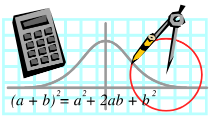

Exemplar 6:
Common Misconceptions on Concepts of Functions

Objectives : Clarify the following concepts of functions.

(a) $f(0) \neq 0$

(b) $f(-a) \neq -f(a)$

(c) $f(ab) \neq a \cdot f(b)$

(d) $f(a+b) \neq f(a) + f(b)$

(e) $f(ab) \neq f(a) \cdot f(b)$

where a and b are constants.

Key Stage : 4

Learning Unit : Functions and Graphs

Materials required : Worksheets

Prerequisite Knowledge : Calculate the value of a given function

Description of the activity:

1. The teacher asks students whether $f(5) = f(2) + f(3)$ is correct in general and the reason.
   (a) Let $f(x) = 2x - 5$. Calculate $f(2+3)$ and $f(2) + f(3)$. The teacher asks students whether $f(2+3) = f(2) + f(3)$ is correct.

   (b) Hence, the teacher asks students that for an arbitrary function $f(x)$ and any constants a and b whether $f(a+b) = f(a) + f(b)$ is always true.

2. The teacher asks students whether $f(-x) = -f(x)$ is always true and asks them to give examples.

3. The teacher distributes Worksheet 1 to students and asks them to finish it by investigating the concepts of functions. The teacher discusses with students and draws the following conclusions.

(a) For any function $f(x)$ and constants *a* and *b*, the followings are NOT ALWAYS true.

(i) $f(0) = 0$
(ii) $f(-a) = -f(a)$
(iii) $f(ab) = a \cdot f(b)$
(iv) $f(ab) = f(a) \cdot f(b)$
(v) $f(a+b) = f(a) + f(b)$

(b) The teacher should emphasize that the above equalities may be correct only for some particular functions. The equalities do not hold for any functions. For example, when $f(x) = 3x$, $f(-a) = -f(a)$. However, $f(-a) \neq -f(a)$ when $f(x) = x^2$.

(c) Similarly, if an equality is only true for some particular values of *a* and *b*, it cannot say that the equality is in general true. For example, when $f(x) = (x-8)(x-4)$, $f(2\times 4) = f(2)\times f(4) = 0$. However, $f(2\times 3) \neq f(2)\times f(3)$. Thus, in general $f(ab) \neq f(a)\cdot f(b)$. In addition, if they have learnt trigonometric functions, when $f(x) = \sin x$, $f(360° + 30°) = f(360°) + f(30°)$, but $f(300° + 30°) \neq f(300°) + f(30°)$. Thus, $f(a+b) \neq f(a) + f(b)$.

4. The teacher should also remind students the meaning of notation $f(x)$ and clarify it should not be equated to $f \cdot x$. The distributive law should not be applied to the concept. This means that in general $f(a+b) \neq f(a) + f(b)$ and $f(a \cdot b) \neq f(a) \cdot f(b)$.

5. The teacher asks students to finish Worksheet 2 at home and gives suitable guidance.

Worksheet 1 :

Common Misconceptions on Concepts of Functions

1. Complete the table by calculating the values of functions.

	$x=-3$	$x=-2$	$x=0$	$x=2$	$x=3$	$x=5$	$x=6$
(i) $f(x)=x^2$							
(ii) $f(x)=x^3$							
(iii) $f(x)=x-1$							
(iv) $f(x)=2x$							
(v) $f(x)=x^2+3$							

2. (a) In the above table, for $f(x)=x^2$, find out the values of $f(2)$, $f(3)$ and $f(5)$. Determine whether $f(5) = f(2)+f(3)$ is correct.

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(b) a and b are constants. Referring to the various functions in the above table, is $f(a+b) = f(a)+f(b)$ always true?

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3. Let a and b be any constant. From the above table, determine which of the followings must be correct. If any of the following is not always true, give an example to illustrate it.

(a) $f(0)=0$

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(b) $f(-a) = -f(a)$

(c) $f(ab) = a \cdot f(b)$

(d) $f(ab) = f(a) \cdot f(b)$

Worksheet 2 :

Common Misconceptions on Concepts of Functions

Let $f(x)$ and $g(x)$ be arbitrary functions and *a*, *b*, *m* and *n* be arbitrary constants.

1. Determine which of the followings are certainly correct and put a "✓" in the appropriate box. If it is not always true, give a counter-example.

		Must be true	Not always true	Counter-example
(a)	$-f(2)=f(-2)$	☐	☐	____________
(b)	$f(m)+f(n)=f(m+n)$	☐	☐	____________
(c)	$g(a+1)=g(a)+1$	☐	☐	____________
(d)	$a+f(b)=f(a+b)$	☐	☐	____________
(e)	$g(a)-g(b)=g(a-b)$	☐	☐	____________
(f)	$f(m+n)=f(n+m)$	☐	☐	____________
(g)	$f(a-1)=f(1-a)$	☐	☐	____________
(h)	$f(ab)=f(ba)$	☐	☐	____________
(i)	$g(2a)=g(a)\times 2$	☐	☐	____________
(j)	$g(5b)=5g(b)$	☐	☐	____________
(k)	$f(mn)=f(m)\times f(n)$	☐	☐	____________
(l)	$[f(a)]^2=f(a)\cdot f(a)$	☐	☐	____________
(m)	$[f(a)]^2=f(a^2)$	☐	☐	____________

2. Which of the following equalities are certainly correct? Substantiate your answer with examples.

(a) $f(0) = 0$

(b) $\dfrac{g(x)}{g(y)} = g\left(\dfrac{x}{y}\right)$

(c) $f\left(\dfrac{1}{a}\right) = \dfrac{1}{f(a)}$

(d) $f\left(\dfrac{1}{a}\right) = \dfrac{f(1)}{f(a)}$

3. Determine which of the following statements is correct and explain your answer.

   (a) If $a = b$, then $f(a) = f(b)$.

   (b) If $f(a) = f(b)$, then $a = b$.

Notes for Teachers:

1. The time required for this activity is about 60 minutes.

2. Answers of the worksheets:

Worksheet 1

1.

	$x = -3$	$x = -2$	$x = 0$	$x = 2$	$x = 3$	$x = 5$	$x = 6$
(i) $f(x) = x^2$	9	4	0	4	9	25	36
(ii) $f(x) = x^3$	–27	–8	0	8	27	125	216
(iii) $f(x) = x - 1$	–4	–3	–1	1	2	4	5
(iv) $f(x) = 2x$	–6	–4	0	4	6	10	12
(v) $f(x) = x^2 + 3$	12	7	3	7	12	28	39

2. (a) Not always true.
   (b) Not always true.

3. (a) No.
   Example: When $f(x) = x - 1$,
   $\because \quad f(0) = -1$
   $\therefore \quad f(0) \neq 0$

   (b) No.
   Example: When $f(x) = x^2$,
   $\because \quad f(-2) = 4$
   and $-f(2) = -4$
   $\therefore \quad f(-2) \neq -f(2)$

   (c) No.
   Example: When $f(x) = x - 1$,
   $\because \quad f(2 \times 3) = f(6)$
   $= 5$

and $2\times f(3) = 2\times 2$

$= 4$

$\therefore \quad f(2\times 3) \neq 2\times f(3)$

(d) No.

Example: When $f(x) = 2x$,

$\because \quad f(2\times 3) = f(6)$

$= 12$

and $f(2)\cdot f(3) = 4\times 6$

$= 24$

$\therefore \quad f(2\times 3) \neq f(2)\cdot f(3)$

Worksheet 2

1.

	Must be true	Not always true	Counter-example
(a) $-f(2) = f(-2)$	☐	☑	$f(x) = x^2$
(b) $f(m) + f(n) = f(m+n)$	☐	☑	$f(x) = x^2$, $m = 3$, $n = 4$
(c) $g(a+1) = g(a)+1$	☐	☑	$g(x) = 3x$, $a = 2$
(d) $a + f(b) = f(a+b)$	☐	☑	$f(x) = 3x$, $a = 2$, $b = 3$
(e) $g(a) - g(b) = g(a-b)$	☐	☑	$g(x) = x^2$, $a = 2$, $b = 3$
(f) $f(m+n) = f(n+m)$	☑	☐	
(g) $f(a-1) = f(1-a)$	☐	☑	$f(x) = 3x$, $a = 3$
(h) $f(ab) = f(ba)$	☑	☐	
(i) $g(2a) = g(a)\times 2$	☐	☑	$g(x) = x^2$, $a = 3$
(j) $g(5b) = 5g(b)$	☐	☑	$g(x) = x^2$, $b = 2$
(k) $f(mn) = f(m)\times f(n)$	☐	☑	$f(x) = 2x$, $m = 2$, $n = 3$

(l) $[f(a)]^2 = f(a)\cdot f(a)$ ☑ ☐ ______________

(m) $[f(a)]^2 = f(a^2)$ ☐ ☑ $f(x)=2x$, $a=3$

2. (a) No.

Example: Let $f(x)=2x+1$

$f(0)=2(0)+1$

$=1\neq 0$

(Students may argue that when $f(x)=x^2$, $f(0)=0$ is correct. The teacher should clarify that it is just a particular case and it may not be true in general. If $f(x)=mx+c$ or $f(x)=ax^2+bx+c$ where a, b, c and m are non-zero constants, then $f(0)\neq 0$.)

(b) No.

Example: Let $g(x)=2x+3$

$$\because \quad \frac{g(3)}{g(6)} = \frac{2\times 3+3}{2\times 6+3} = \frac{9}{15} = \frac{3}{5}$$

also $g(\frac{3}{6}) = g(\frac{1}{2})$

$= 2\times\frac{1}{2}+3$

$=4$

$$\therefore \quad \frac{g(3)}{g(6)} \neq g(\frac{3}{6})$$

(Similarly, when students argue that $\frac{g(3)}{g(6)} = g\left(\frac{3}{6}\right)$ is correct for $g(x)=x^2$, the teacher should also clarify that it is just a particular case and it may not be true in general. If $g(x)=mx+c$ or $g(x)=ax^2+bx+c$ where a, b, c and m are non-zero constants, then $\dfrac{g(x)}{g(y)} \neq g\left(\dfrac{x}{y}\right)$.)

(c) No.

Example: Let $f(x) = 2x$

$\because \quad f\left(\dfrac{1}{3}\right) = 2 \times \dfrac{1}{3}$

$= \dfrac{2}{3}$

also $\dfrac{1}{f(3)} = \dfrac{1}{2 \times 3}$

$= \dfrac{1}{6}$

$\therefore \quad f\left(\dfrac{1}{3}\right) \neq \dfrac{1}{f(3)}$

(d) No.

Example: Let $f(x) = x + 2$

$\because \quad f\left(\dfrac{1}{3}\right) = \dfrac{1}{3} + 2$

$= 2\dfrac{1}{3}$

also $\dfrac{f(1)}{f(3)} = \dfrac{1+2}{3+2}$

$= \dfrac{3}{5}$

$\therefore \quad f\left(\dfrac{1}{3}\right) \neq \dfrac{f(1)}{f(3)}$

3. (a) True, from the definition of function.

(b) Not always true. For example, if $f(x) = x^2$, $f(3) = 9$, $f(-3) = 9$. Obviously, $f(3) = f(-3)$, but $3 \neq -3$.

3. The teacher should emphasize that the above equality may hold for some values but not all. The words ‘certainly correct’ in the questions mean that the functions hold for all values of the domain.

4. The teacher may further discuss with students which kind of functions can satisfy the above equalities and let students know more about the characteristic of them. In general,

   (a) If $f(x) = ax$, $f(x) = ax^2$, $f(x) = ax^3 \ldots$, etc. (where a is a constant), then $f(0) = 0$.

   (b) If $f(x)$ is an odd function such as $f(x) = ax$, $f(x) = ax^3$, $f(x) = ax^5$ …, etc. (where a is a constant), then $f(-b) = -f(b)$.

   (c) If $f(x) = mx$ (where m is a constant), then $f(ab) = a \cdot f(b)$.

   (d) If $f(x) = mx$ (where m is a constant), then $f(a+b) = f(a) + f(b)$.

   (e) If $f(x) = x$, $f(x) = x^2$, $f(x) = x^3 \ldots$, etc., then $f(ab) = f(a) \cdot f(b)$.