Exemplar 10:
The effect of transformation on the graph of functions (3)

Objectives : 1. Understand the effect of reflection on graphs of functions and the corresponding changes on the algebraic form of function
2. Understand the relation between the graphs of the functions $y = f(kx)$, $y = kf(x)$ and that of the function $y = f(x)$

Key Stage : 4

Learning Unit : Functions and Graphs

Materials required : 1. Spreadsheet software such as *Microsoft Excel* (*graph.xls*), *Graphmatica* and worksheets
2. Graph papers and the transparency printed with the graphs of functions $y = x^2$ and $y = x^3$

Prerequisite Knowledge : 1. Understand the graph of function $f(x) = x^2$ and $f(x) = x^3$
2. Understand the general form of quadratic and cubic functions

Description of the activity:

1. The teacher revises with students the relation between the translation of the graphs of functions and the corresponding algebraic expressions.

Changes on the function $f(x)$	Transformation on the graph of function
$y = f(x) + k$, where $k > 0$	Translate upwards by k units
$y = f(x) - k$, where $k > 0$	Translate downwards by k units

Changes on the function $f(x)$	Transformation on the graph of function
$y = f(x+h)$, where $h>0$	Translate to the left by *h* units
$y = f(x-h)$, where $h>0$	Translate to the right by *h* units

The teacher may use examples such as $y = f(x)$ when $f(x) = x^3$, the functions $y = (x-1)^3$ and $y = x^3 - 1$ to check whether students master the idea of translations or not.

2. The teacher asks students to guess the relation between the graphs of original function $y = x^3$ and the function $y = -x^3$ (i.e. $y = -f(x)$). The teacher gives students the transparency with the graph of function $y = x^3$ printed on it and arranges students in groups to explore the graph of function $y = -x^3$. Students are then asked to explain the relation between these two graphs by checking the coordinates of the corresponding points.

3. The teacher uses the *Excel* file *graph.xls* to explore the relation between graphs of various quadratic or cubic functions (such as $y = x^3 + 3x^2 - 1$) and their reflections in the *x*-axis and the corresponding changes on the algebraic expressions. Students may attend to the following points:
   (a) the number of points where the new and the original graphs cut the *x*-axis and their coordinates;
   (b) the number of points where the new and the original graphs cut the *y*-axis and their coordinates;
   (c) The coordinates of corresponding points on the new and the original graphs;
   (d) The relation between the coefficients of terms in the new and original algebraic expressions.

4. The teacher discusses with students the relation between the graphs of the original and the new functions, if the function $y = f(x)$ where $f(x) = x^3 + 3x^2 - 1$ is changed to $y = (-x)^3 + 3(-x)^2 - 1$ (i.e. $y = f(-x)$). The teacher may distribute the transparency with the graph of function $y = x^3 + 3x^2 - 1$ printed on it to let students explore the corresponding change on the graph. Students are requested to discuss with their classmates on the relation between the two graphs by looking at the coordinates of corresponding points.

5. The teacher concludes that when the graph of function $y = f(x)$ is reflected in the *y*-axis, it becomes $y = f(-x)$. When the graph of function $y = f(x)$ is reflected in the *x*-axis, it becomes $y = -f(x)$. The teacher uses the *Excel* file *graph.xls* to

explore the relation among the algebraic expression, tabular and graphical representations of various quadratic and cubic functions if they are reflected in the x, y-axes. Comparisons are to made on:

(a) the coordinates of the intersecting points with the x / y-axis before and after reflections in the y-axis;

(b) the differences in the graphs of functions between the reflection in x-axis and that in the y-axis;

(c) the differences in algebraic expressions of the function when the function is reflected in the x-axis and that in the y-axis.

6. The teacher may play the following games with students to consolidate their understanding on the reflection transformation:

   (a) put the transparency printed with the graph of the function $y = x^3 + 3x^2 - 1$ on different positions of the graph paper. Then, the teacher asks students to find the corresponding image when the function is reflected in the x-axis / the y-axis;

   (b) give students the transparency with graphs of functions after transformation printed on it. They are asked to identify the transformation undergone.

7. The teacher explains the transformation $y = -f(x)$ is a special case of function $y = kf(x)$. Meanwhile, the teacher discusses with students the relation between the new graph and the original graph when the function from $y = f(x)$ is changed to $y = 2f(x)$. The teacher uses polynomial functions such as $f(x) = x^2$ for discussion. If students have learned the trigonometric functions, the teacher could use $f(x) = \sin x$, etc. for students to discuss the transformation. Alternatively, the teacher uses the tabular or graphical representation of the transformation $y = 2f(x)$ to discuss the changes even though students do not have any ideas of trigonometric functions.

8. The teacher may use the graph plotting software *Graphmatica* to illustrate the graph of function $y = kf(x)$ when stretched vertically for $k > 1$ and that of $y = kf(x)$ when compressed vertically for $0 < k < 1$.

9. Similarly, the teacher may use the software *Graphmatica* to explore the changes on the graphs of functions $y = f(x)$ and $y = f(kx)$ for $k > 1$. The teacher then discusses with students:

   (a) the similarity between the two graphs;

   (b) the features between the two graphs and the coordinates of intersecting points with the x-axis;

(c) if there is/are maximum value(s) of y for $y = f(x)$, compare the maximum values between $y = f(kx)$ and $y = f(x)$;

(d) if the function $y = f(x)$ is a periodic function, compare the period between the functions $y = f(kx)$ and $y = f(x)$;

(e) compare between the graphs of $y = f(kx)$ for $k > 1$ and $0 < k < 1$ to find the similarity and difference by going through the steps (a) to (d).

10. Finally, the teacher draws the following conclusions:

Algebraic form of the function		The graph of the function
$y = kf(x)$	$k > 1$	Stretch across y-axis by k times
	$0 < k < 1$	Compress across y-axis by k times
$y = f(kx)$	$k > 1$	Stretch across x-axis by k times
	$0 < k < 1$	Compress across x-axis by k times

11. The teacher distributes the Worksheet for consolidation. Students are requested to:

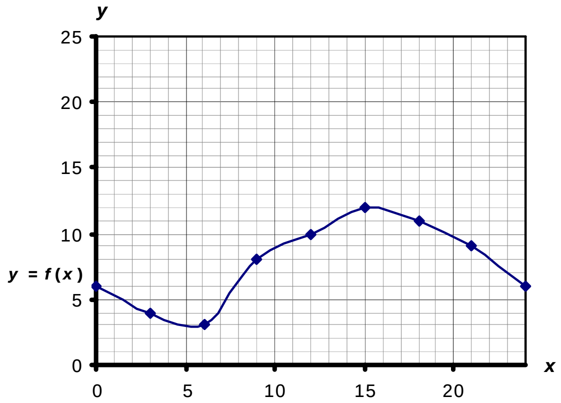
(a) determine the algebraic expressions and the transformations involved from the given changes on the graphs of functions;

(b) sketch the graphs of functions after transformations from the given changes on the algebraic expressions.

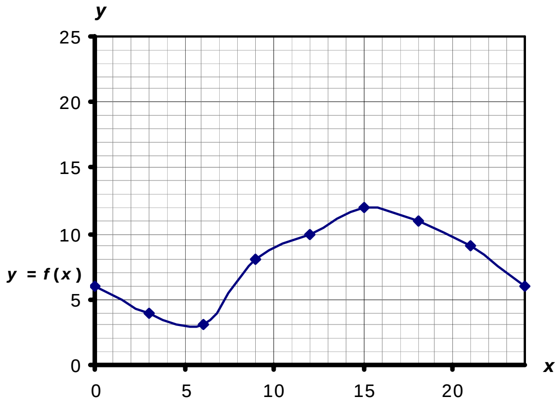
Worksheet

1. The figure below shows the graph of the function $y = f(x)$. In parts (a) to (e), sketch the graphs of functions after the given transformation.

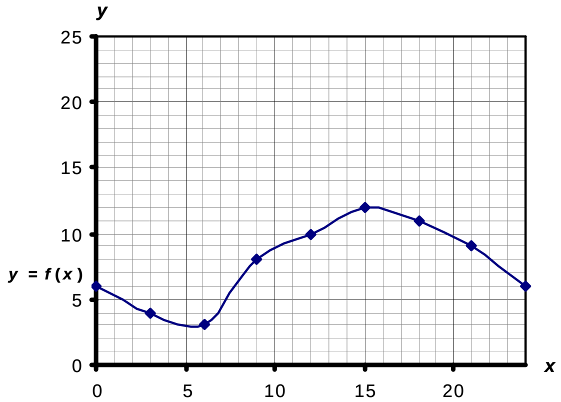
(a) Sketch the graph of function $y = 2f(x)$.



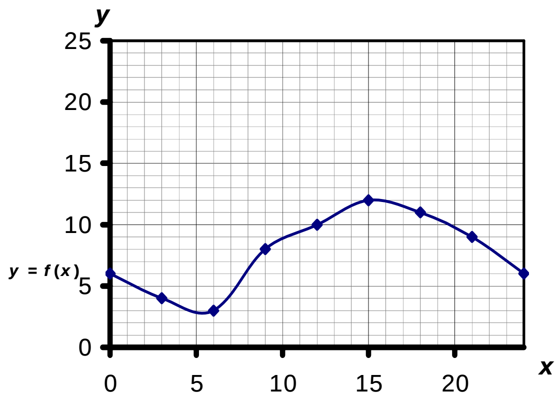
(b) Sketch the graph of function $y = \frac{1}{2} f(x)$.



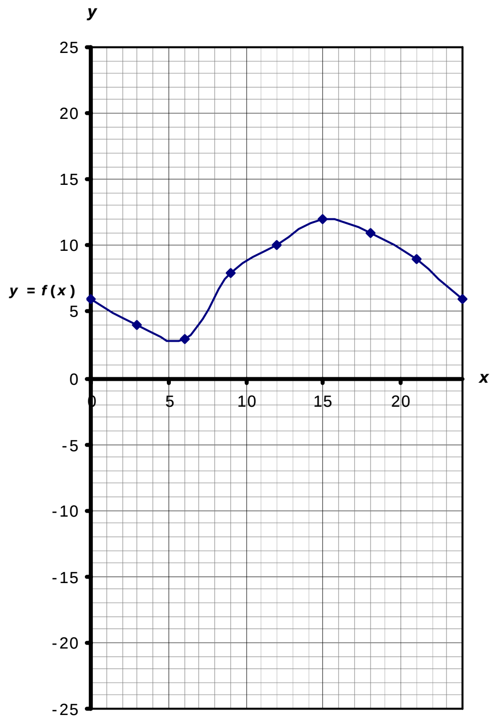
(c) Sketch the graph of function $y = f(2x)$.



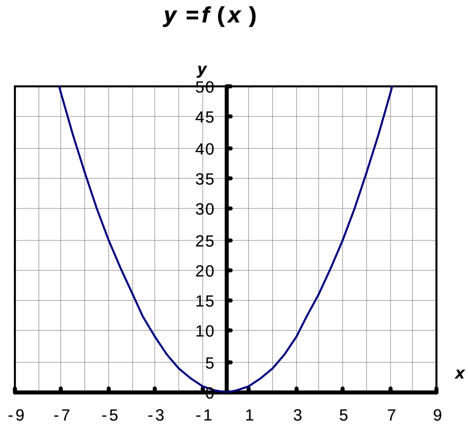
(d) Sketch the graph of function $y = f(\frac{x}{2})$.



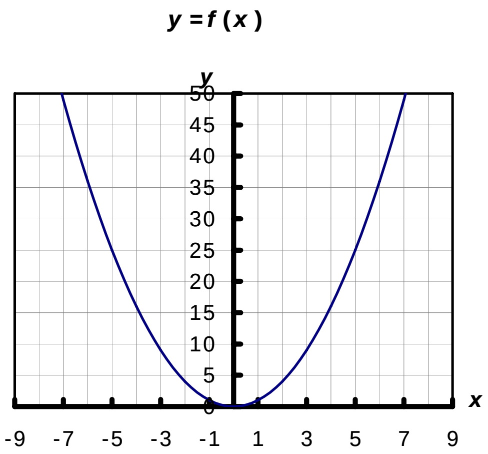
(e) Sketch the graph of function $y = -f(x)$.



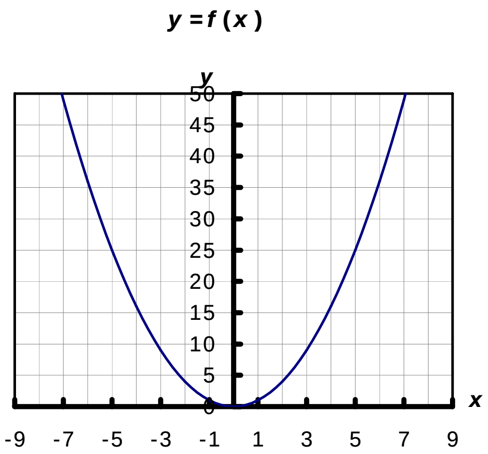
2. The figure on the right shows the graph of quadratic function $f(x) = x^2$. Based on the given information, sketch the corresponding graphs after the transformation.



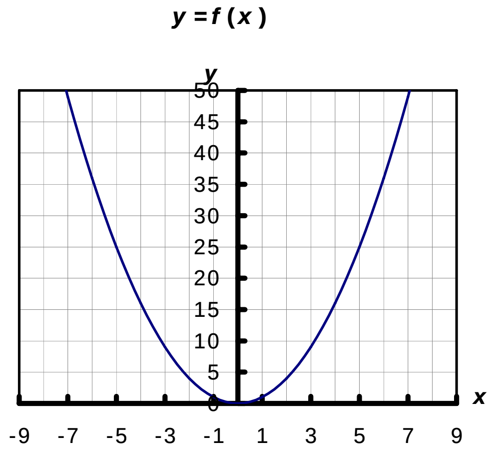
(a) Sketch the graph of function $y = f(2x)$.



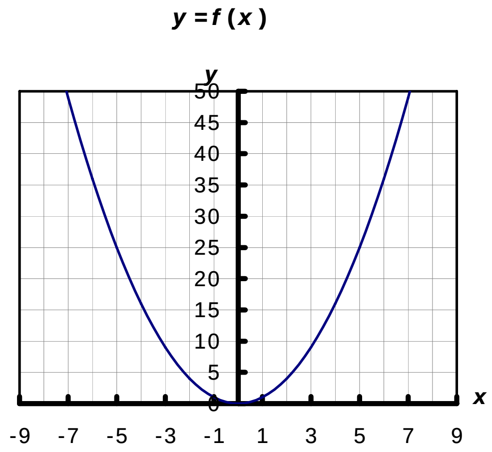
(b) Sketch the graph of function $y = 2f(x)$.



(c) Sketch the graph of function $y = 2f(x) + 10$.



(d) Sketch the graph of function $y = 2f(x + 5)$.



(e) Compare the graphs of functions $y = 2f(x) + 10$ and $y = 2f(x + 5)$ in parts (c) and (d) above to check whether they are the same or not. In other words, is the function $y = kf(x) + k \cdot h$ the same as the function $y = kf(x + h)$?

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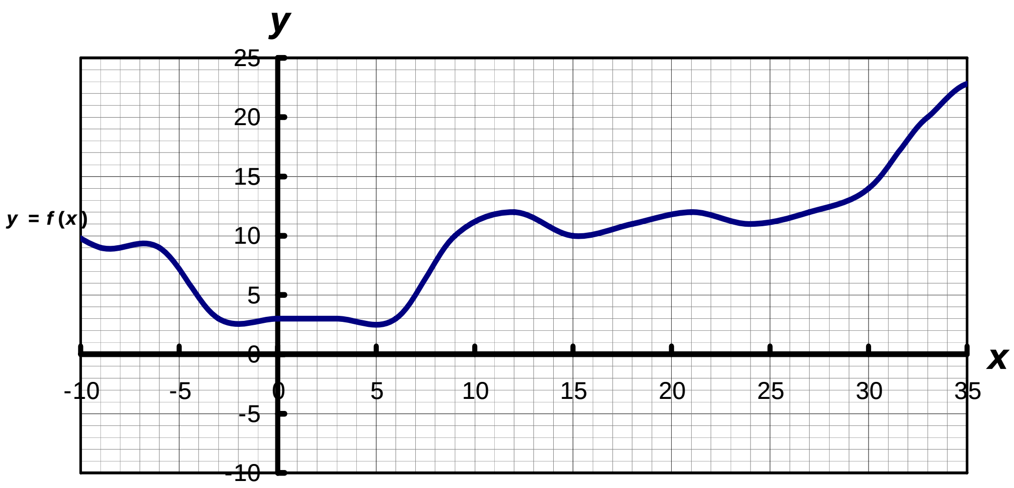
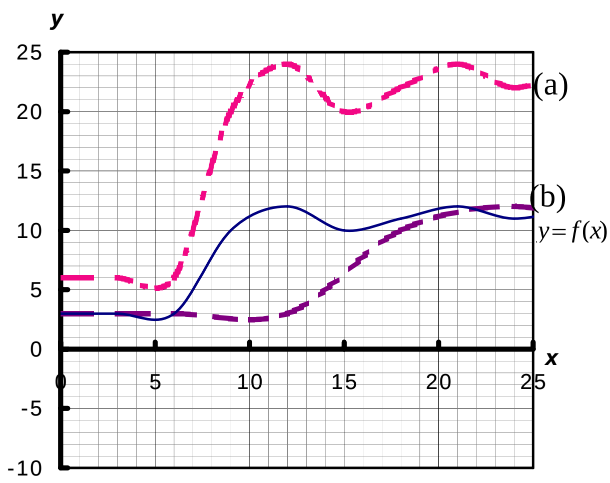
3. Determine the corresponding algebraic expressions from the given graphs of function after the transformation.

The original graph of function $y = f(x) \quad (-10 \le x \le 35)$



The graphs of function after transformation $(0 \le x \le 25)$

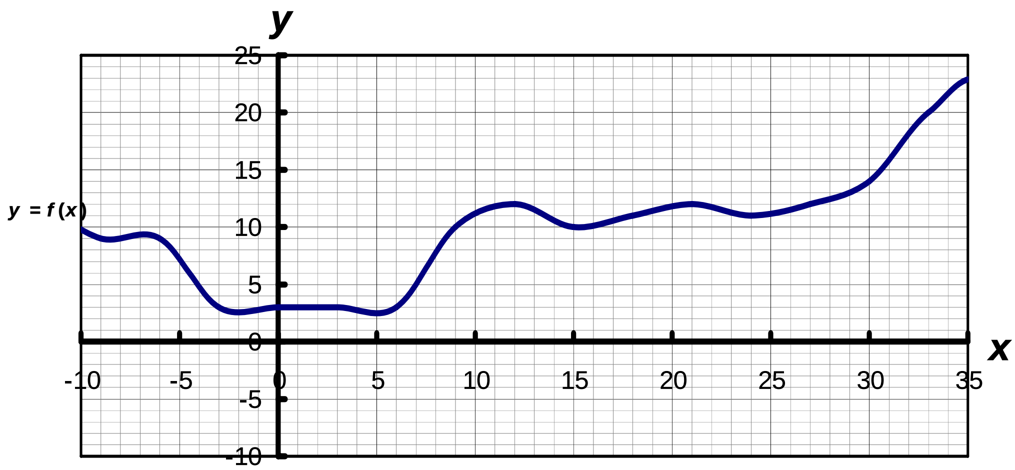


(i) By considering the relation between the graph (a) and the function $y = f(x)$, write down the equation of function of graph (a). ______________________

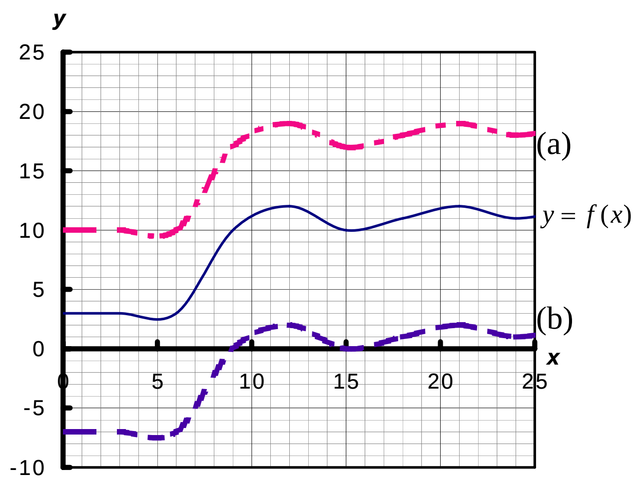
(ii) By considering the relation between the graph (b) and function $y = f(x)$, write down the equation of function of graph (b). ______________________

4. Determine the corresponding algebraic expressions from the given graphs of function after the transformation.

The original graph of function $y = f(x)$ $(-10 \le x \le 35)$



The graphs of function after transformation $(0 \le x \le 25)$

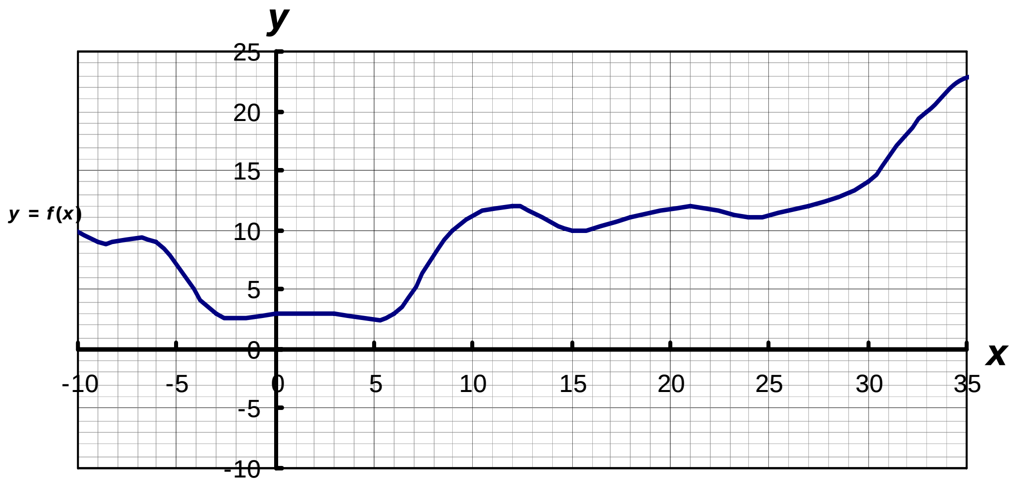


(i) By considering the relation between the graph (a) and the function $y = f(x)$, write down the equation of function of graph (a). ________________________

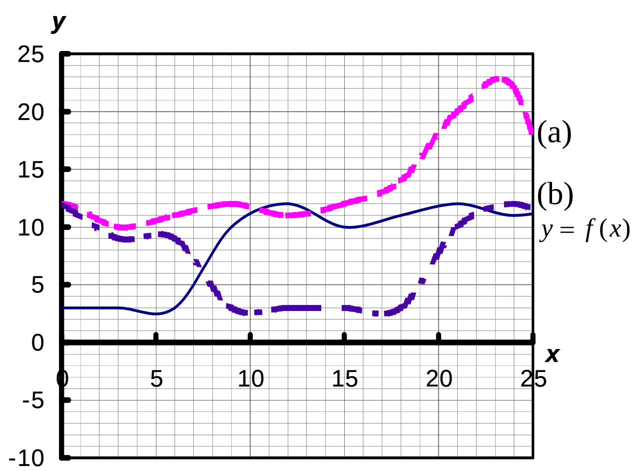
(ii) By considering the relation between the graph (b) and the function $y = f(x)$, write down the equation of function of graph (b). ______________________

5. Determine the corresponding algebraic expressions from the given graphs of function after the transformation.

The original graph of function $y = f(x)$ $(-10 \le x \le 35)$



The graphs of function after transformation $(0 \le x \le 25)$

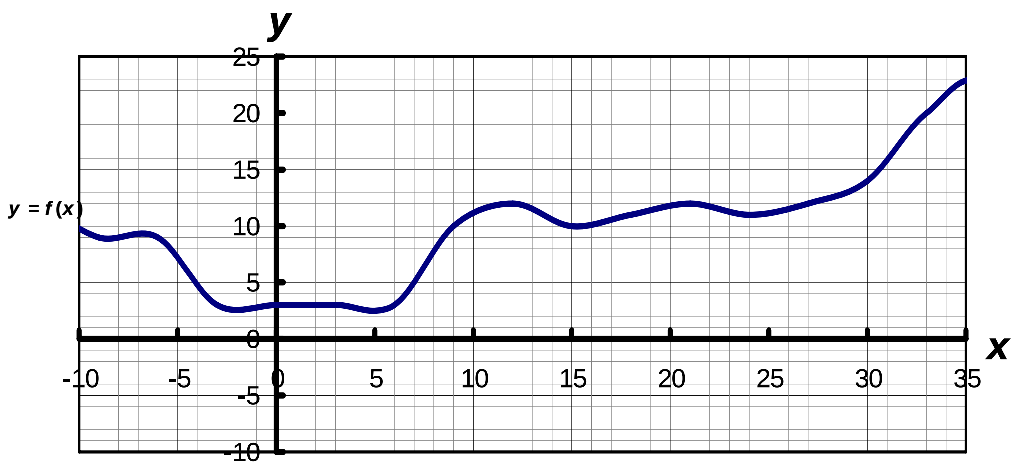


(i) By considering the relation between the graph (a) and the function $y = f(x)$, write down the equation of function of graph (a). ________________________

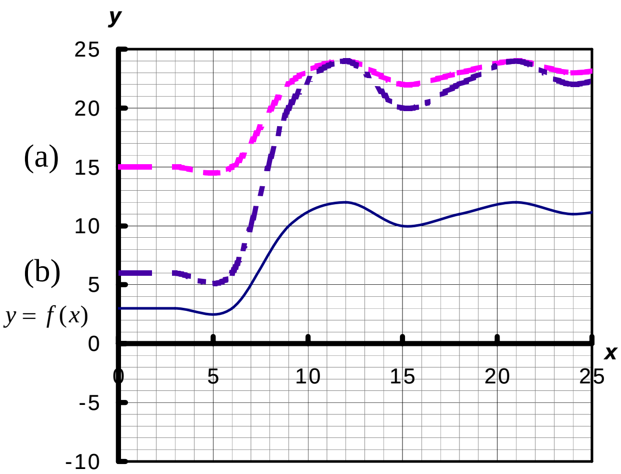
(ii) By considering the relation between the graph (b) and the function $y = f(x)$, write down the equation of function of graph (b). ________________________

6 . Determine the corresponding algebraic expressions from the given graphs of function after the transformation.

The original graph of function $y = f(x)$ $(-10 \le x \le 35)$



The graphs of function after transformation $(0 \le x \le 25)$



(i) By considering the relation between the graph (a) and the function $y = f(x)$, write down the equation of function of graph (a). ________________________

(ii) By considering the relation between the graph (b) and function $y = f(x)$, write down the equation of function of graph (b). ________________________

Notes for teacher:

1. The time required for the activities in this exemplar is approximately 70 to 80 minutes.

2. The teacher should note that when $y = x^2$ is used as an example to demonstrate reflectional transformation, there is no difference between both the graphs and algebraic forms of $y = f(x)$ and $y = f(-x)$. Thus, $y = x^3$ is used in this exemplar to illustrate reflection in the *y*-axis. Nevertheless, as the algebraic expressions and the graphs of $y = -x^3$ and $y = (-x)^3$ are identical, it is unsuitable to use them to contrast the differences between $y = f(-x)$ and $y = -f(x)$. Thus, the cubic function $y = x^3 + 3x^2 - 1$ is used in this exemplar to illustrate reflection in the *x*-axis.

3. It is easier for students to compare the coordinates of the points where the graph of the original function and the graph of the function after transformation cut the *x*-axis. Alternatively, the teacher may consider using other quadratic or cubic functions for discussion at the very beginning.

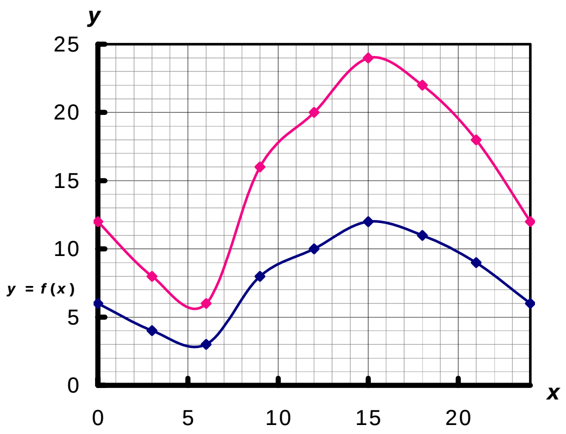
4. Using transparencies by students to illustrate the effect of reflection will help them consolidate the concept of reflection learnt at lower forms. Through comparing coordinates of points between the 2 graphs, students have a better understanding between the algebraic forms and the graphical representations. In addition, using transparency will help student save the time required to draw the graphs repeatedly. The teacher may make copies of different graphs for students to explore the graphs after reflection in the class game activity. The functions such as $y = x^3 + 3x^2 - 1$ and $y = x^3 + x^2 + 1$ may be adopted to vary the level of the difficulty of the activity.

5. When the teacher uses trigonometric functions such as $y = \sin x$ or $y = \cos x$ to perform the enlargement or reduction on the graphs and compare their changes before and after the transformations. It should be noted that students may not have the concept of $\sin 120°$, etc. The teacher may consider just to compare the coordinates of points of the graphs $y = f(x)$, $y = f(kx)$ and $y = kf(x)$ without specifying whether it is a graph of sine or cosine. Of course, the teacher may also use polynomial functions to illustrate the difference between graphs. Nevertheless, the changes on graphs before and after transformation will not be as obvious as those of the trigonometric functions.

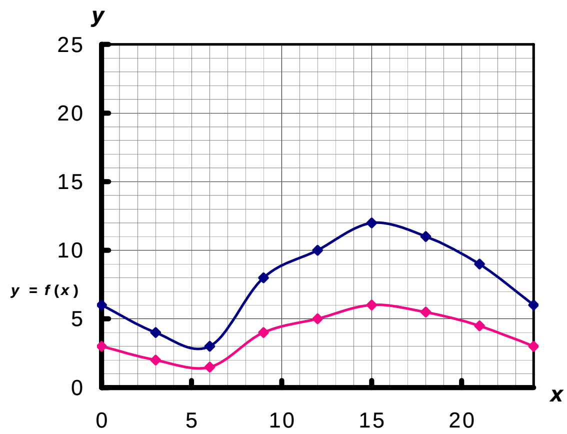
6. Suggested answers for worksheets are as follows:

1.

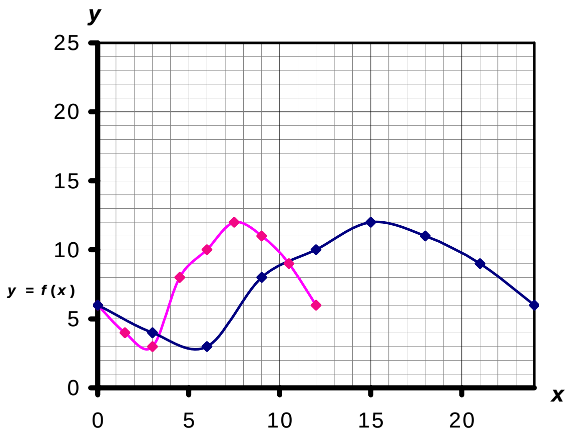
(a) Sketch the graph of function $y = 2f(x)$.



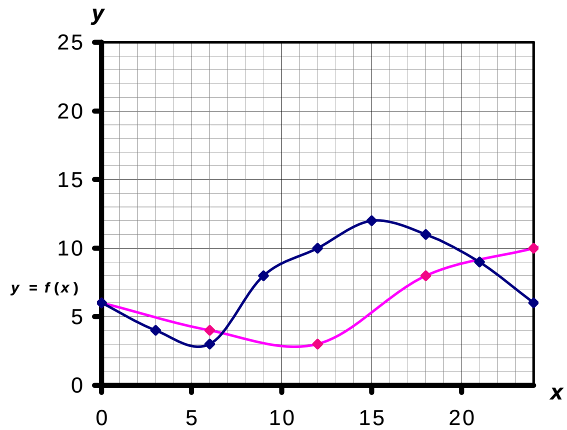
(b) Sketch the graph of function $y = \frac{1}{2}f(x)$.



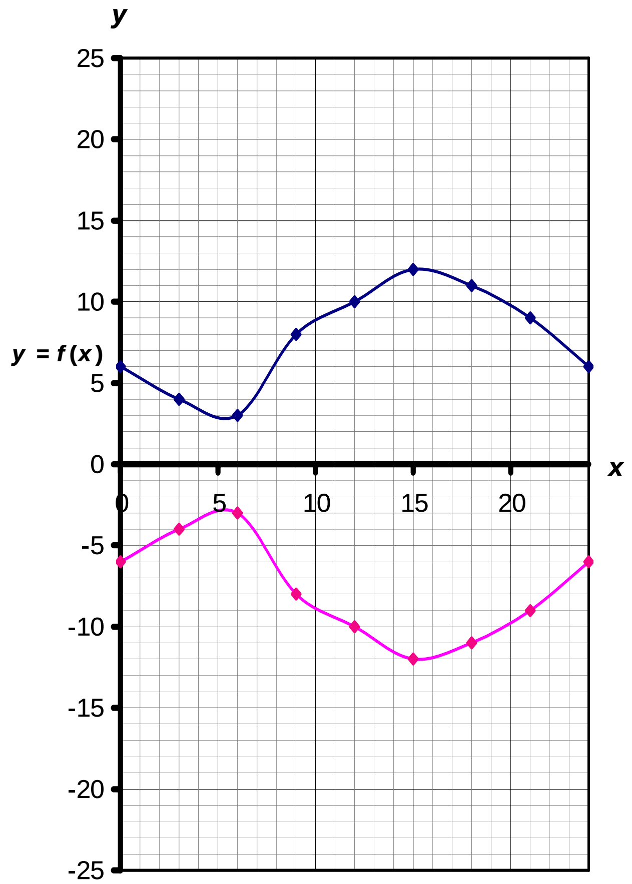
(c) Sketch the graph of function $y = f(2x)$.



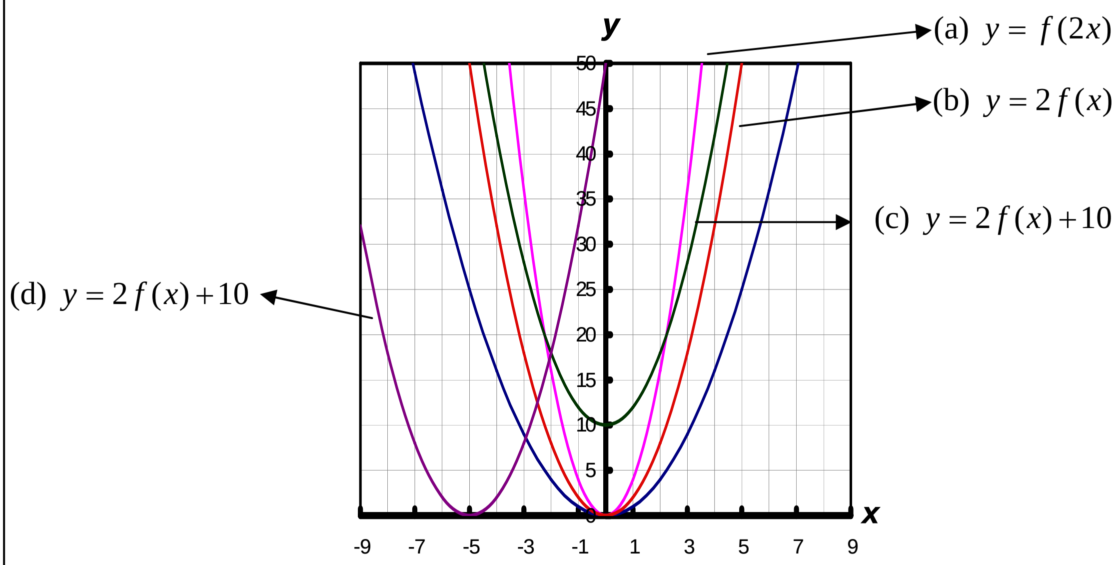
(d) Sketch the graph of function $y = f(\frac{x}{2})$.



(e) Sketch the graph of function $y = -f(x)$.



2.



(e)

Not equal. In general,

$kf(x+h) \neq kf(x) + k \cdot h$.

3. (i) $y = 2f(x)$

   (ii) $y = f\left(\frac{x}{2}\right)$

4 (i) $y = f(x) + 7$

   (ii) $y = f(x) - 7$

5. (i) $y = f(x + 12)$

   (ii) $y = f(x - 12)$

6. (i) $y = f(x) + 12$

   (ii) $y = 2f(x)$