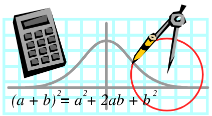


Exemplar 5:

The tabular, graphical and algebraic representations of functions (2)

Objective	**:**	Compare and understand the representations of functions through different examples
Key Stage	**:**	4
Learning Unit	**:**	Functions and Graphs
Materials required	**:**	Worksheets and calculators
Prerequisite Knowledge	**:**	Basic ideas of functions

Description of the activity:

1. The teacher revises the basic ideas of functions and different representations of functions with students.

2. Students are asked to study the “function” given in Question 1 of Worksheet 1. The teacher reminds students to check whether the given is a function. This helps students consolidate their concepts about functions.

3. Students are asked to complete Parts (a), (b) and (c) in Worksheet.

4. Students are divided into groups to compare and discuss their answers. Some groups are invited to present their answers to the whole class. The teacher comments and guides their discussions whenever appropriate.

5. After students finish doing Question 1, the teacher discusses with students how the table and graph in Question 2 are constructed by iteration and demonstrates that the algebraic expression provided is correct by verifying the first few terms.

6. The teacher guides students to understand and appreciate the strengths and recognize the limitations of different representations of functions in doing

Worksheet 1, including:

(a) The relation between two variables in real-life situations often starts with a table of values of the two variables and it is hoped to develop the symbolic form gradually. The tabular representation only shows some discrete values of the variables and does not reflect the relation of these 2 variables and all their values.

(b) The graphical representation of a function is usually worked out from a set of tabular values. From the trends of points in the graph, the symbolic form is then formulated. The graphical representation could cover more values than the tabular representation. However, the relation of these 2 variables is confined to the range shown. Outside the range, the relation between these 2 variables cannot be determined.

(c) In general, the symbolic form can give a more comprehensive representation on the relation between the 2 variables. Nevertheless, in real-life situation, there is no symbolic form to represent the relation between the values of the two variables. Various tools are required to find the symbolic form (e.g. the general term of the Fibonacci sequence in Question 2). It is more accurate but sometimes tedious when using symbolic form to find the output. (For example, it is easier to find the 11^{th} term in Question 2 from the tabular representation than from symbolic representation, but it is not difficult to find values from the symbolic form in Question 1.)

7. Students are divided into groups to discuss Questions 1 (a) and (b) on table 2.1 in Worksheet 2.

8. Some groups are invited to present their answers on table 2.1. If students cannot give the correct answers or cannot justify with reason, the teacher may give clues, guidelines and examples. For example, the teacher can remind students to pay attention to the given condition and the definition of function to identify whether a relation is a function or not. This may include:

(a) It can only judge whether "y is the function of x" only on each given x in the table gives a unique y value or not. It cannot be determined whether a function is a relation for values outside the given values. For example, Table 2 in Question 1, when x is an integer and within the given range, "y is a function of x". However, it is not sure whether "y is a function of x" for any real number x.

(b) In addition, even when "y is a function of x", it cannot guarantee that "x is a function of y".

9. The teacher asks students to complete other questions in Question 1 and checks their answers.

10. Students are asked to finish Questions 2 and 3. From the activities, teacher should initiate students to use the definition of a function and apply the definition accurately to determine whether "y is a function of x" or "x is a function of y" from the given algebraic graphical representations.

Worksheet 1 :

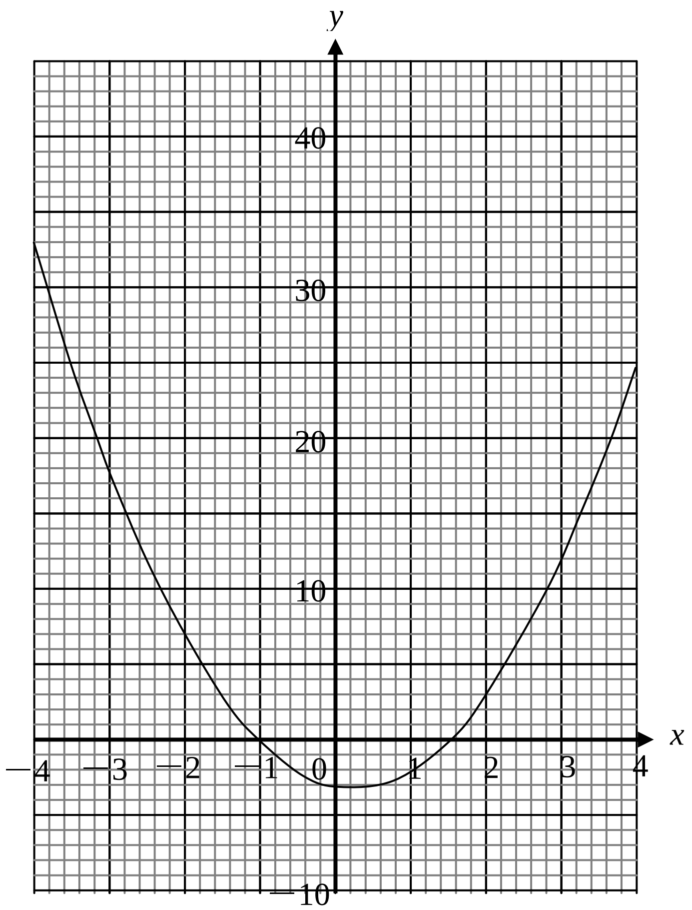
Comparison among Different Representations of Function

1. Consider a function $f(x) = 2x^2 - x - 3$.

(I) The tabular form of the function $f(x)$ is given as below:

x	...	–4	–3	–2	–1	0	1	2	3	4	...
$f(x)$	...	33	18	7	0	–3	–2	3	12	25	...

(II) The graph of the function $y = f(x)$ for $-4 \le x \le 4$ is given:



(III) The algebraic form of the function is:

$$f(x) = 2x^2 - x - 3.$$

(a) Find the value of $f(x)$ when

(i) $x = -3$.

(ii) $x = 2.5$.

(iii) $x = 3.7$.

(iv) $x = 6$.

(b) Put a tick "✓" under the option(s) in the following table to show which representation(s) you have used to ge the answers above.

	Tabular	Graphical	Algebraic
$x = -3$			
$x = 2.5$			
$x = 3.7$			
$x = 6$			

(c) Explain your choice(s).

2. A function T is defined below for all positive integers:

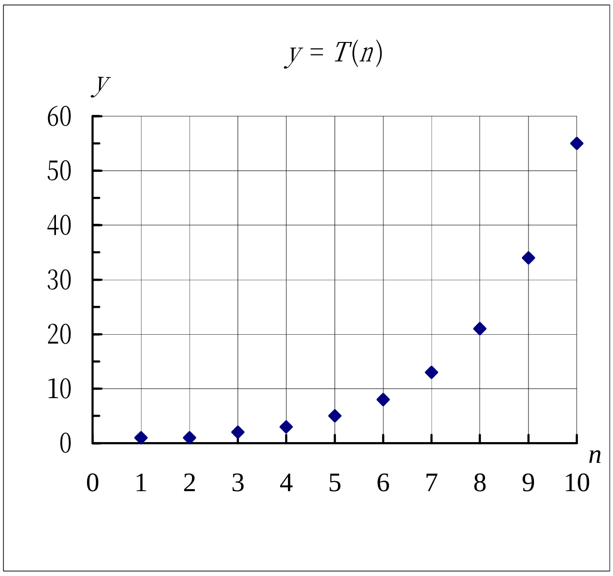
$$\begin{cases} T(1) = T(2) = 1 \\ T(n+2) = T(n) + T(n+1) \quad \text{for } n \geq 1. \end{cases}$$

Obviously, all positive integers of n determine a unique value of $T(n)$, where $T(n)$ is the n^{th} term.

(I) The tabular form of the function T is given as

n	1	2	3	4	5	6	7	8	9	10	...
$T(n)$	1	1	2	3	5	8	13	21	34	55	...

(II) The graphical form of the function T is given as



(III) The algebraic form of the function T is given as

$$T(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

(a) Find the value(s) of following terms:

(i) the 11th, 12th and 13th terms.

(ii) the 41st and 42nd terms.

(iii) the 43rd, 44th and 45th terms.

(b) (i) Which form will you use to get the answers in Part(a)? Put a tick "✓" to show your option(s) in the following table

	Tabular	Graphical	Algebraic
The 11th, 12th and 13th terms			
The 41st and 42nd terms			
The 43rd, 44th and 45th terms			

(c) Explain you choice(s).

Worksheet 2 : Recognition of functions in different representations

1. Consider the following tables.

   Suppose the range under consideration are $0 \le x \le 8$ and $0 \le y \le 8$.

Table 2.1

x	0	1	2	3	4	5	6	7	8
y	0	1	2	3	4	5	6	7	8

Table 2.2

x	0	1	2	3	4	5	6	7	8
y	0	1	2	2	4	5	6	7	8

Table 2.3

x	0	1	2	3	4	2	4	6	8
y	0	1	4	2	5	3	6	7	8

Table 2.4

x	0	1	2	3	4	5	6	7	8
y	0	1	2	3	4	2	4	6	8

Table 2.5

x	0	1	2	3	4	5	6	7	8
y	0	1	2	3	8	4	7	5	6

Suppose the range under consideration are $0 \le x \le 8$ and $0 \le y \le 8$. Determine whether

(a) it is correct for the statement "*y* is a function of *x*".

(b) it is correct for the statement "*x* is a function of *y*".

Put a "✓" in the appropriate box.

		y is a function of *x*.				*x* is a function of *y*.			
		if							
	x is	integer	integer	real no.	real no.	integer	integer	real no.	real no.
	y is	integer	real no.	integer	real no.	integer	real no.	integer	real no.
Table 2.1	True								
	Not sure								
	False								
Table 2.2	True								
	Not sure								
	False								
Table 2.3	True								
	Not sure								
	False								
Table 2.4	True								
	Not sure								
	False								
Table 2.5	True								
	Not sure								
	False								

2. In the following equations, determine whether y is a function of x and give reason(s) for your judgment.

   (a) $y = 2x + 1$,

   (b) $y = x^2 + 3$,

   (c) $y^2 = x^2 + 3$,

   (d) $y^3 = x^2 - x + 5$.

3. In the following diagrams, determine whether y is a function of x and give reason(s) for your judgment.

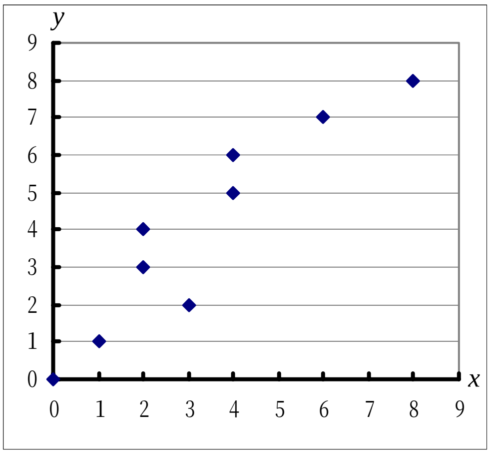


Fig 3.1

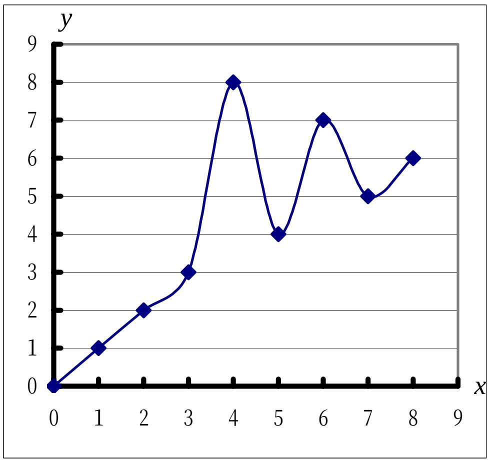


Fig 3.2

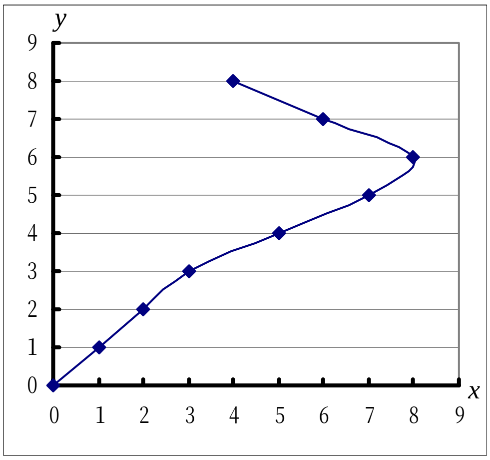


Fig 3.3

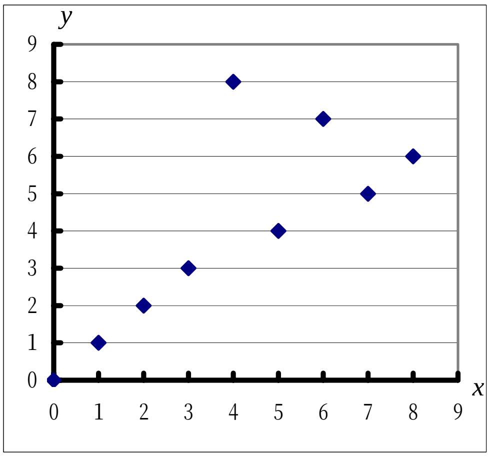


Fig 3.4

Notes for Teachers:

1. The time suggested for the activities is:

   Worksheet 1 40 minutes

   Worksheet 2 40 – 60 minutes

2. Answer to Worksheet 1.

Question 1

(a)

X	–3	2.5	3.7	6
$f(x)$	18	7	20.68 or around 20	63

(b)

x	Tabular	Graphical	Algebraic
x = –3	✓	✓	✓
x = 2.5		✓	✓
x = 3.7		✓	✓
x = 6			✓

(c) Reasons for the choices are as follows:

(i) It is clearly that the answer is in the table. Some students may also choose to read it from the diagram or calculate from the algebraic form. However, it is more time consuming for the latter two methods.

(ii) The answer is not in the table. It is easy to read the answer from the given graph. (Nevertheless, some students may prefer to calculate the value from the algebraic form and the answer is more accurate.)

(iii) The answer in fact can be found either from graphical or algebraic forms. The teacher should remind students they have to pay attention to the accuracy when using the graphical form to get the answer. The teacher may then bring out the idea of significance of scale chosen for the x and y-axes on the accuracy of the answer.

(iv) As the value of function when $x = 6$ cannot be read neither from the tabular nor the graphical forms, the algebraic form must be used to get the answer. (The teacher may bring out the idea of importance of the range of x value. If $x = 6$ is included in the graph, it should be easier to get the answer graphically.)

Question 2

(a)

n	11	12	13	14	15	41	42	43	44	45
$T(n)$	89	144	233	377	610	165580141	267914296	433494437	701408733	1134903170

(b)(i)

	Tabular	Graphical	Algebraic
The 11th, 12th and 13th terms	✓		
The 41st and 42nd terms			✓
The 43rd, 44th and 45th terms	✓		✓

(b)(ii)Reasons for the choices are as follows:

- To find the 11th, 12th and 13th terms:

  It is clearly simplest to get the 11th, 12th and 13th terms by continuing the table given with the recurrence relation $T(n+2) = T(n) + T(n+1)$.

- To find the 41st and 42nd terms:

  It is time consuming and easy to make mistakes to get the 41st and 42nd terms through iteration. Rather it is easier to get the values with a calculator from the algebraic expression.

- To find the 43rd, 44th and 45th terms:

  A combination of using algebraic expression and iteration is optimal in this case. The 41st and 42nd terms are found from the algebraic expression in (b)(ii) with a calculator. Then the 43rd, 44th and 45th terms are obtained by iteration using the recurrence relation.

3. The method chosen by students are open. The teacher should guide students to justify their answers rather than merely using wild guess.

4. In Question 2 in Worksheet 1, the table is obtained by iterating the recurrence relation $T(n+2) = T(n) + T(n+1)$ starting from $T(1) = T(2) = 1$, i.e.,

$$T(3) = T(1) + T(2) = 1 + 1 = 2$$

$$T(4) = T(2) + T(3) = 1 + 2 = 3$$

$$\vdots$$

5. In Question 2 in Worksheet 1, the algebraic expression is obtained by solving the second order difference equation $T(n+2) = T(n) + T(n+1)$ with the initial conditions $T(1) = T(2) = 1$.

   To solve the equation, the auxiliary equation $t^2 - t - 1 = 0$ is considered. Roots of this auxiliary equation are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Therefore

$$T(n) = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad \ldots (*)$$

   for constants *A* and *B*. To find the values of *A* and *B*, put *n* = 1 and 2 into the above relation (*). Two linear equations in *A* and *B* are obtained. Solve this system of simultaneous equations. It happens that

$$A = \frac{1}{\sqrt{5}}, \quad B = -\frac{1}{\sqrt{5}}$$

   So finally $T(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ for all positive integers *n*.

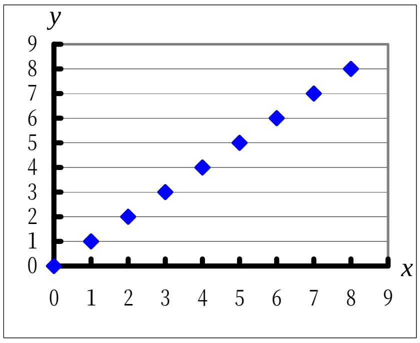
6. Answer for Question 1 in Worksheet 2:

Question 1(a) & (b)

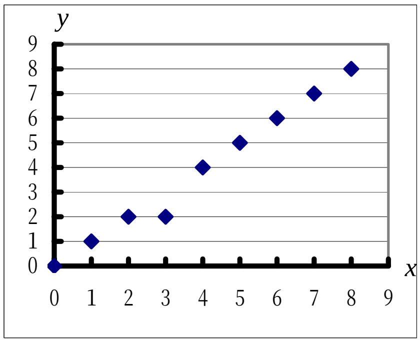
		y is a function of x.				x is a function of y.			
		If							
	x	integer	integer	real no.	real no.	integer	integer	real no.	real no.
	y	integer	real no.	integer	real no.	integer	real no.	integer	real no.
Table 2.1	True	✓	✓			✓		✓	
	Not sure			✓	✓		✓		✓
	False								
Table 2.2	True	✓	✓						
	Not sure			✓	✓				
	False					✓	✓	✓	✓
Table 2.3	True					✓		✓	
	Not sure						✓		✓
	False	✓	✓	✓	✓				
Table 2.4	True	✓	✓						
	Not sure			✓	✓				
	False					✓	✓	✓	✓
Table 2.5	True	✓	✓			✓		✓	
	Not sure			✓	✓		✓		✓
	False								

7. The graphs corresponding to the Tables 2.1 to 2.5 are as follows:

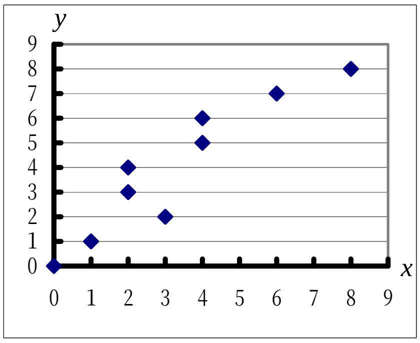
(a)



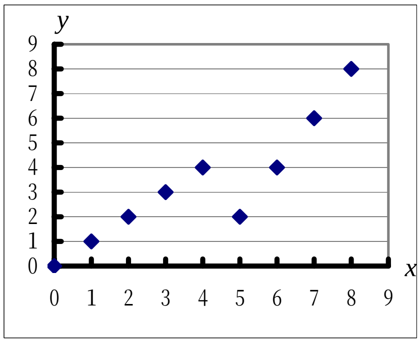
(b)



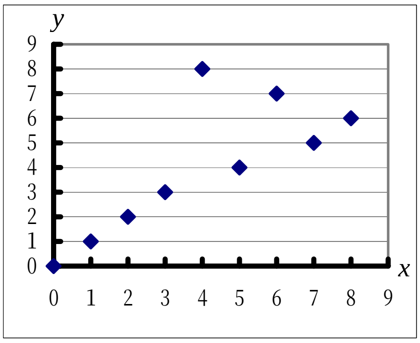
(c)



(d)

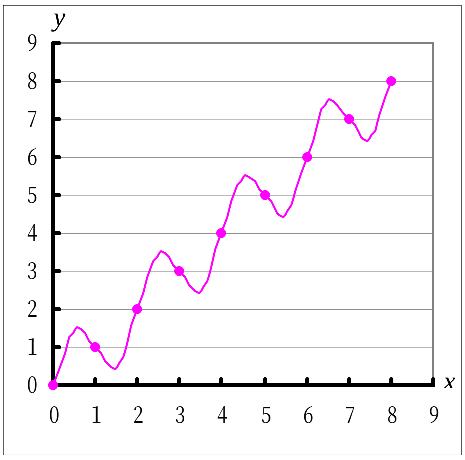


(e)

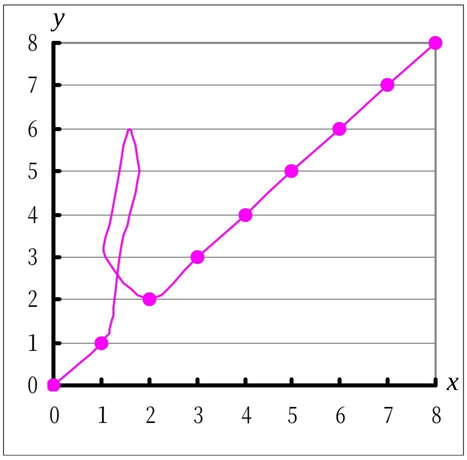


8. When x and y are real numbers, below are possible graphs corresponding to Table 2.1:

y is a function of *x*

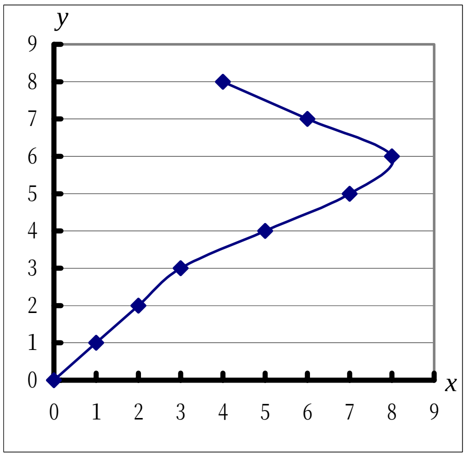


y is not a function of *x*

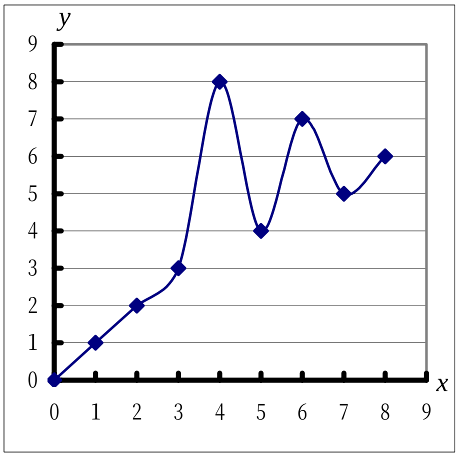


Similarly, the following graphs could also be drawn corresponding to Table 2.5:

y is a function of *x*



y is not a function of *x*



Therefore, the teacher should remind students that they have to be aware of the conditions given in questions and the basic definition of functions when they have to determine whether a relation is a function. For example, Table 2.1 in Question 1 shows that y is a function of x when x takes integral values. However, it is not sure whether 'y is a function of x' when x is a real number.

9. The teacher may use these 2 examples to illustrate the limitation of tabular representation of functions and raise the need to find the graphical and symbolic representations of a function.

10. Suggested answer to Questions 2 and 3 in Worksheet 2.

2. (a) Yes.

   For each *x*, there is exactly one corresponding value of 2*x*+3 and hence the unique value of *y* is determined. Hence *y* is a function of *x*.

   (b) Yes.

   For each *x*, there is exactly one corresponding value of x^2+3 and hence the unique value of *y* is determined. Hence *y* is a function of *x*.

   (c) No.

   For each *x*, there is exactly one corresponding value of x^2+3. But for each positive number *a*, two values of *y* satisfy the equation $y^2=a$. When $x=1$, $x^2+3=4$ and so the corresponding *y* is not uniquely determined. Hence, *y* is NOT a function of *x*.

   (d) Yes.

   For each *x*, there is exactly one corresponding value of x^2-x+5. For each positive number *a*, only one value of *y* satisfies the equation $y^3=a$. Hence *y* is a function of *x*.

3. Diagram 3.1: *y* is not a function of *x*. When *x* = 4, there are more than one *y* output values.

   Diagram 3.2: *y* is a function of *x*.

   Diagram 3.3: *y* is not a function of *x*. When $4\le x\le 8$, there are more than one *y* output values corresponds to each *x* input value.

   Diagram 3.4: *y* is a function of *x*.

11. In similar problems of diagrams 3.3 and 3.4 in Question 3, the teacher could further consolidate students' understanding on the range of *x* to determine whether '*y* is a function of *x*'.