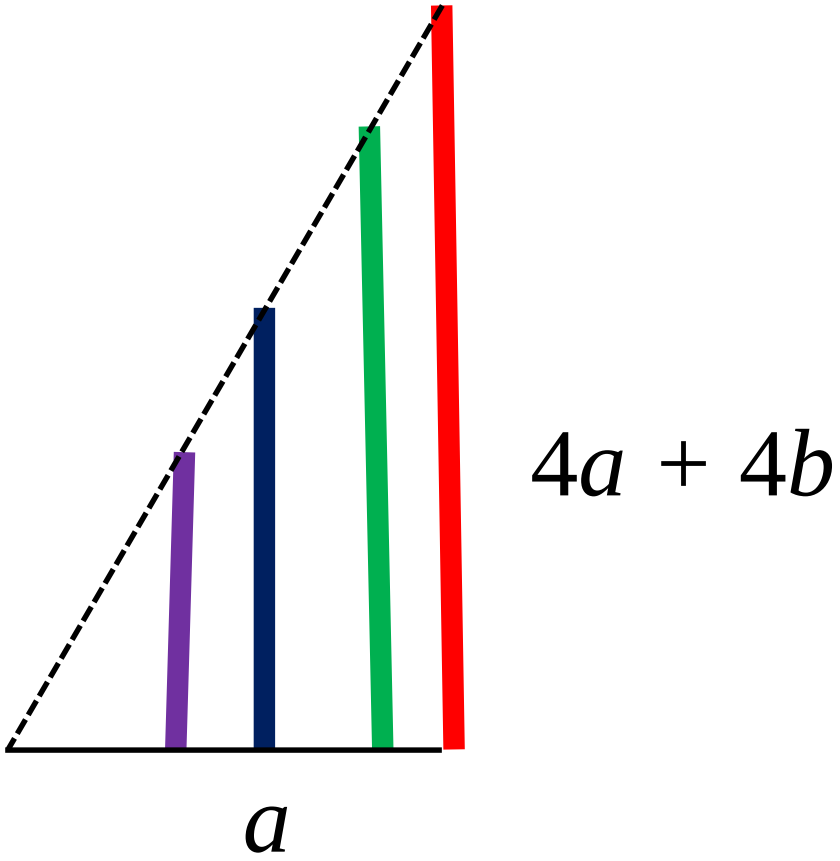




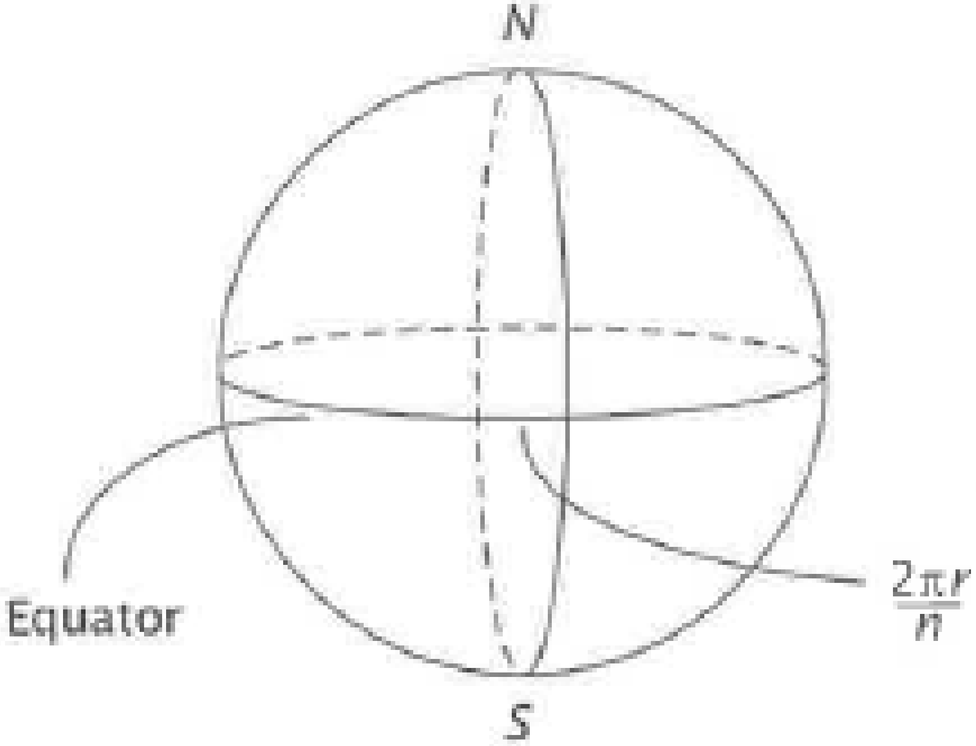
2a
2b
a
4a + 4b
a

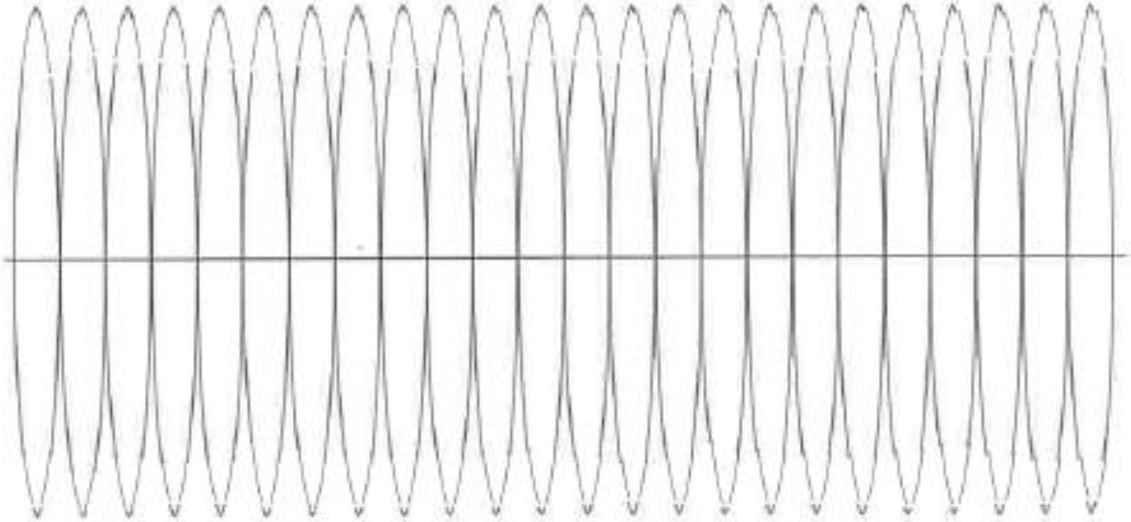
$$\text{Area} = (a)(4a + 4b)/2 = 2a(a + b)$$



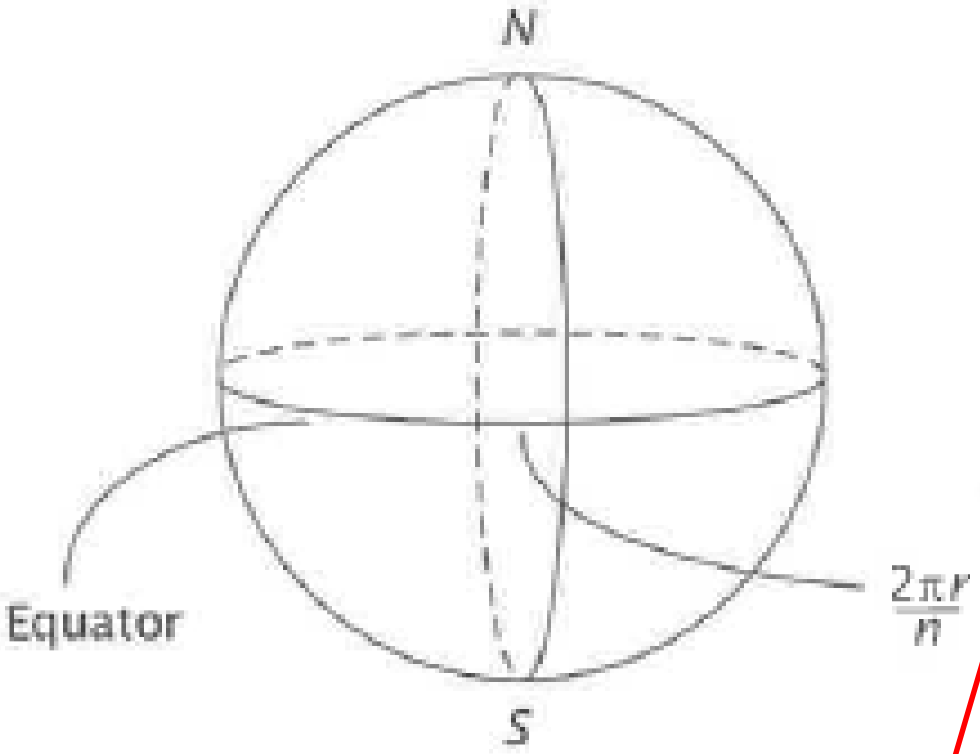


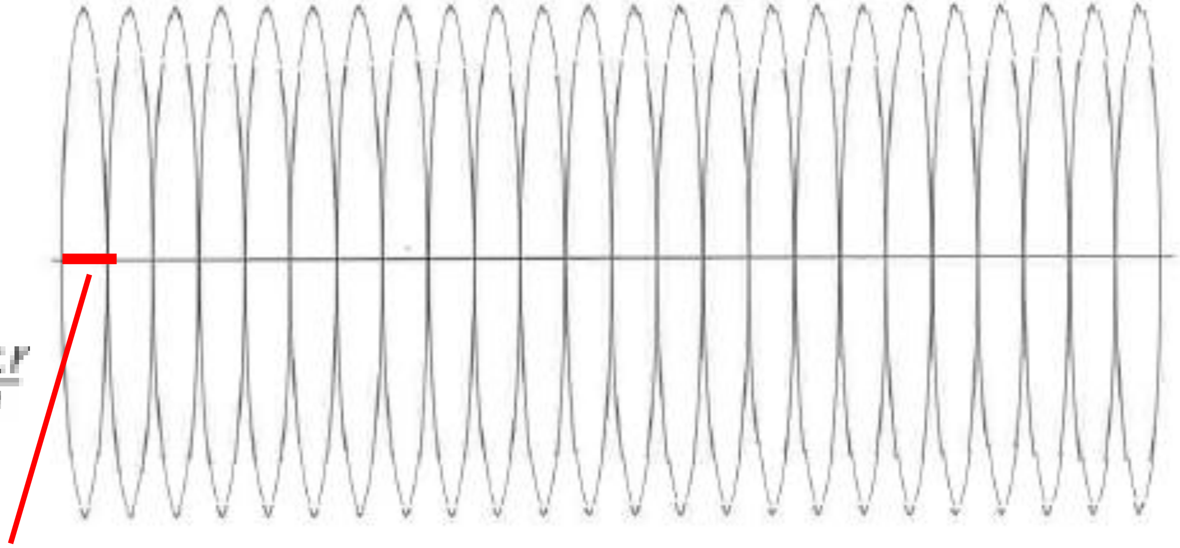
Area of a sphere





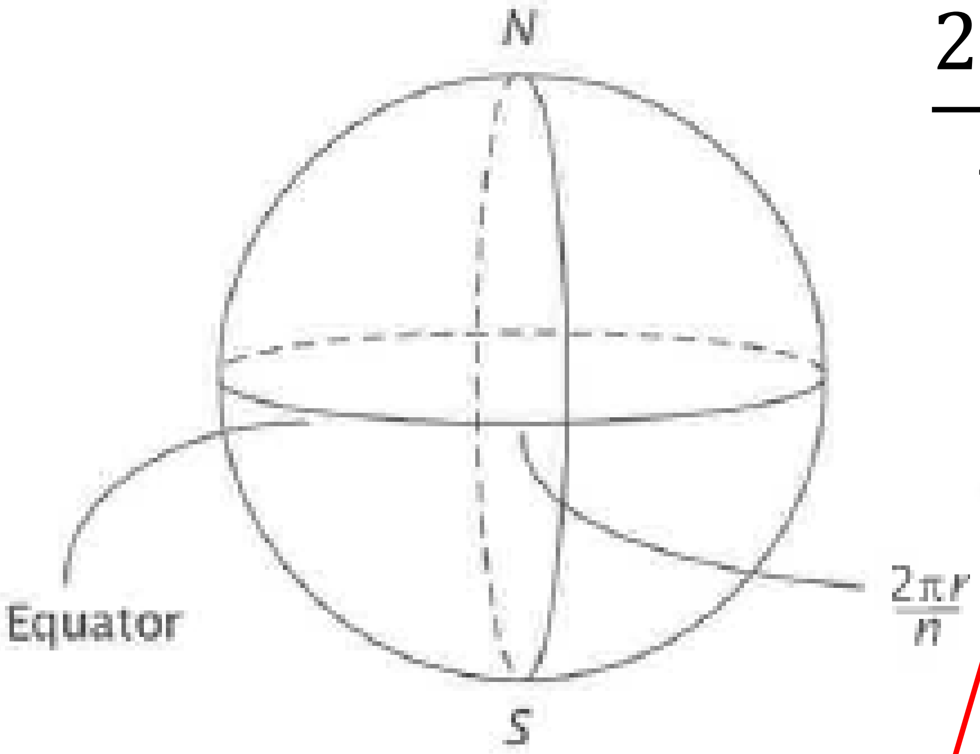
Area of a sphere

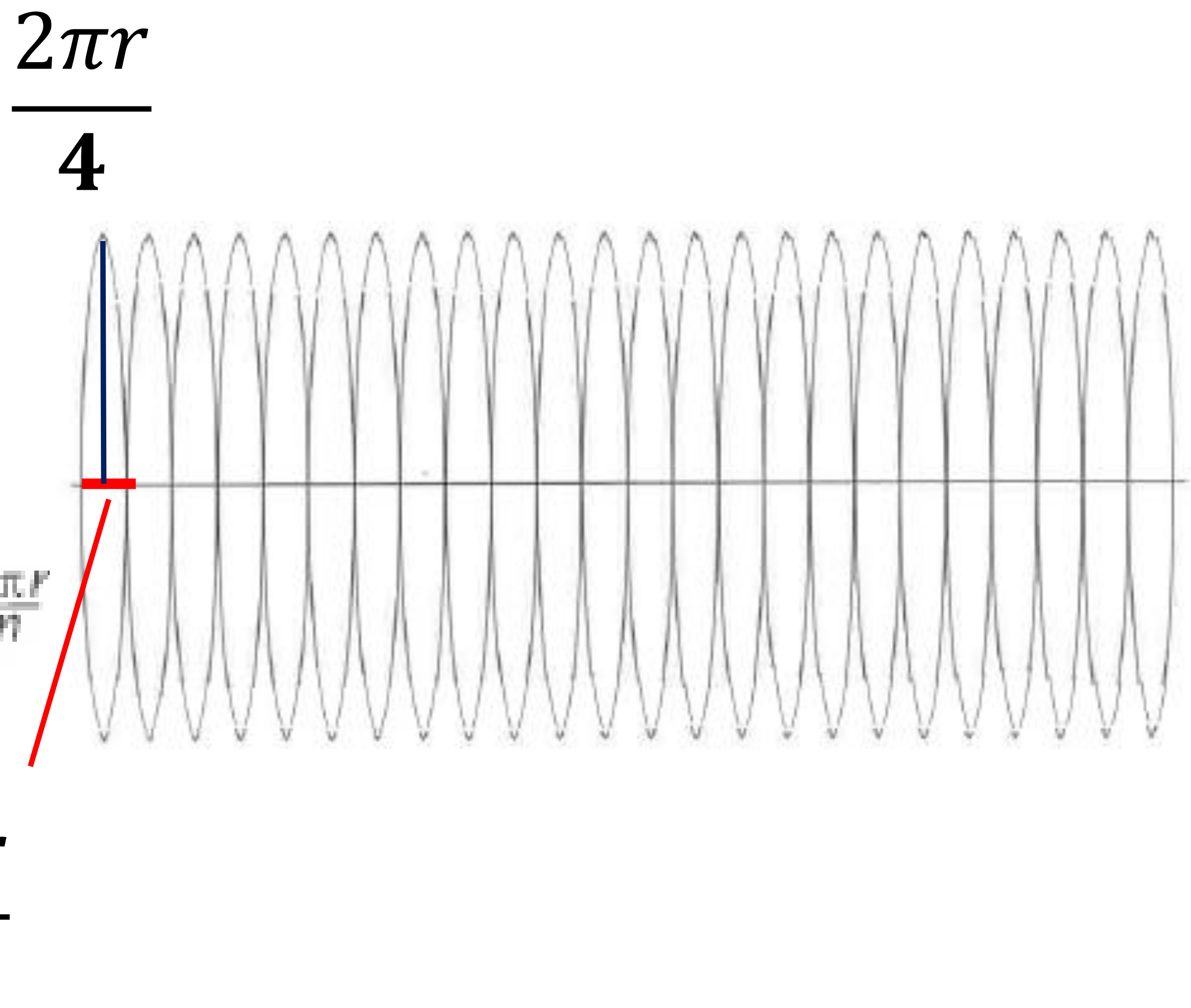




$$\frac{2\pi r}{n}$$

Area of a sphere

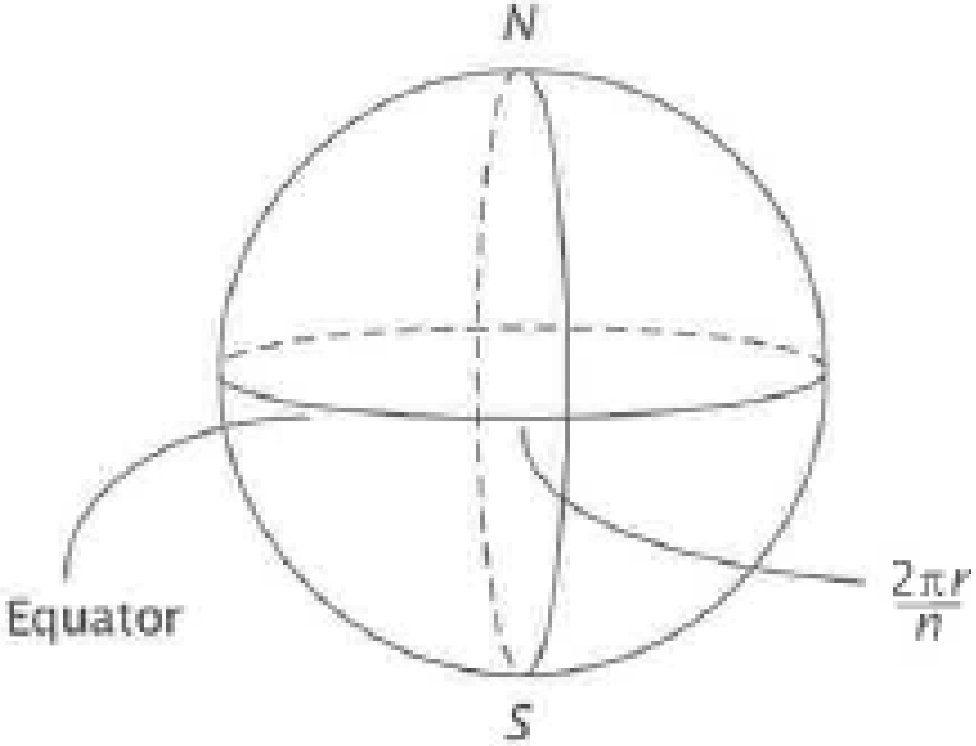


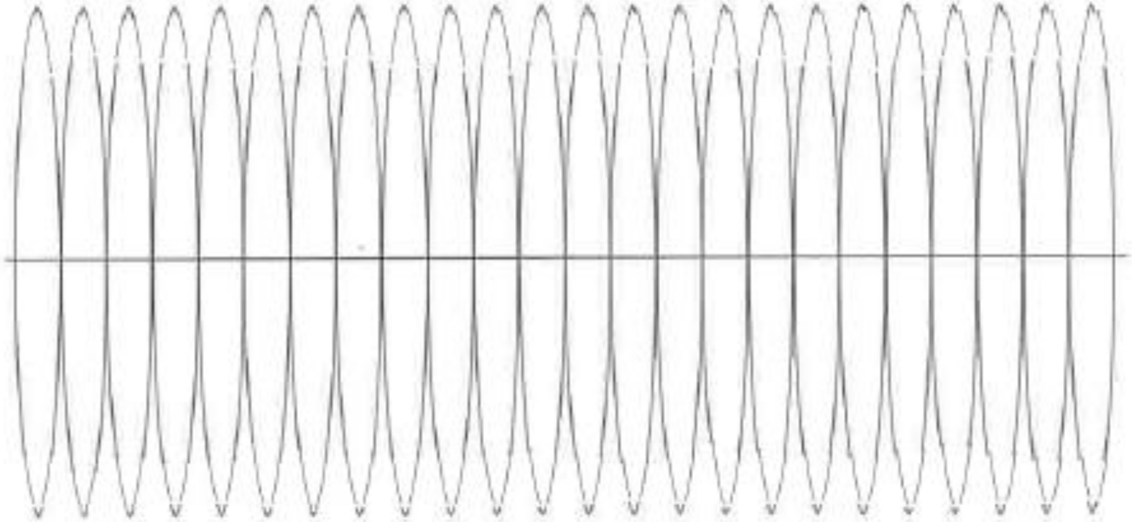


Area of a sphere

$$= 2 \times \frac{1}{2} \times \frac{2\pi r}{n} \times \frac{2\pi r}{\mathbf{4}} n$$

$$= \pi^2 r^2$$





Sorites paradox
(Paradox of heap)

Sorites paradox (Paradox of heap)

1 grain of wheat does not make a heap.
If n grains don't make a heap, then $n + 1$ grains don't.

Therefore,
1 million grains don't make a heap.

Surprise examination paradox

A teacher tells her students that the examination will be held on one weekday in the following week but that the examination will be a surprise to the students.

Proof by Mathematical Induction

(A) For any positive real numbers x and any positive integer n, $(1+x)^n \geq 1+nx$

Proof by Mathematical Induction

(A) For any positive real numbers x and any positive integer n, $(1+x)^n \geq 1+nx$

Proof:

It is obviously true for $n = 1$.

For any positive integer k, if $(1+x)^k \geq 1+kx$

$$\begin{aligned}(1+x)^{k+1} &\geq (1+x)(1+kx)\\ &= 1+(k+1)x+kx^2\\ &\geq 1+(k+1)x\end{aligned}$$

Proof by Mathematical Induction

(A) For any positive real numbers x and any positive integer n, $(1+x)^n \geq 1+nx$

(B) For any positive real numbers x and any positive integer n, $(1+x)^n > nx$

(A) is *stronger* than (B) as (B) can be deduced from (A).

But can we prove (B) by MI?

Proof by Mathematical Induction

Proof:

It is obviously true for $n = 1$.

For any positive integer k, if $(1+x)^k > kx$

$(1+x)^{k+1} > (1+x)kx$

$$(1+x)kx > (k+1)x$$

$$\Leftrightarrow x(k + kx - k - 1) > 0$$

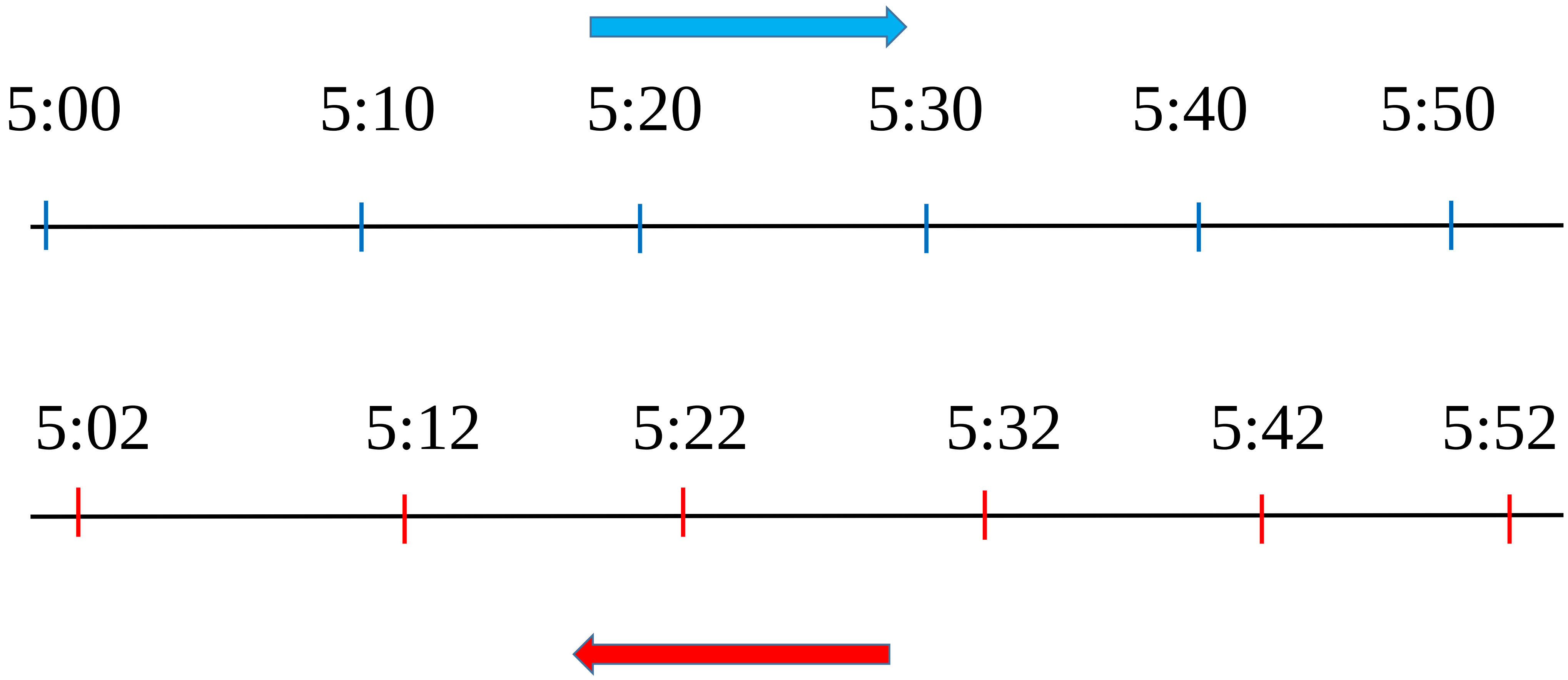
$$\Leftrightarrow kx > 1$$ *Not necessarily true!*

Proof by Mathematical Induction

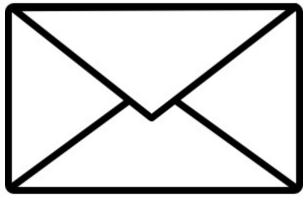
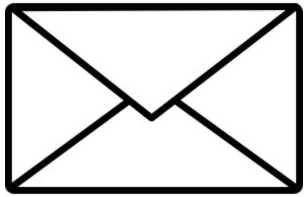
$$\begin{array}{ccc} P_A(k) & \Rightarrow & P_A(k+1) \\ \Downarrow & & \Downarrow \\ P_B(k) & & P_B(k+1) \end{array}$$

Unfair subway paradox

Unfair subway paradox



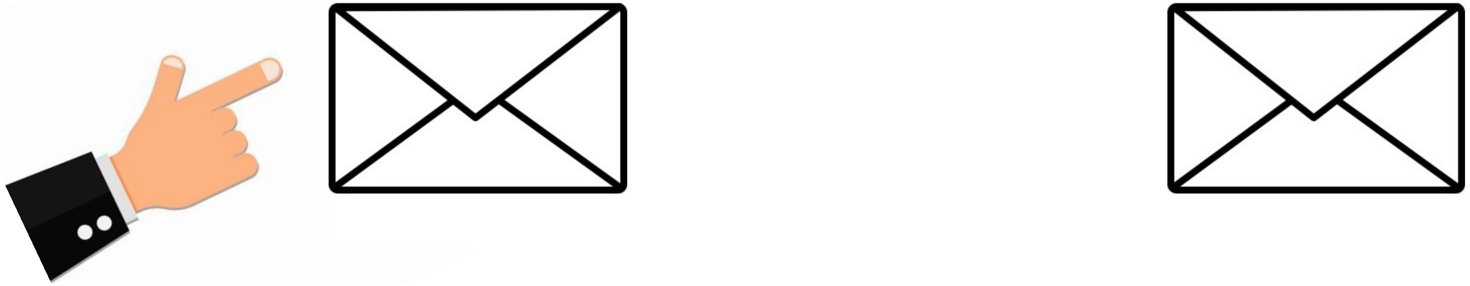
Two-evelopes paradox

One contains twice as much money as the other

Switch or not?

Two-evelopes paradox

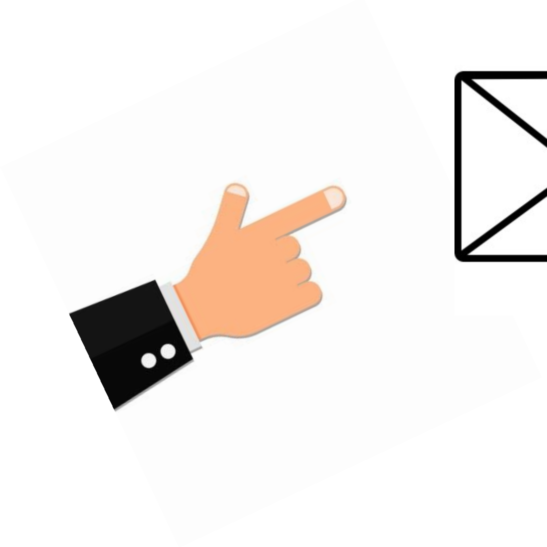


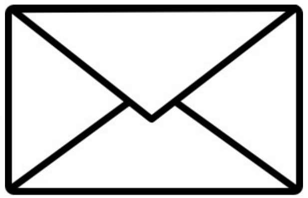
One contains twice as much money as the other

Switch or not?

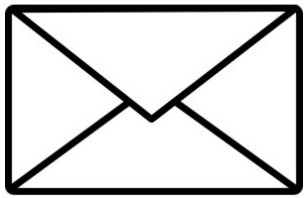
By symmetry, no need to switch!

Two-evelopes paradox



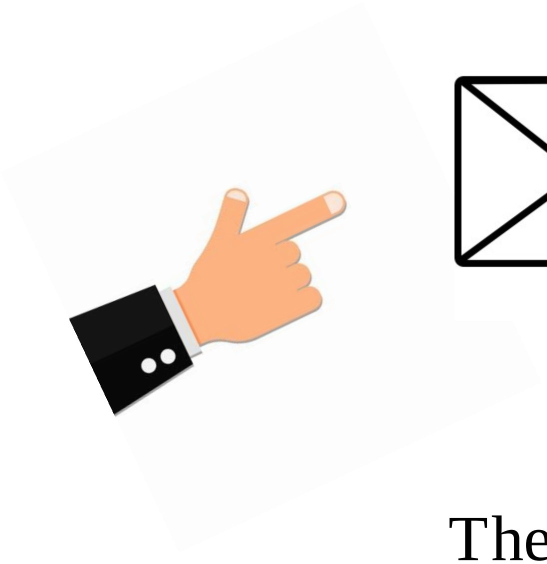


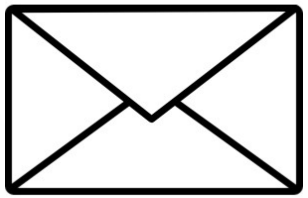
$\$M$



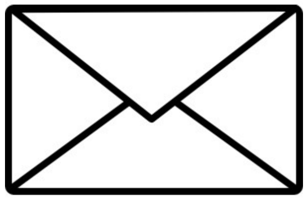
$\$2M \text{ or } \$\frac{M}{2}$

Two-evelopes paradox





$\$M$



$\$2M$ or $\$\dfrac{M}{2}$

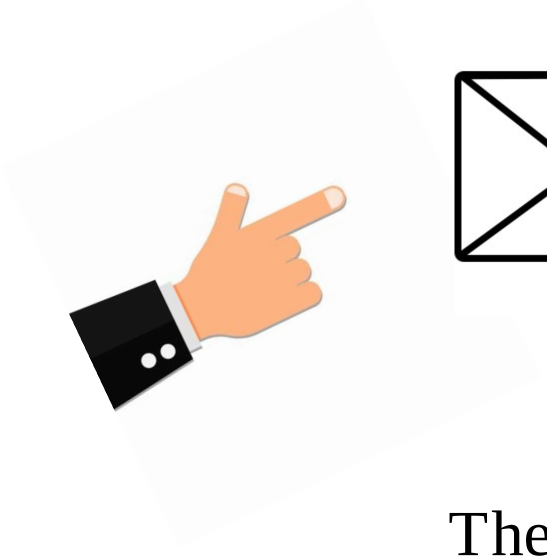
The expected value of money in the other evelope

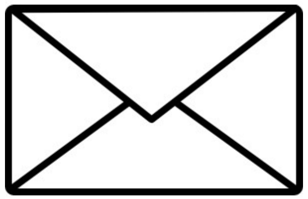
$$= \$(2M \times \frac{1}{2} + \frac{M}{2} \times \frac{1}{2})$$

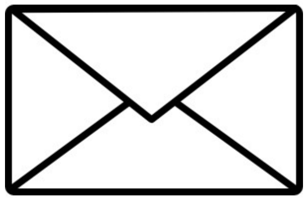
$$= \$\frac{5M}{4} > \$M$$

Should switch!

Two-evelopes paradox







$\$M$

$\$2M$ or $\$\frac{M}{2}$

1 Equivocation!

The expected value of money in the other evelope

$$= \$(2M \times \frac{1}{2} + \frac{M}{2} \times \frac{1}{2})$$

$$= \$\frac{5M}{4} > \$M$$

2 Switch again!

Two-evelopes paradox

This paradox has been mentioned in a talk by the Fields medallist Martin Hairer at the Heidelberg Laureate Forum 2017.

https://plus.maths.org/content/two-envelopes-problem-resolution

Doomsday argument

How long will our human race survive?

Doomsday argument

How long will our human race survive?




Doomsday argument

A: 1 – 20 balls

B: 1 – 2000 balls

Doomsday argument

A: 1 – 20 balls

B: 1 – 2000 balls

From *A* or from *B*?

Doomsday argument

A: 1 – 20 balls

B: 1 – 2000 balls

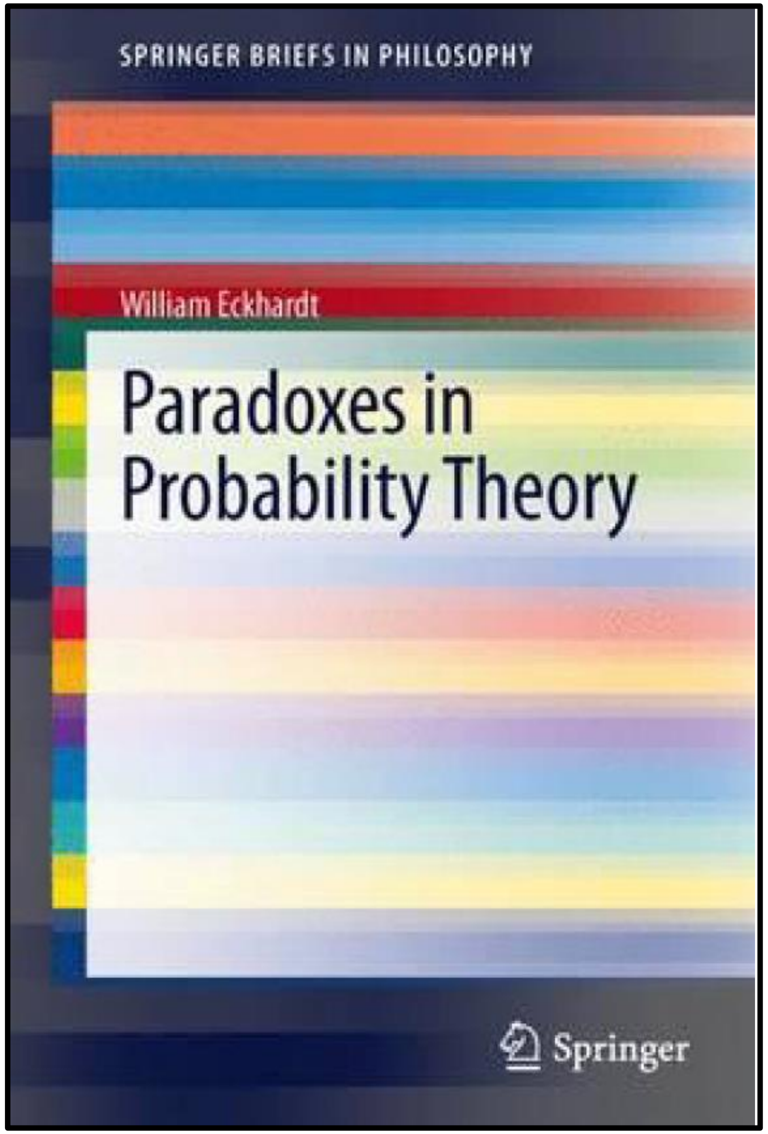
From A or from B?

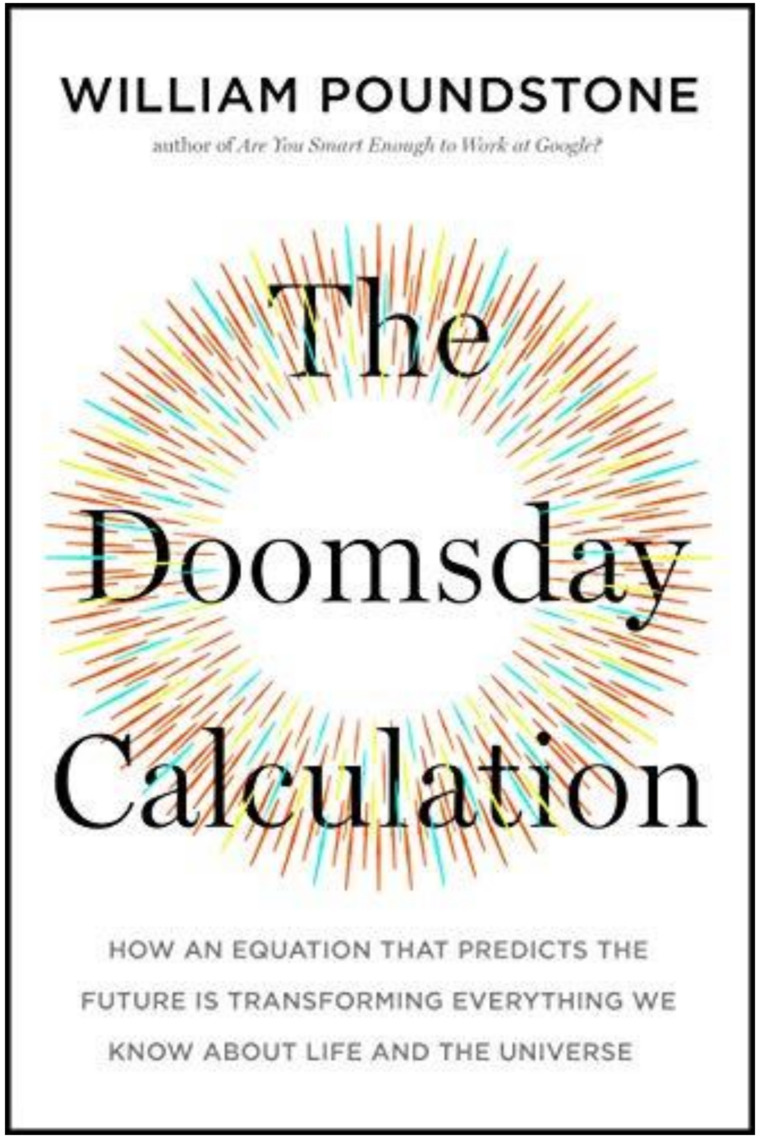
$$P(B|3)$$

$$=\frac{P(3|B)P(B)}{P(3|B)P(B)+P(3|A)P(A)}$$

$$=\frac{\frac{1}{2000}}{\frac{1}{2000}+\frac{1}{20}}$$

$$=\frac{1}{101}$$


SPRINGER BRIEFS IN PHILOSOPHY
William Eckhardt
Paradoxes in Probability Theory
Springer


WILLIAM POUNDSTONE
author of Are You Smart Enough to Work at Google?
The Doomsday Calculation
HOW AN EQUATION THAT PREDICTS THE FUTURE IS TRANSFORMING EVERYTHING WE KNOW ABOUT LIFE AND THE UNIVERSE

Hilbert’s hotel paradox

Infinity and the Mind

the science and philosophy of the infinite

With a new preface by the author

RUDY RUCKER

Gödel incompleteness theorems

$\boxed{\text{This sentence is not provable in the system S}}$

… Mid-2020