Exemplar 11:
Effects of Transformation on Functions

Objectives : Through daily life examples to consolidate

1. concepts of functions
2. different representations of functions
3. the effect of transformation on functions

Key Stage : 4

Learning Unit : Functions and Graphs

Materials required : Spreadsheet software such as *Microsoft Excel(cellphone.xls)***,** calculators and worksheets

Prerequisite Knowledge :

1. Recognize concepts of function
2. Use symbols to represent quantities and functions
3. Recognize different representations of functions (symbolic, tabular and graphical)

Description of the activity:

1. The teacher may revise the concepts of functions with students either at the beginning, or after completing Worksheet 1.

2. 【Worksheet 1】 (Concepts of functions)

   The teacher distributes Worksheet 1 to students, then discusses with students whether there is any function appearing in the given mobile phone service plan. If so, the teacher may further raise which one the independent variable is while which one the dependent variable is.

   The teacher may guide students to think from the points whether the value of a quantity “determines” the value of another quantity and thus gives a unique value of this quantity [1] (i.e. it must be one-to-one or many-to-one, but is never one-to-many).

3. 【Worksheet 2】 (Different Representations of Functions)

(a) Activity 2.1 provides a table of values as part of the mobile phone plan. The teacher may ask students to provide a complete plan that satisfies the given charge and describes clearly the charge for any call minutes (called the complete "**Charge Function**"), and then represents this charge function graphically.

(b) After students complete Activity 2.1 of Worksheet 2, the teacher may discuss with students the answers. They may conclude that there are various types of plans such as *piecewise fixed charging*, *piecewise linear increasing charging*, *continuous smooth increasing charging* and their combinations [2], etc. satisfying the given table. Consequently, it is expected that students understand the generality of the definition on functions through the example of mobile phone service plan (e.g. functions are not confined to be expressed in algebraic form or to be continuous, etc.) and their limitation (e.g. functions cannot be one-to-many).

(c) It is expected that students' concepts on the definition of functions and different representations of functions may be consolidated through Activity 2.2. This part can be treated as a teacher-guided discussion. The teacher may discuss with students other representations such as tabular and algebraic forms so as to investigate their advantages and disadvantages. This part is rather abstract. The teacher may use and print this part selectively.

4. 【Worksheet 3】 (Effects of transformation on the graphs of functions)

(a) In Worksheet 3, students may understand the practical aspect of transformation of functions through the example of mobile phone service plan. **Transformation of functions can be considered as a mathematical modeling to interpret the mobile phone plan. On the other hand, the mobile phone plan can act as a prototype of**

[1] Refer to Notes for Teachers 5
[2] Refer to point 8 in Notes for Teachers

transformation of functions. This also deepens students' impression on the transformation of functions and initiates their interest in learning the transformation of functions.

(b) The charge function and a corresponding table of a certain mobile phone service plan are given in Activity 3.1. The algebraic expression (i.e. symbolic form) of the charge function is not given in the worksheet [3].

The teacher asks students to use spreadsheet software *Microsoft Excel* to open the file *cellphone.xls* and complete the worksheet (or to check the answers). From the results, it is expected that students can investigate the correspondence between different plans and different transformations as well as the effects on the graphs.

(c) After students complete Activity 3.1, the teacher may invite students to present their answers before their classmates. Student may then make use of the table in Activity 3.2 to conclude the results in Activity 3.1, i.e. the correspondence between different plans and different transformations. The teacher may point out whether a plan influences the independent or the dependent variable as well as what the influence is so as to guide students to find the difference between $f(x+h)$ and $f(x)+k$, etc.

(d) Finally the teacher provides with students an algebraic form and asks students to work out the corresponding mobile phone plan (e.g. which plan does $F\left(\frac{x}{m}\right)-k$ correspond to?) in order to assess how well students understand the content.

(e) For less able students, the teacher may use Activity 3.1[A] – [D] separately. After students complete Activity [A], the teacher discusses with students their results and follows up with further examples to make sure that students thoroughly understand the concept. The teacher then proceeds to Activity [B], and then [C] and [D]. The table in Activity 3.2 could be used as a summary and final rounding up of the activities.

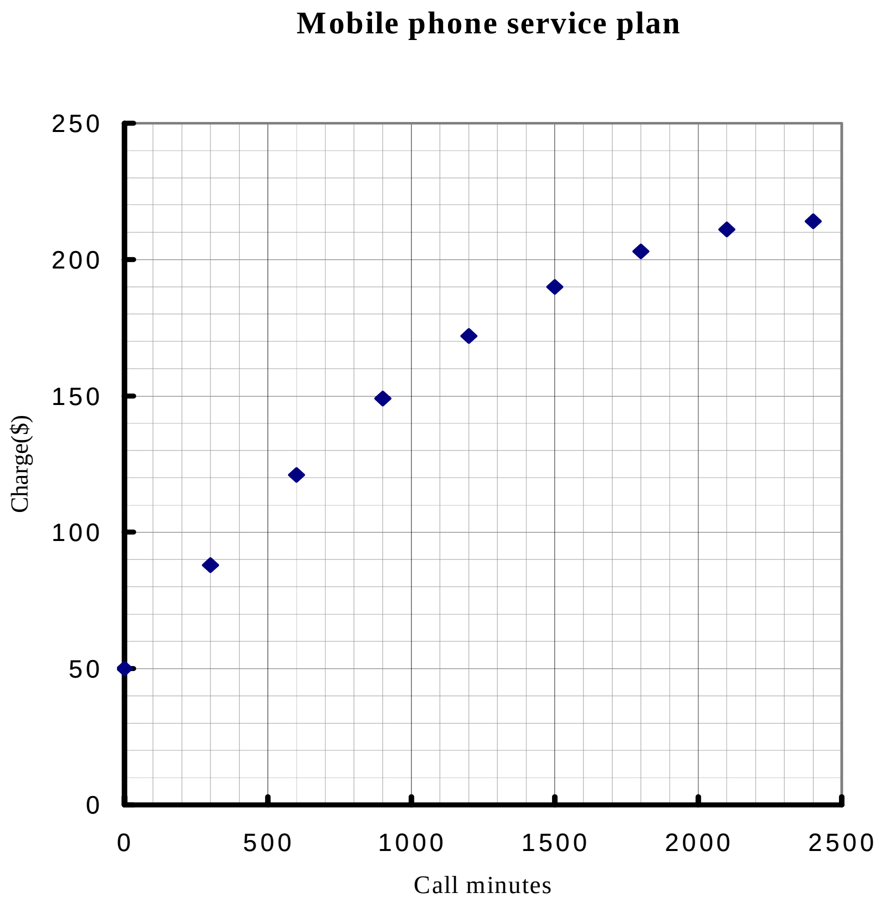
[3] Refer to point 11 in Notes for Teachers

Background

The call charges of a mobile phone service plan are given as follows:

Call minutes	0	300	600	900	1200	1500	1800	2100	2400
Charge (dollars)	50	88	121	149	172	190	203	211	214

Graphically,



There are different kinds of promotional offer by the company:

[A] a rebate of *m* dollars;

[B] a discount of *r* %;

[C] *n* free minutes;

[D] stretching the call minutes by *p* % (e.g. for a 10% stretch implies that a 300-minute charge on its normal scale gives you 330 minutes of actual call minutes).

Worksheet 1

The call charges of a mobile phone service plan are given as follows:

Call minutes	0	300	600	900	1200	1500	1800	2100	2400
Charge ($)	50	88	121	149	172	190	203	211	214

1. In the above mobile phone service plan, is charge a function of usage? Briefly explain.

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2. If charge is a function of call minutes, state which quantity is the independent variable and which one is the dependent variable?

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Worksheet 2

Activity 2.1

The call charges of a mobile phone service plan are given as follows:

Call minutes	0	300	600	900	1200	1500	1800	2100	2400
Charge ($)	50	88	121	149	172	190	203	211	214

1. The above mobile phone service plan only provides charges of call in certain minutes (e.g. the charge for 250 call minutes is not stated). Propose a complete charge function (i.e. a complete plan that describes clearly the charge for any call minutes).

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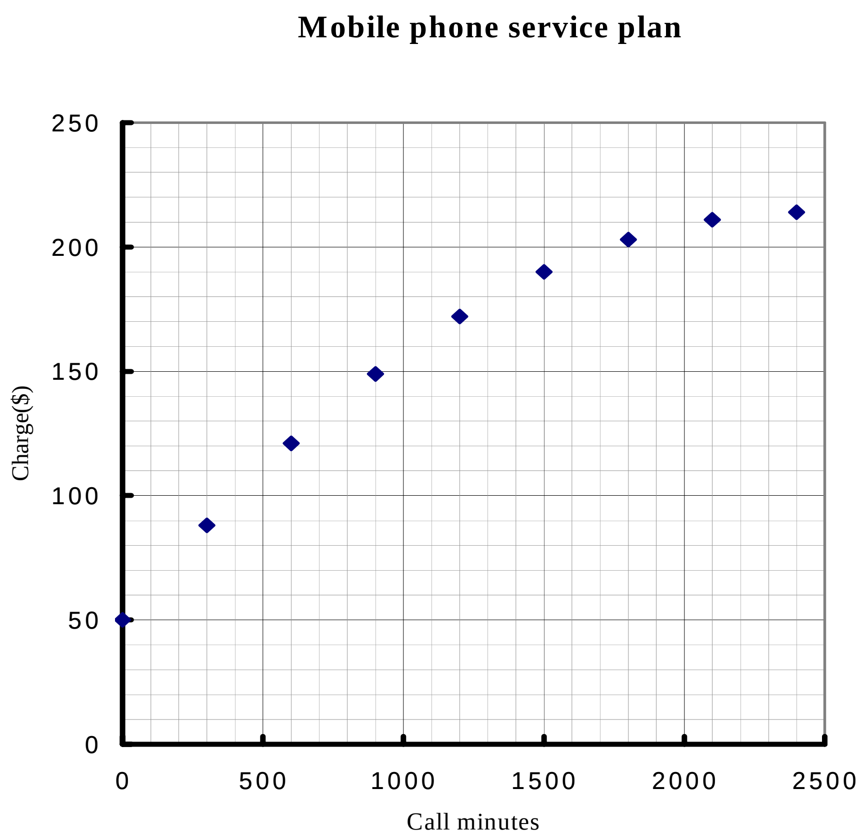
2. Draw the graph of the proposed charge function in the given coordinate plane.



Activity 2.2 (Teacher guided discussion)

1. (a) According to your charge function, what is the charge for 1050 call minutes?

   ____

   (b) According to your charge function, what is the charge for 48 000 call minutes?

   ____

2. Write down your charge function in the table below.

Call minutes					…
Charge ($)					…

3. Write down your charge function in algebraic form.

   ____

4. Do you find it difficult to answer the above questions? Describe and elaborate briefly why it is difficult to do so.

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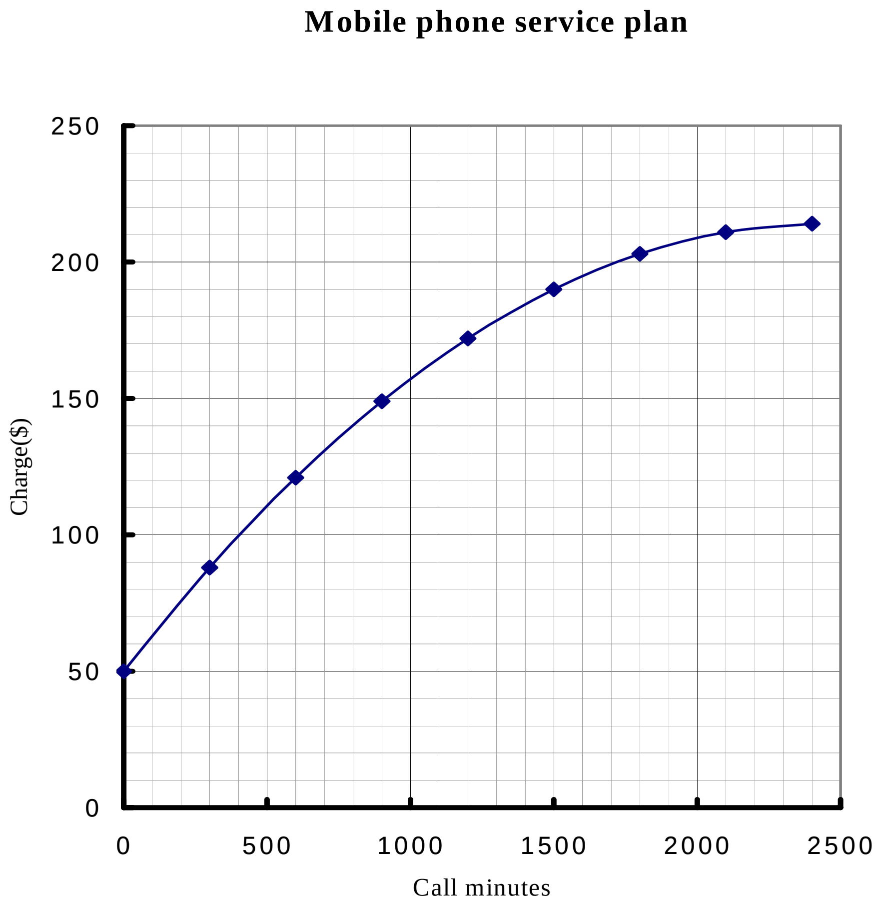
Worksheet 3

Activity 3.1

The graph of the charge function and a corresponding table of a certain mobile phone service plan are given below.



Call minutes	Charge ($)
0	50
300	88
600	121
900	149
1200	172
1500	190
1800	203
2100	211
2400	214

Fill in the tables below and draw the corresponding graphs according to the given promotional offers. Check your results using the *Excel* file *cellphone.xls*.

[A] A rebate of *m* dollars per month

1. Fill in the tables for the new charges:

(a) $m = 40$

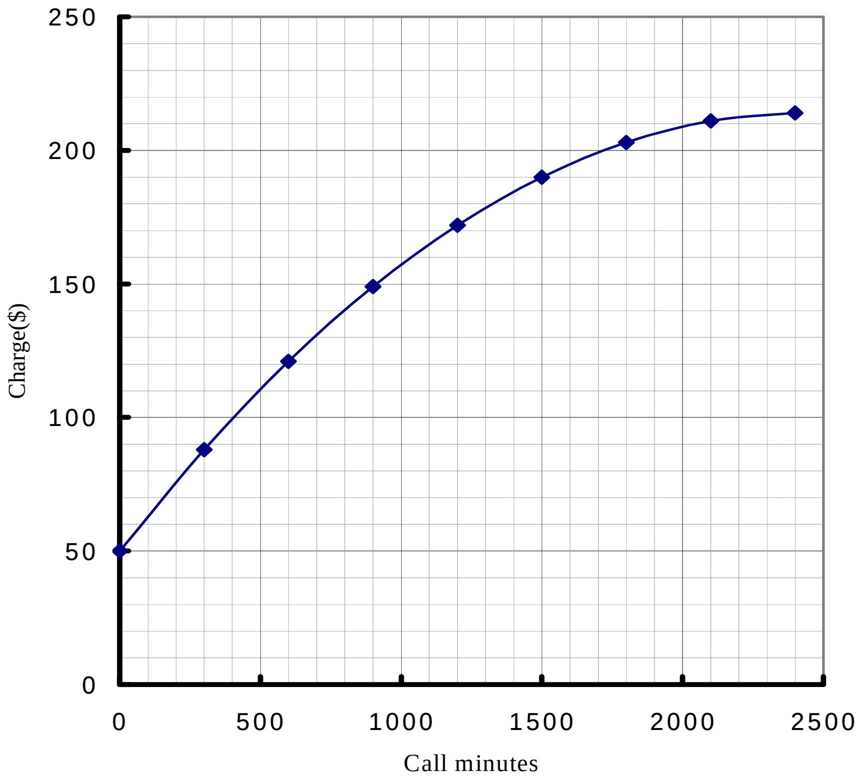
Call minutes	Original charge ($)	New charge ($)
0	50	
600	121	
1200	172	
1800	203	
2400	214	

(b) $m = 20$

Call minutes	Original charge ($)	New charge ($)
0	50	
600	121	
1200	172	
1800	203	
2400	214	

2. Draw the graphs for the corresponding new charges:

Mobile phone service plan



3. Compared with the graph of the original plan, what are the changes of the graphs corresponding to the promotional offer of a rebate of $20 as well as that of a rebate of $40?

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4. What are the relations between the charge function *F* of the original plan and the charge functions of the promotional offer of a rebate of $20 as well as that of a rebate of $40 respectively?

[B] A discount of *r* % off

1. Fill in the tables for the new charges:

(a) $r = 50$

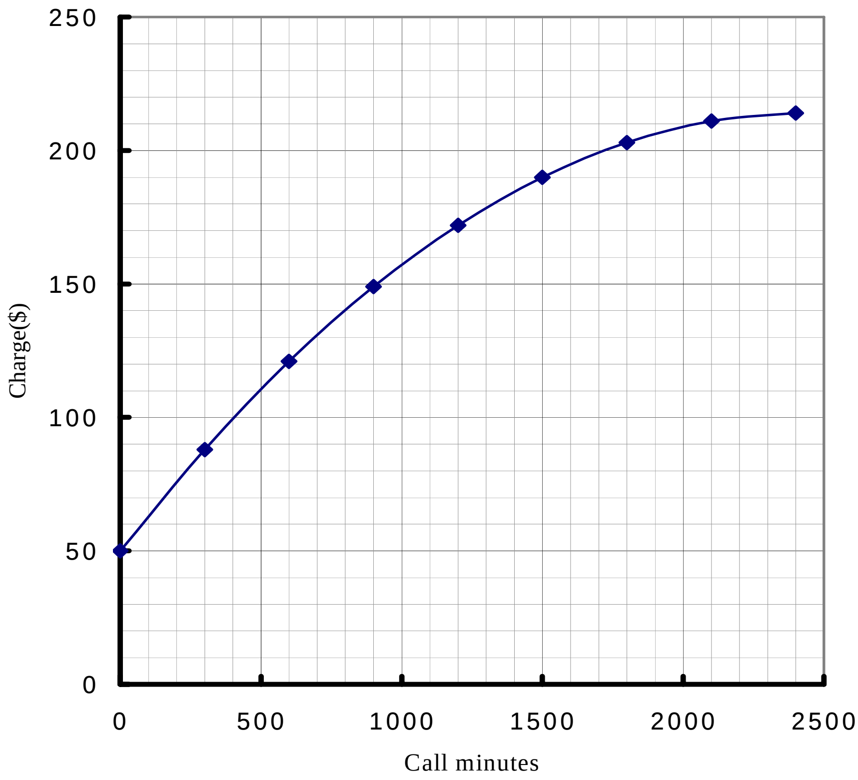
Call minutes	Original charge ($)	New charge ($)
0	50	
600	121	
1200	172	
1800	203	
2400	214	

(b) $r = 20$

Call minutes	Original charge ($)	New charge ($)
0	50	
600	121	
1200	172	
1800	203	
2400	214	

2. Draw the graphs for the corresponding new charges:

Mobile phone service plan



3. Compared with the graph of the original plan, what are the changes of the graphs corresponding to the promotional offer of a discount of 20% as well as that of a discount of 50%?

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4. What are the relations between the charge function *F* of the original plan and the charge functions of the promotional offer of a discount of 20% as well as that of a discount of 50% respectively?

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[C] *n* free minutes

1. Fill in the tables for the new charges:

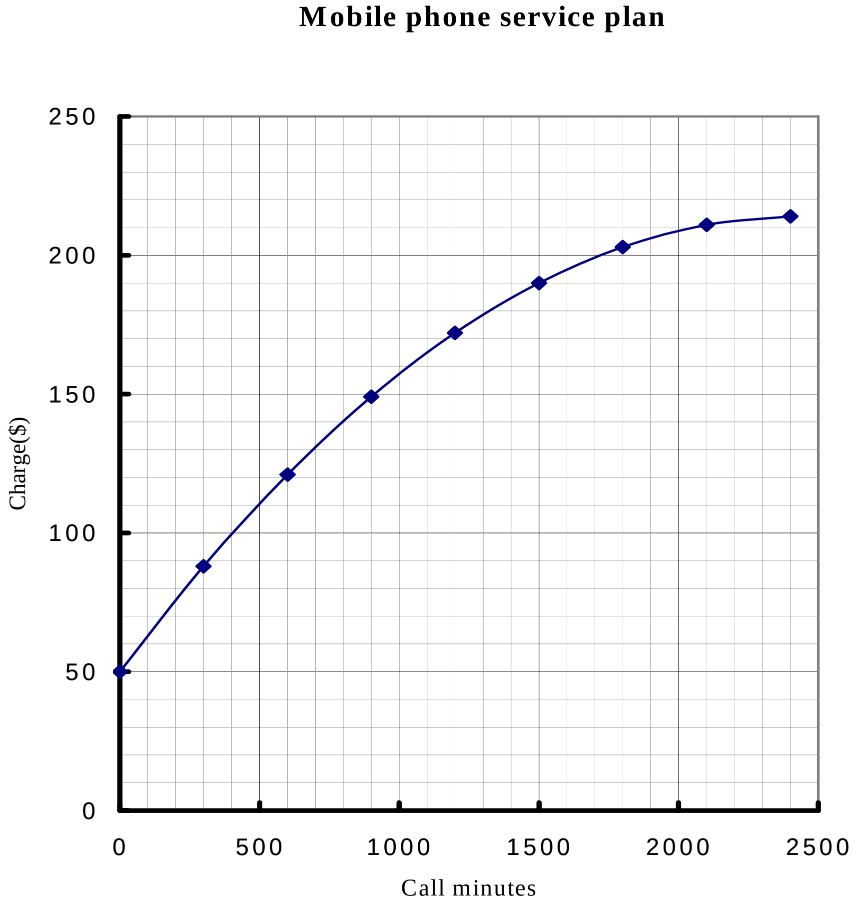
(a) $n = 600$

Call minutes	Original charge ($)	New charge ($)
0	50	
600	121	50
1200	172	
1800	203	172
2400	214	

(b) $n = 300$

Call minutes	Original charge ($)	New charge ($)
0	50	
600	121	88
1200	172	
1800	203	190
2400	214	

2. Draw the graphs for the corresponding new charges:



3. Compared with the graph of the original plan, what are the changes of the graphs corresponding to the promotional offer with the first 300 minutes free as well as that with the first 600 minutes free?

4. What are the relations between the charge function *F* of the original plan and the charge functions of the promotional offer with the first 300 minutes free as well as that with the first 600 minutes free respectively?

[D] Stretching the call minutes by *p* %

(e.g. for a 20% stretch, a 300-minute charge on its normal scale gives you 360 minutes of actual call minutes; for a 50% stretch, a 300-minute charge on its normal scale gives you 450 minutes of actual call minutes;).

1. Fill in the tables for the new charges:
   As it is tedious to fill in the original tables, you may try to fill in the tables below instead.

(a) $p = 50$

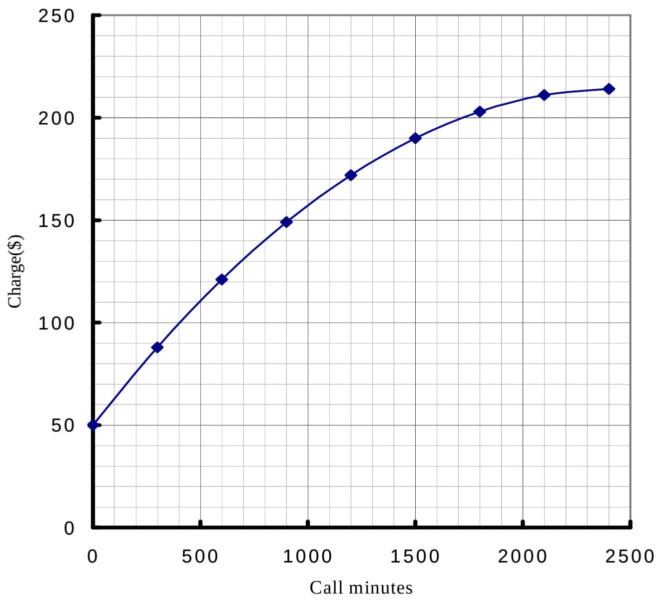
New call minutes	Original call minutes	New charge ($)
0	0	50
450	300	88
	600	121
	1200	172
	1800	203
	2400	214

(b) $p = 20$

New call minutes	Original call minutes	New charge ($)
0	0	50
	300	88
	600	121
1540	1200	172
	1800	203
	2400	214

2. Draw the graphs for the corresponding new charges:

Mobile phone service plan



3. Compared with the graph of the original plan, what are the changes of the graphs corresponding to the promotional offer of stretching the call minutes by 20 % as well as that of stretching the call minutes by 50 %?

4. What are the relations between the charge function *F* of the original plan and the charge functions of the promotional offer of stretching the call minutes by 20 % as well as that of stretching the call minutes by 50 %?

Activity 3.2

1. Suppose the monthly mobile phone call is x minutes, the corresponding charge is $F(x)$ dollars. F is called the "Charge Function". Complete the following table.

Promotional offer	Quantity directly influenced is Independent variable x or dependent variable $F(x)$	Express in F the corresponding transformation of functions
[A] A rebate of m dollars		
[B] A discount of r %		
[C] The first n minutes free		
[D] Sketching the call minutes by p %		

2. (a) Write down the promotional offer corresponding to $0.6F(x)$?

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(b) Write down the promotional offer corresponding to $F(x-10)-20$?

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(c) Write down the promotional offer corresponding to $F\left(\frac{x}{1.2}\right)-20$?

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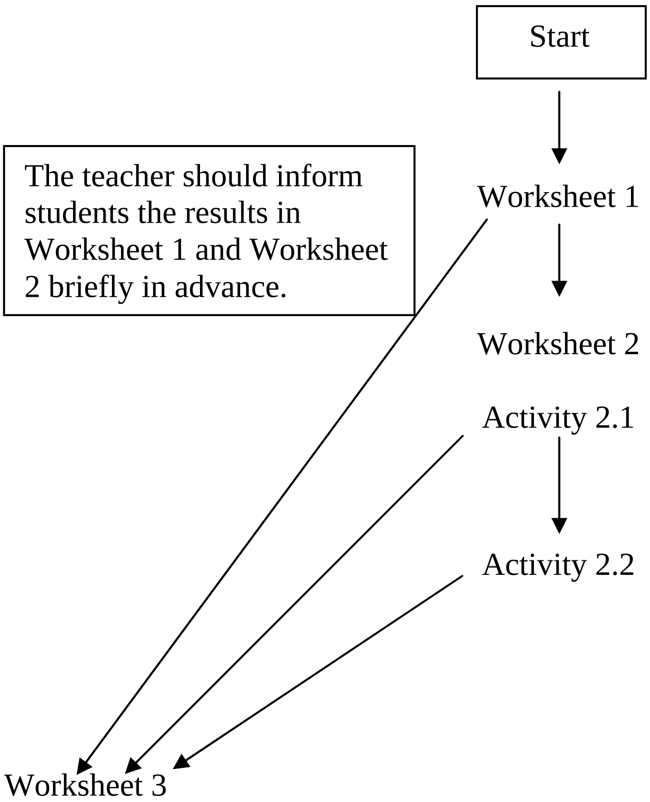
Notes for Teachers:

【General】

1. It should be aware that in reality there are only a few different types of mobile phone service plans, and in addition most charge functions are discontinuous. The teacher should understand that the plans in the examples might not be the same as in real-life situations. Nevertheless, these plans can be used as a model to understand daily-life situations.

2. A suggested sequence in using the worksheets:



3. The time estimated for this activity is as follows:

Worksheet 1 + Discussion on concepts of functions	about 20 minutes
Worksheet 1 + Worksheet 2	about 35 minutes
Worksheet 3	about 80 - 100minutes

【Worksheet 1】

4. In Worksheet 1, it is expected that, through students' commonly encountered daily examples, they obtain a better understanding on the meaning of functions

as well as the difference between independent and dependent variables. In addition, the example may consolidate the concepts of functions.

5. The teacher should alert students that when a quantity “determines” the value of another quantity uniquely as stated, it may not involve any causality in daily life sense. Thus, charge is a function of call minutes, and on the other hand call minutes may also be a function of charge.

   The teacher may ask students, at the appropriate moment, to discuss whether the roles of independent variable and dependent variable can be interchanged. The answer is open. It depends on whether the charge function is one–to–one (e.g. *continuously smooth increasing charging*) or many–to–one (e.g. *piecewise fixed charging*). The answer to the former case is positive whereas the latter is negative.

   Besides, since x is a function of y does not imply that y is a function of x, the teacher should avoid to use phases like “x corresponding to y”, “function between x and y” etc.

6. Answers to Worksheet 1

1. Charge is a function of call minutes. Since under a selected plan, the call minutes determine the charge uniquely. 2. Usage is the independent variable while charge is the dependent variable.

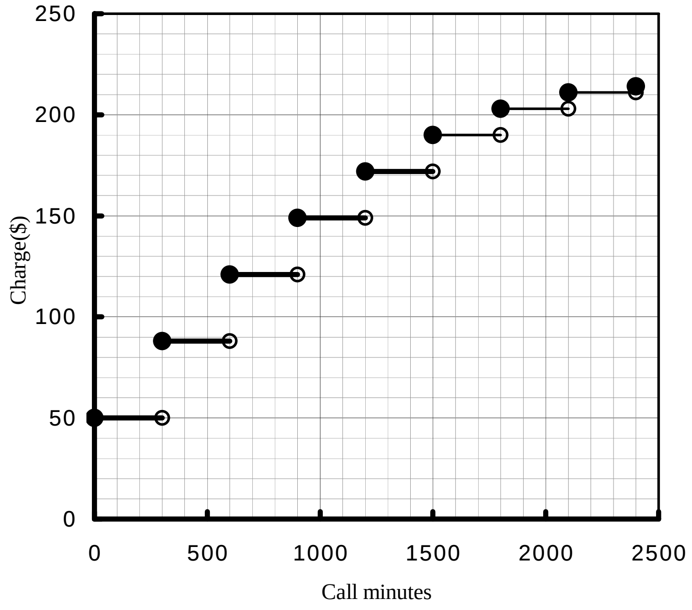
【Worksheet 2】

7. It is better for the teacher to discuss with students how to determine various ways of charging in-between the given points and how to determine the charge for call minutes outside the range of the graph on completing Worksheet 2.

8. The answer to Activity 2.1 is not unique, the followings are suggested answers. The teacher may point out the meaning and difference between small dots and small circles.

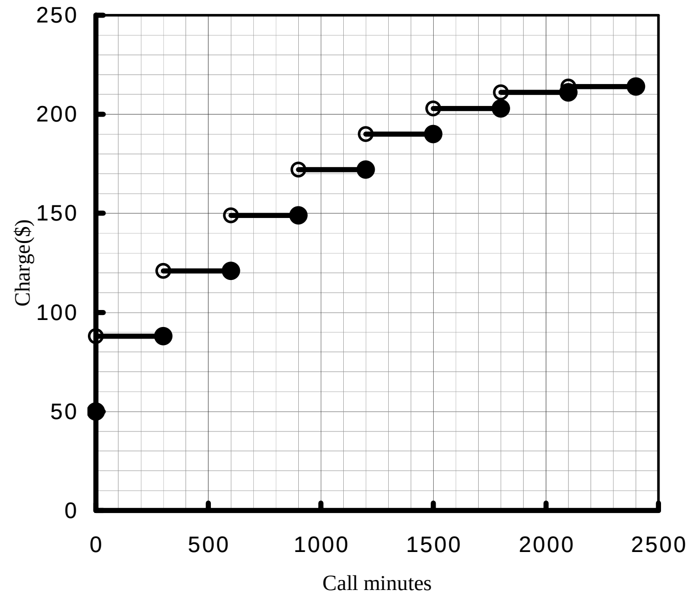
Piecewise forward fixed charging

Mobile phone service plan

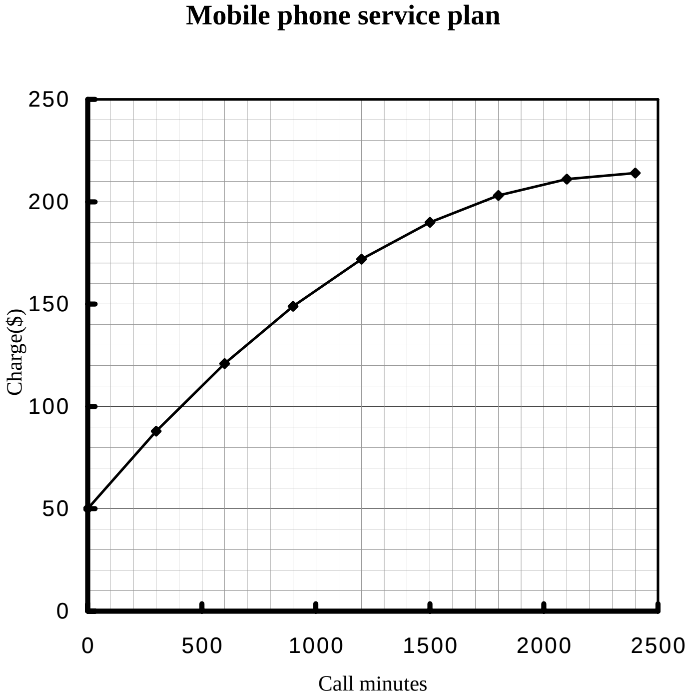


Piecewise backward fixed charging

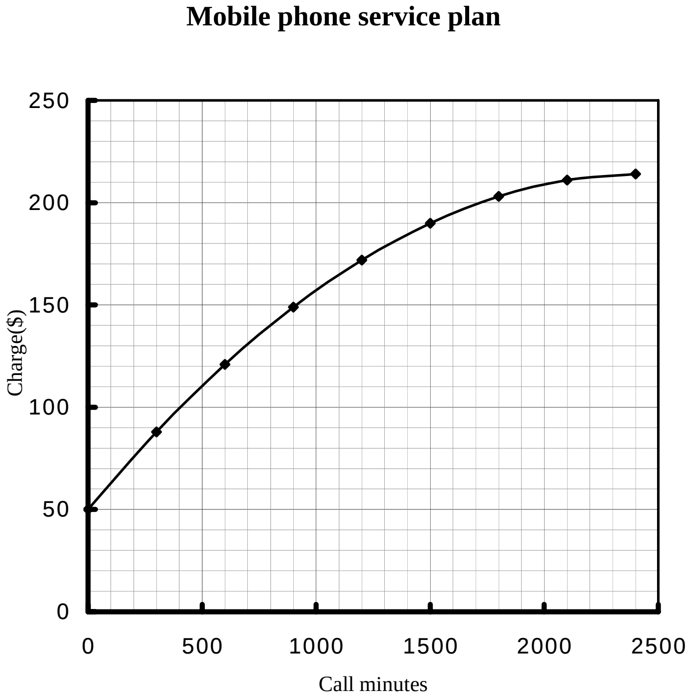
Mobile phone service plan



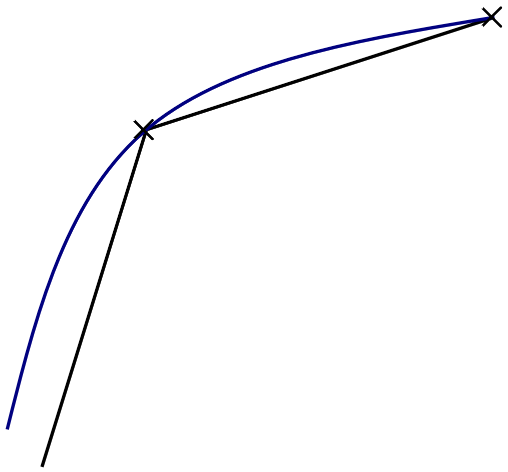
Piecewise linear increasing charging (linked adjacent points by straight lines)



Continuously smooth increasing charging (linked adjacent points by a smooth curve)



Note that the graphs of piecewise linear increasing charge function and continuously smooth increasing charge function look very similar, but the difference becomes obvious after enlarging the vertical axis. The following diagram is part of the overlapping regions of the graphs after enlargement.



9. The followings are some suggested answers to Activity 2.2. If the teacher shows the following algebraic expressions to students, it is suggested that appropriate explanations have to be supplemented because students may not be familiar with such representations.

Piecewise forward fixed charging

Tabular form

Call minutes	$0 \le x < 300$	$300 \le x < 600$	$600 \le x < 900$	$900 \le x < 1200$	$\cdots$
Charge ($)	50	88	121	149	$\cdots$

Algebraic form

$$F(x) = \begin{cases} 50 & \text{if } \quad 0 \le x < 300 \\ 88 & \quad 300 \le x < 600 \\ 121 & \quad 600 \le x < 900 \\ 149 & \quad 900 \le x < 1200 \\ \vdots & \quad \vdots \end{cases}$$

Piecewise backward fixed charging

Tabular form

Call minutes	$x = 0$	$0 < x \le 300$	$300 < x \le 600$	$600 < x \le 900$	$900 < x \le 1200$	$\cdots$
Charge ($)	50	88	121	149	172	$\cdots$

Algebraic form

$$F(x)=\begin{cases}50 & \text{if} \quad x=0\\ 88 & 0<x\le 300\\ 121 & 300<x\le 600\\ 149 & 600<x\le 900\\ \vdots & \vdots\end{cases}$$

Piecewise linear increasing charging

It is rather difficult to write down all values of usage and charge in a single table. Some of the values are shown as below:

Call minutes	150	300	450	600	750	900	1050
Charge ($)	69	88	104.5	121	135	149	160.5

Algebraic form

$$F(x)=\begin{cases}\dfrac{88-50}{300-0}(x-0)+50 & \text{if} \quad 0\le x<300\\[2ex] \dfrac{121-88}{600-300}(x-300)+88 & 300\le x<600\\[2ex] \dfrac{149-121}{900-600}(x-600)+121 & 600\le x<900\\[2ex] \dfrac{172-149}{1200-900}(x-900)+149 & 900\le x<1200\\ \vdots & \vdots\end{cases}$$

Continuously smooth increasing charging

It is rather hard to write down all values of call minutes and charge in a single table. Some of the values are shown as below:

Call minutes	150	300	450	600	750	900	1050
Charge ($)	69	88	104.5	121	135	149	160.5

Algebraic form

$$F(x)=50+\frac{81}{2}\left(\frac{x}{300}\right)-\frac{5}{2}\left(\frac{x}{300}\right)^2 \text{ for } 0\le x\le 2400$$

(If there is time and students are more able, the teacher may discuss the underlying principle of "use more, save more" in the plan. Regarding the formula, it involves more advanced theory (the tabular form actually provides a second order difference equation and $F(x)$ is the solution). The teacher should not ask students to deduce and need not teach the technique.)

10. It should be noted that students might not be able to write down the corresponding tables and algebraic forms of the graphs they give. In fact, there may be no such correspondences. Answers are not the first priority. It is most important to let students understand the significance and limitations of different representations.

【Worksheet 3】

11. The algebraic expression for the charge function in Worksheet 3 is supposed to be $F(x) = 50 + \frac{81}{2}\left(\frac{x}{300}\right) - \frac{5}{2}\left(\frac{x}{300}\right)^2$ for $0 \le x \le 2400$. Note that x should not be larger than $31 \times 24 \times 60 = 44640$.

12. In this exemplar, when the usage is 0 minute, the charge is assumed to be $50. This can be explained by the minimum charge of the service plan. In reality, one usually has to pay a minimum charge once he joins a plan even he does not really use any services provided.

    In [C] even there is 300 free minutes, one has to pay for the call minutes of the first 300 minutes because there is a minimum charge.

13. Suggested answers to Activity 3.1:

[A] A rebate of *m* dollars

1.

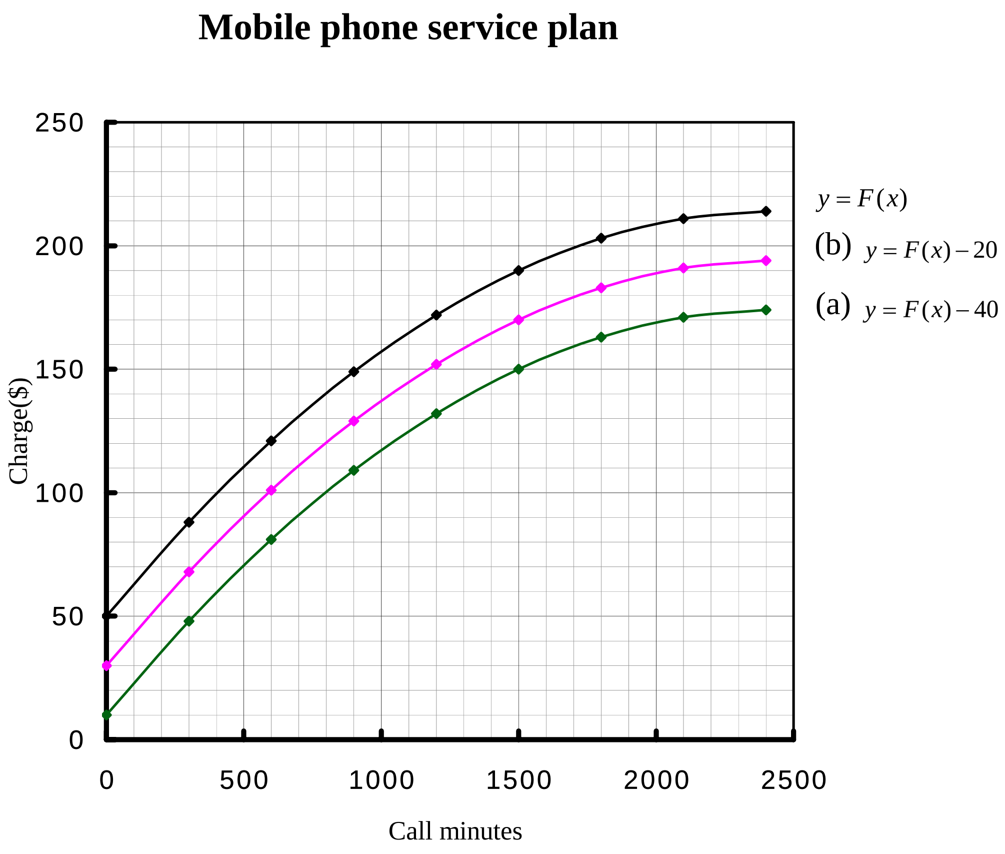
(a) $m = 40$

Call minutes	Original charge ($)	New charge ($)
0	50	10
600	121	81
1200	172	132
1800	203	163
2400	214	174

(b) $m = 20$

Call minutes	Original charge ($)	New charge ($)
0	50	30
600	121	101
1200	172	152
1800	203	183
2400	214	194

2.



3. The graphs of rebates of $20 and $40 offers are the graphs with the original graphs translated downwards by 20 and 40 units respectively.

4. The charge functions of rebates of $20 and $40 offers are $F(x) - 20$ and $F(x) - 40$ respectively.[4]

[4] Formally speaking, the charge function *G* of a rebate of $20 offer is $G(x) = F(x) - 20$.

[B] A discount of r % off

1.

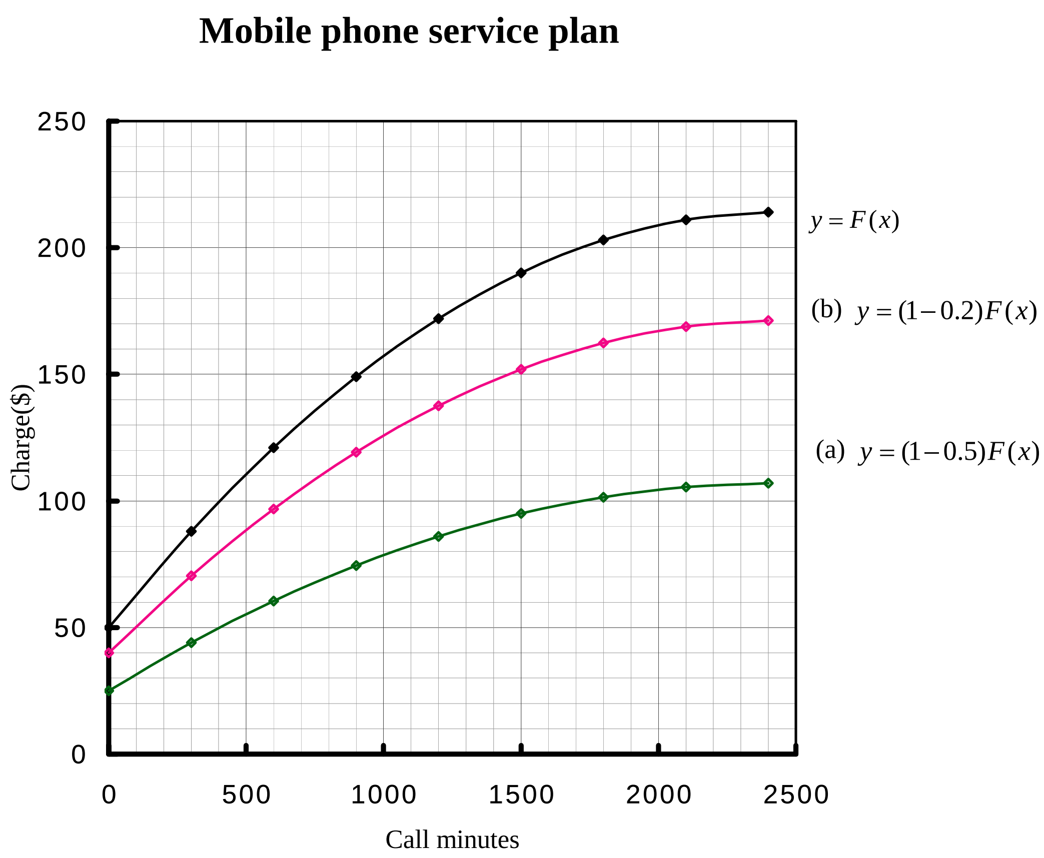
(a) $r = 50$

Call minutes	Original charge ($)	New charge ($)
0	50	25
600	121	60.5
1200	172	86
1800	203	101.5
2400	214	107

(b) $r = 20$

Call minutes	Original charge ($)	New charge ($)
0	50	40
600	121	96.8
1200	172	137.6
1800	203	162.4
2400	214	171.2

2.



3. The graphs of discounts of 20 % and 50% offers are the graphs with heights 80% and 50% of that of the original graph.

4. The charge functions of discounts of 20% and 50% offers are $0.8F(x)$ and $0.5F(x)$ respectively.[5]

[5]Formally speaking, the charge function G of a discount of 20% offer is $G(x) = 0.8F(x)$.

[C] n free minutes

Since initially F is not defined for $-300 \le x < 0$, the charge at 0 call minute is not fixed and the graph of $y = F(x-300)$ in the range $0 \le x < 300$ is also unavailable. A reasonable assumption is to set a basic charge of $50. So the rows with ** in the tables and the horizontal line in the graphs below do not entirely represent $F(x-300)$.

(a) $n = 600$

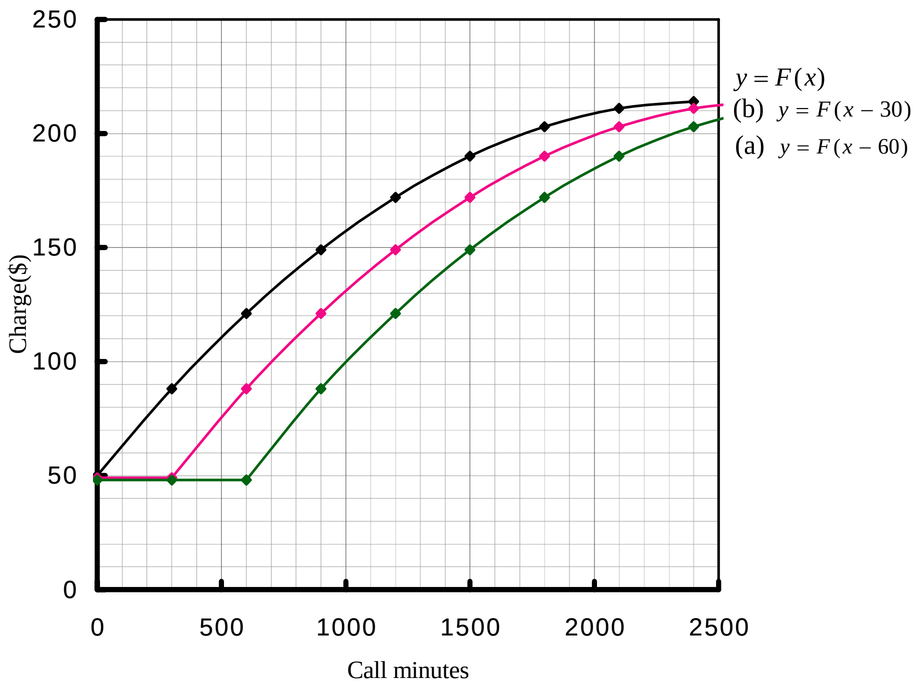
Call minutes	Original charge ($)	New charge ($)
0	50	50**
600	121	50
1200	172	121
1800	203	172
2400	214	203

(b) $n = 300$

Call minutes	Original charge ($)	New charge ($)
0	50	50**
600	121	88
1200	172	149
1800	203	190
2400	214	211

2.

Mobile phone service plan



3. The graphs of the first 300 minutes and 600 minutes free offers are the graphs with the original graphs right translated by 300 and 600 units.

4. The charge functions with the first 300 minutes and 600 minutes free offers are $F(x-300)$ and $F(x-600)$ respectively.[6]

[6]Formally speaking, the charge function G with the first 30 minutes free offer is $G(x) = F(x-300)$ and $G(x) = F(x-600)$ excluding the region (0-300 minutes) and (0-600 minutes) respectively.

[D] Stretching the call minutes by p %

(e.g. for a 20% stretch, a 300-minute charge on its normal scale gives you 360 minutes of actual call minutes; for a 50% stretch, a 300-minute charge on its normal scale gives you 450 minutes of actual call minutes and so on .)

(a) $p = 50$

New call minutes	Original call minutes	New charge ($)
0	0	50
450	300	88
900	600	121
1800	1200	172
2400	1800	203
3600	2400	214

(b) $p = 20$

New call minutes	Original call minutes	New charge ($)
0	0	50
360	300	88
720	600	121
1540	1200	172
2160	1800	203
2880	2400	214

If the original table of call minutes is used, it is rather difficult to find the charge. The method is suggested below:

For instance, suppose the actual call minutes are 300, in 1(a) the corresponding call minutes according to the new offer is $300 \div 120\% = 300 \div 1.2 = 250$ minutes. Then from the original graph the charge is $82. The calculated result is as follows:

1. (a) $p = 50$

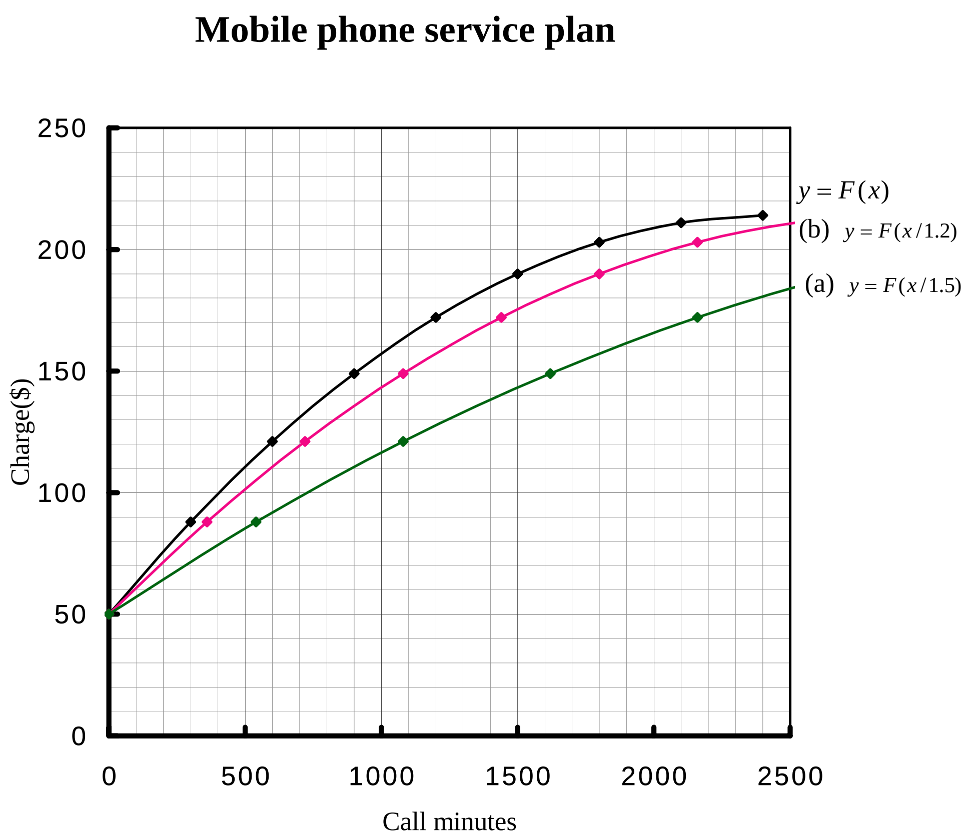
Call minutes	Original charge ($)	New charge ($)
0	50	50
300	88	75.9
600	121	99.6
900	149	121.0
1200	172	140.2
1500	190	157.2
1800	203	172.0
2100	211	184.6
2400	214	194.9

(b) $p = 20$

Call minutes	Original charge ($)	New charge ($)
0	50	50
300	88	82.0
600	121	110.6
900	149	135.6
1572	172	157.2
1753	190	175.3
1900	203	190.0
2012	211	201.2
2089	214	208.9

This method is more complicated. Thus the tables in questions 1(a) and 1(b) in which students need to find the corresponding call minutes from the given charge is adopted in the exemplar.

2.



3. The graphs of stretching the call minutes by 20 % and 50% offers are the graphs with the original graphs stretching by 20% and 50%.

4. The charge functions of stretching the call minutes by 20 % and 50% offers are $F\left(\frac{x}{1.2}\right)$ and $F\left(\frac{x}{1.5}\right)$ respectively.[7]

[7] Formally speaking, the charge function G of sketching the call minutes by 20% offer is $G(x) = F\left(\frac{x}{1.2}\right)$.

14. Answers to Activity 3.2:

1. Suppose the call minutes are x and the corresponding charge is $F(x)$ dollars. Then F is called the "Charge Function".

Promotional offer	Quantity directly influenced is Independent variable x or dependent variable $F(x)$	Express in F the corresponding transformation of functions
[A] A rebate of m dollars	$F(x)$	$F(x)-m$
[B] A discount of r %	$F(x)$	$(1-r\%)F(x)$
[C] The first n minutes free	x	$F(x-n)$
[D] Sketching the call minutse time by p %	x	$F\left(\frac{x}{1+p\%}\right)$

2. (a) $0.6F(x)$ represents a 40% discount.

(b) $F(x-100)-20$ represents the offer with the first 100 minutes free and then a \$20 rebate on the calculated charge. For instance, 0 call minute is charged \$30; 400 call minutes are charged \$68; 700 call minutes are charged \$101.

(c) $F\left(\frac{x}{1.2}\right)-20$ represents stretching the call minutes by 20% first, the charge is calculated based on this as actual call minutes and then a \$20 rebate is offered. For instance, 0 call minute is charged \$30; 300 call minutes are charged \$62; 360 call minutes are charged \$68.

15. The teacher should notice that the effects in transformation of straight lines are rather confusing. It is not easy to distinguish translation vertically and horizontally for straight lines. In the other words, it is difficult for students to compare the difference between the graphs of $f(x)+k$ and $f(x+k)$ for a linear function $f(x)$. For graphs of polynomial functions (including straight lines and quadratic graphs), it is also not easy to distinguish stretching vertically and horizontally. In other words, it is difficult for students to compare the difference between the graphs of $f(kx)$ and $kf(x)$, for a polynomial function $f(x)$.

16. Reference

"You Make the Call", Algebraic Thinking Math Project.

http://www.pbs.org/teachersource/mathline/lessonplans/atmp/call/call_procedure.shtm

17. The teacher may make use of graphical software such as *Winplot* or *Graphmatica*, etc. to investigate the effects of transformation on graphs and that in algebraic forms of functions.

18. Website (for further information)
(a) http://illuminations.nctm.org/mathlets/grapher/index.htm/

(b) http://www.mathsnet.net/graphs/findfunction.html

(c) http://math.Exeter.edu/rparris/winplot.html

(d) http://graphmatica.com