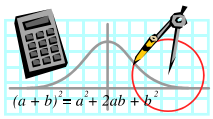


Exemplar 7:
Graphs of Quadratic Functions

Objective : To explore various graphs of quadratic function and the properties of the function

Key Stage : 4

Learning Unit : Functions and Graphs

Materials required : Worksheets; *Excel file*(*QUAD.XLS*); *Winplot* and an OHP transparency with the graph of $y = x^2$ printed on it

Prerequisite Knowledge : To give the equations of vertical lines in the coordinate plane

Description of the activity:

1. The teacher distributes Worksheet 1 and opens the file "*QUAD.XLS*". The teacher asks students to attempt Questions 1-3 on the worksheet.

2. The teacher then helps students to conclude the effects of the value of *a* on the graph of the quadratic function $y = ax^2 + bx + c$.

3. The teacher asks students to attempt Questions 4-5 in order to explore the effects of the values of *b* and *c* on the graphs of the functions.

4. The teacher discusses with students the effects of changing the values of *b* and *c* on the graph of $y = ax^2 + bx + c$. Then the teacher further discusses Question 6 with students to conclude the effects of changing values of *a*, *b* and *c* on the graph of $y = ax^2 + bx + c$. These observations include:
   (a) No matter how *a* changes, for $a \neq 0$, the graph of the function $y = ax^2 + bx + c$ will be a parabola with symmetric properties;
   (b) The direction of opening and the steepness of the shape of the curve depend only on the values of *a*;
   (c) The *y*-intercept of the curve depends only on the values of *c*;

(d) The coordinates of the vertex changes as the values of b and c change.

In discussing the above observations, the teacher may invite students to explain the phenomenon, including why the graph is not a curve when a = 0, etc.

5. The teacher distributes Worksheet 2 and use the software *Winplot* to investigate the features of the graph of $y = (x + h)^2$ and $y = (x + h)^2 + k$.

6. The teacher asks students to complete Worksheet 2 according to the instructions given and helps them to draw conclusions. These include:
   (a) If the quadratic function is expressed in the form $y = a(x + h)^2$:
      (i) the value of a will determine the direction of openness and whether the curve has the maximum or minimum value;
      (ii) the axis of symmetry is $x = -h$ and
      (iii) the coordinates of the vertex is $(-h, 0)$;
   (b) If the quadratic function is in the form $y = a(x + h)^2 + k$, there is no difference on the shape and the axis of symmetric between this curve and that of $y = a(x + h)^2$. However, the coordinates of the vertex of the former curve is changed to $(-h, k)$.

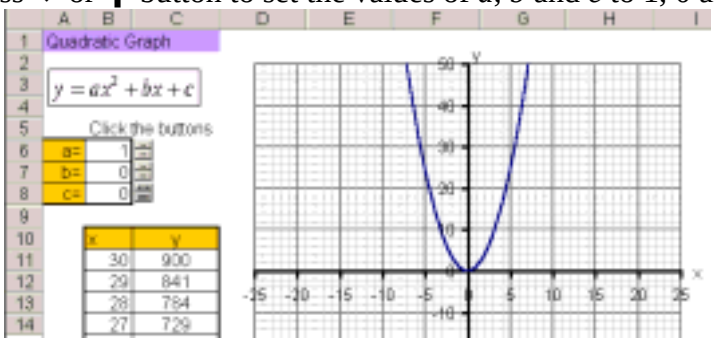
   In discussing the above observations, the teacher may invite students to explain the phenomenon, such as why the perfect square expressions facilitate the determination of the coordinates of vertex and why the direction of openness is determined solely by the value of a. The teacher may also relate the perfect square expressions with the coordinates of vertex to the observation in Worksheet 1 as "the vertex of the curve changes according to values of b and c".

7. When students are aware of the relation between the vertex, the axis of symmetry and the form of the quadratic function $y = (x + h)^2 + k$, the teacher may introduce the method of completing the square and its significance.

Worksheet 1 : Features of Quadratic Graphs (1)

Questions:

1. Press ↓ or ↑ button to set the values of *a*, *b* and *c* to 1, 0 and 0 respectively.



The figure above shows the graph of $y =$ _____.

2. Keep the values of *b* and *c* to 0. Press ↑ button to increase the value of *a* gradually from 1 to 4. Observe the changes on the graph and complete the table below.

Value of *a*	Direction of Opening	Shape	Coordinates of the Vertex	Symmetry	Nature
2		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
3		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
4		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve

(a) Describe briefly the changes on the graph.

_____

(b) Is there any change in the location of the vertex?

_____

(c) Is there any change in the direction of opening?

_____

3. Keep the values of b and c to 0. Press ↓ button to decrease the value of a gradually from 4 to −4. Observe the changes on the graph and complete the table below.

Value of a	Direction of Opening	Shape	Coordinates of the Vertex	Symmetry	Nature
0.5		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
0.2		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
0		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
−1		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
−2		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
−3		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
−4		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve

(a) Describe briefly the changes on the graph.

_____

(b) Is there any change in the location of the vertex?

_____

(c) If $a > 0$, the graph opens _____ (upwards/downwards).
If $a < 0$, the graph opens _____ (upwards/downwards).

4. Keep the values of a and c to 1 and 0 respectively. Press ↑ button to increase the value of b from 0 to 12, and then press ↓ button to decrease its value gradually to –8. Observe the changes on the graph and complete the table below.

Value of b	Direction of Opening	Shape	Coordinates of the Vertex	Symmetry	Nature
4		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
8		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
12		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
–4		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
–8		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve

5. Keep the values of a to 1 and the value of b to 0. Press ↑ button to increase the value of c gradually from 0 to 12, and then press ↓ button to decrease its value gradually to -8. Pay attention to the changes in the graph and complete the table below.

Value of c	Direction of Opening	Shape	Coordinates of the Vertex	Symmetry	Nature
4		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
8		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
12		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
–4		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve
–8		Flatter/Steeper/Unchanged		Yes/No	Straight line/Curve

6. In Questions 2 to 5, we found that:

(a) No matter how we alter the value of a ($a \neq 0$), the graphs of $y = ax^2 + bx + c$ preserve the following properties:

_____

(b) What happens to the direction of the opening of the graph if the value of a changes from positive to negative?

_____

(c) The greater the magnitude of a (ignoring its sign), the _____ (flatter/steeper) its shape is.

(d) When $a = 0$, the graph is a _____. The reason is:

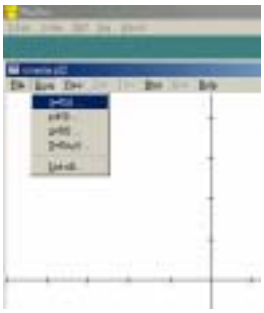
_____

(e) When the values of b and c are kept constant, the _____ (steepness of shape/location of vertex/y-intercept) of the graph will change; but _____ (steepness of shape/location of vertex/y-intercept) will not change.

(f) If we alter the value of b while the value of a and c are kept constant, the _____of the graph will change; but the _____ will not change.

(g) If we alter the value of c while the value of a and b are kept constant, the _____of the graph will change; but the _____ will not change.

Worksheet 2：Features of Quadratic Graph (2)



Activity 1

Procedure:

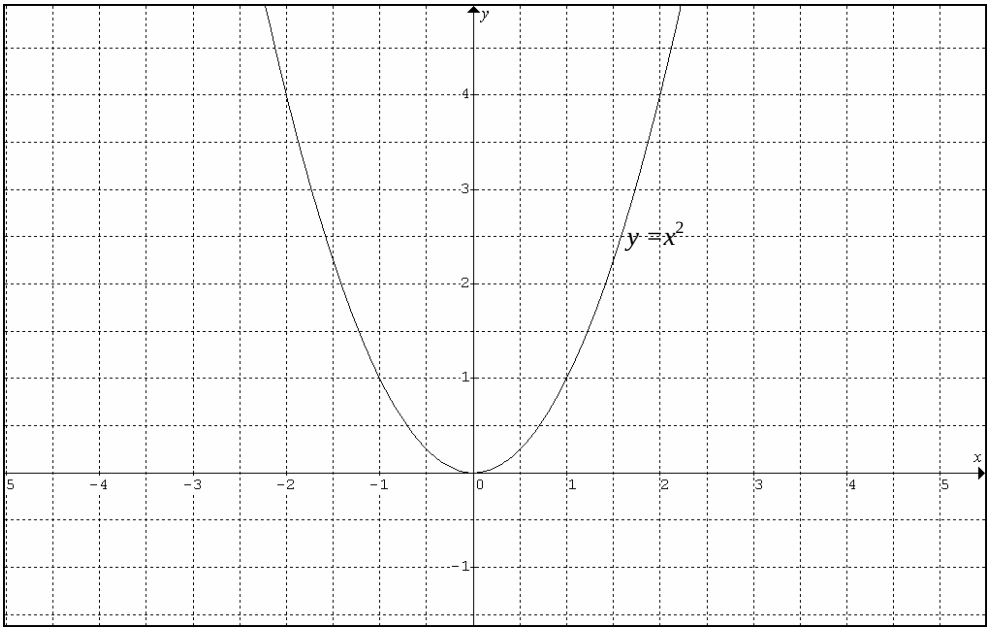
1. Launch the algebraic software *Winplot*.
2. Click "**2-dim**".
3. Click "**Equa**", " **y=f(x)…**".
4. Sketch the graph of $y = x^2$ by inputting " x ^ 2".
5. Repeat step 3 and input "(x－2) ^ 2" to sketch the graph of $y = (x-2)^2$.
6. Repeat step 3 and input "(x－4) ^ 2" to sketch the graph of $y = (x-4)^2$.
7. Compare the graphs obtained. What do you observe?

_____

8. Is there any change in the shape of the graph?

_____

9. Sketch the graph of $y = (x-3)^2$ in the figure below.



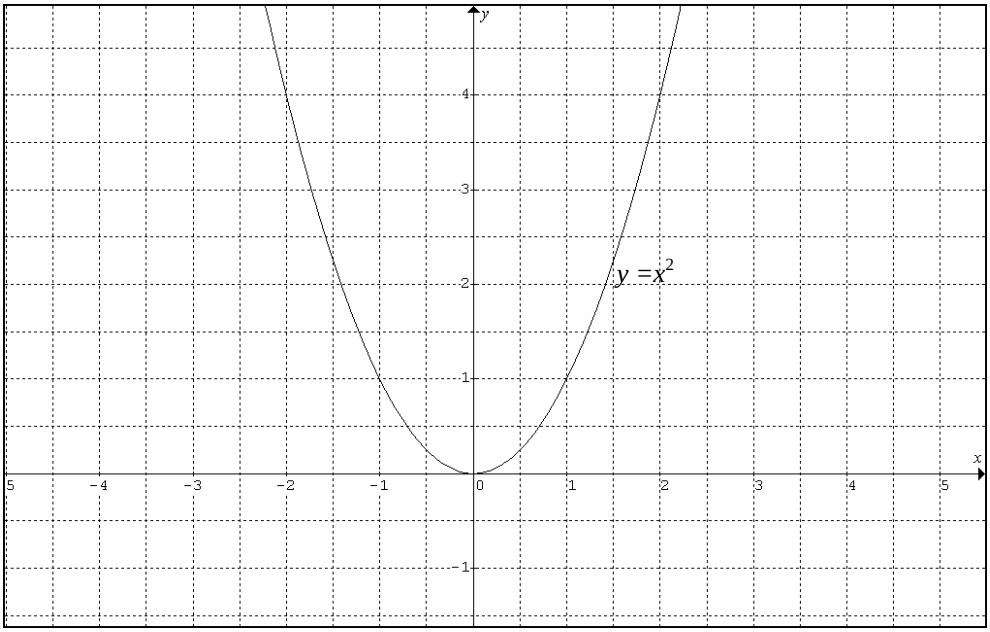
10. Use *Winplot* to sketch the graph of $y = (x-3)^2$. Compare the graph thus obtained with the graph in step 9. Are they the same?

_____

11. Your conclusions:

_____

12. Sketch the graphs of $y = (x+1)^2$ and $y = (x+3)^2$ in the figure below.



13. Use *Winplot* to sketch the graphs of $y = (x+1)^2$ and $y = (x+3)^2$. Compare the graphs thus obtained with the graphs in step 12. Are they consistent with each of them?

_____

14. Observe the above graphs and complete the following table:

	$y = (x-3)^2$	$y = (x+1)^2$	$y = (x+3)^2$
Equation of the axis of symmetry			
Coordinates of the vertex			
The max / min value of y			

15. From the results above, what are your conclusions?

_____

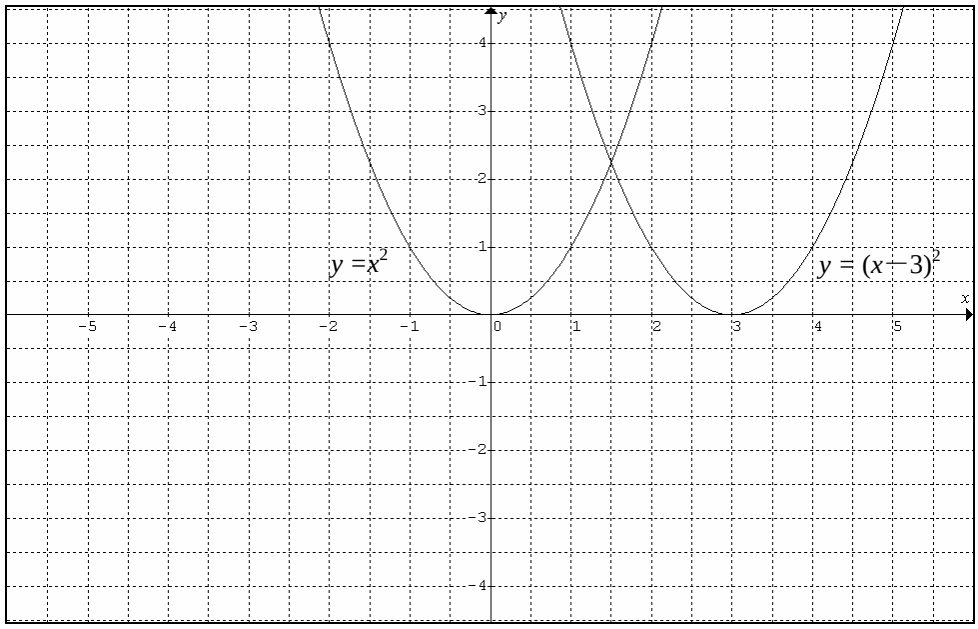
_____

_____

Activity 2

Procedure:

1. Sketch the graph of the function $y = (x-3)^2 + 1$ in the figure below.



2. Compare your graph with your classmates. Do you get the same graph?

_____

3. Use *Winplot* to verify your answer and explore the features of the graph when the values of h and k in the function $y = (x+h)^2 + k$ change. How do the values of h and k in $y = (x+h)^2 + k$ affect the position of the curve (such as the equation of the axis of the symmetry and the location of the vertex)?

_____

_____

4. Identify the vertex, axis of symmetry and maximum/minimum value of each of the following quadratic functions and complete the table below.

	Coordinates of Vertex	Axis of Symmetry	Maximum/ minimum value
(a) $y = (x-3)^2 - 4$			(max / min)* value =_____
(b) $y = (x+3)^2 - 4$			(max / min)* value =_____
(c) $y = (x-3)^2 + 4$			(max / min)* value =_____

	Coordinates of Vertex	Axis of Symmetry	Maximum/ minimum value
(d) $y=(x+3)^2+4$			(max / min)* value =_____
(e) $y=(x+2)^2-5$			(max / min)* value =_____
(f) $y=(x-12)^2+5$			(max / min)* value =_____
(g) $y=(x-7)^2$			(max / min)* value =_____
(h) $y=x^2+11$			(max / min)* value =_____
(i) $y=-x^2+3$			(max / min)* value =_____
(j) $y=-2x^2$			(max / min)* value =_____
(k) $y=5(x-8)^2+1$			(max / min)* value =_____
(l) $y=3(x-8)^2+1$			(max / min)* value =_____
(m) $y=-3(x-1)^2-4$			(max / min)* value =_____
(n) $y=-(x+5)^2+7$			(max / min)* value =_____
(o) $y=3(5-x)^2-6$			(max / min)* value =_____

*Circle the appropriate one.

Use *Winplot* to verify your answer.

5. In conclusion, the axis of symmetry and the vertex of graph of the function $y=a(x+h)^2+k$ (where a, h and k are constants) are _____ and _____ respectively.

Notes for Teachers:

1. The time required for the activity is about 40-50 minutes.

2. The teacher has to inform students that the values of *a*, *b* and *c* must be adjusted by using 🡻 or 🡹 button. Students must not input the values required directly into the cell. Otherwise, the file does not work properly.

3. When the file *QUAD.XLS* is used, the teacher should explain the relation among the algebraic, tabular and graphical representation of the function. Students may also try to calculate by the algebraic form of the function, some of the *y*-values with the given *x*-values in the tabular form. They could then check their calculated values with the values shown in the computer. The calculation is important for students to see their relations; otherwise, they only look at the changes of the graphs but do not understand the reasons behind.

4. When the teacher uses Worksheet 1, he/she may consider to use the main questions of the tables without asking students to finish Questions 2 to 5. However, it is important to make the conclusion as stipulated in Question 6.

5. Some students may not understand the meaning of 'the steepness of the curve'. They may confuse and consider the graphs of $y = x^2$, $y = x^2 + 4$ (Question 5) and $y = x^2 + 4x$ (Question 4) to have different steepness. The teacher needs to explain the meaning or uses the OHP transparency of the graph of $y = x^2$ for illustration whenever necessary.

6. As the application file *QUAD.XLS* will only show one curve at a time, it is difficult for students to compare graphs of different functions. Hence, *Winplot* is used in Worksheet 2 so that several graphs can be shown in one screen. This will enhance students to relate the differences between the algebraic forms and their graphical representations.

7. *Winplot* is a freeware. Teachers may download it from the following website: http://math.exeter.edu/rparris.

8. The suggested answer for the Worksheet 1 is as follows:

1. The figure above shows the graph of $y =$ x^2.

2.

Value of a	Direction of Opening	Shape	Coordinates of the Vertex	Symmetry	Nature
2	upward	Steeper	(0, 0)	Yes	Curve
3	upward	Steeper	(0, 0)	Yes	Curve
4	upward	Steeper	(0, 0)	Yes	Curve

(a) The greater the value of a the steeper the shape of the graph.

(b) No change.

(c) No change.

3.

Value of a	Direction of Opening	Shape	Coordinates of the Vertex	Symmetry	Nature
0.5	upward	Flatter	(0, 0)	Yes	Curve
0.2	upward	Flatter	(0, 0)	Yes	Curve
0	/	/	/	/	Straight line
–1	downward	steeper	(0, 0)	Yes	Curve
–2	downward	steeper	(0, 0)	Yes	Curve
–3	downward	steeper	(0, 0)	Yes	Curve
–4	downward	steeper	(0, 0)	Yes	Curve

(a) As the value of a changes from positive to negative, the direction of opening changes from upward to downward. When the value of a is negative, the smaller the value of a is, the steeper the shape of the curve will be.

(b) No change.

(c) If $a > 0$, the graph opens upwards (upwards/downwards).
If $a < 0$, the graph opens downwards (upwards/downwards).

4.

Value of b	Direction of Opening	Shape	Coordinates of the Vertex	Symmetry	Nature
4	upward	Unchanged	(–2, –4)	Yes	Curve
8	upward	Unchanged	(–4, –16)	Yes	Curve
12	upward	Unchanged	(–5, –25)	Yes	Curve
–4	upward	Unchanged	(2, –4)	Yes	Curve
–8	upward	Unchanged	(4, –16)	Yes	Curve

5.

Value of c	Direction of Opening	Shape	Coordinates of the Vertex	Symmetry	Nature
4	upward	Unchanged	(0, 4)	Yes	Curve
8	upward	Unchanged	(0, 8)	Yes	Curve
12	upward	Unchanged	(0, 10)	Yes	Curve
-4	upward	Unchanged	(0, –4)	Yes	Curve
-8	upward	Unchanged	(0, –8)	Yes	Curve

6. (a) The graph is a symmetric curve.

(b) From "upward" changes to "downward".

(c) The greater the magnitude of a (ignoring its sign), the steeper (flatter/steeper) its shape is.

(d) When $a = 0$, the graph is a straight line. The reason is:
When $a = 0$, $y = ax^2 + bx + c$ will change to $y = bx + c$ which is a straight line.

(e) If we alter the value of a while the values of b and c are kept constant, the shape and the vertex (shape/vertex/y-intercept) of the graph will change, but the y-intercept (shape/vertex/y-intercept) of the graph will not change.

(f) If we alter the value of b while the values of a and c are kept constant, the vertex of the graph will change, but the shape and the y-intercept of the graph will not change.

(g) If we alter the value of c while the value of a and b are kept constant, the vertex and the y-intercept of the graph will change, but the shape of the graph will not change.

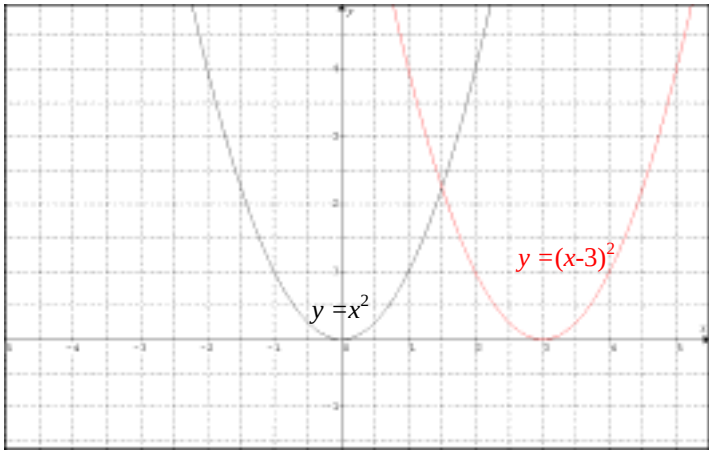
9. The suggested answer for the Worksheet 2 is as follows:

Activity 1

7. Compared to the graphs to $y = x^2$, $y = (x-2)^2$ and $y = (x-4)^2$ move to the right by 2 and 4 units respectively.

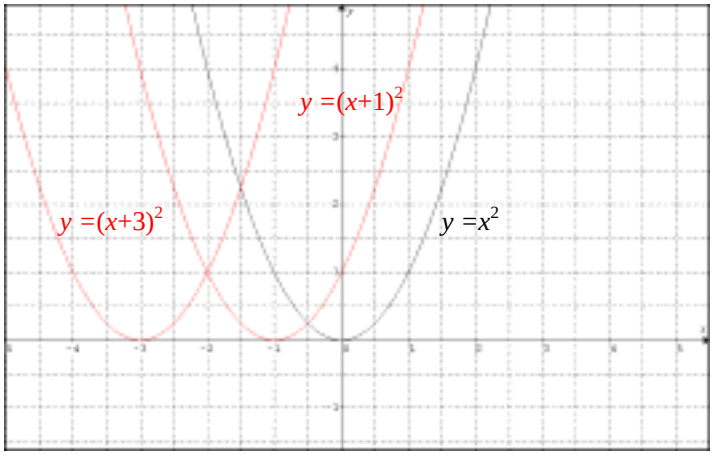
8. No change.

9. Sketch the graph of $y = (x-3)^2$ in the figure below.



11. Compared to the graph of $y = x^2$, the graph of $y = (x-a)^2$ ($a > 0$) moves a units to the right and the shape does not change.

12. Sketch the graphs of $y = (x+1)^2$ and $y = (x+3)^2$ in the figure below.



14.

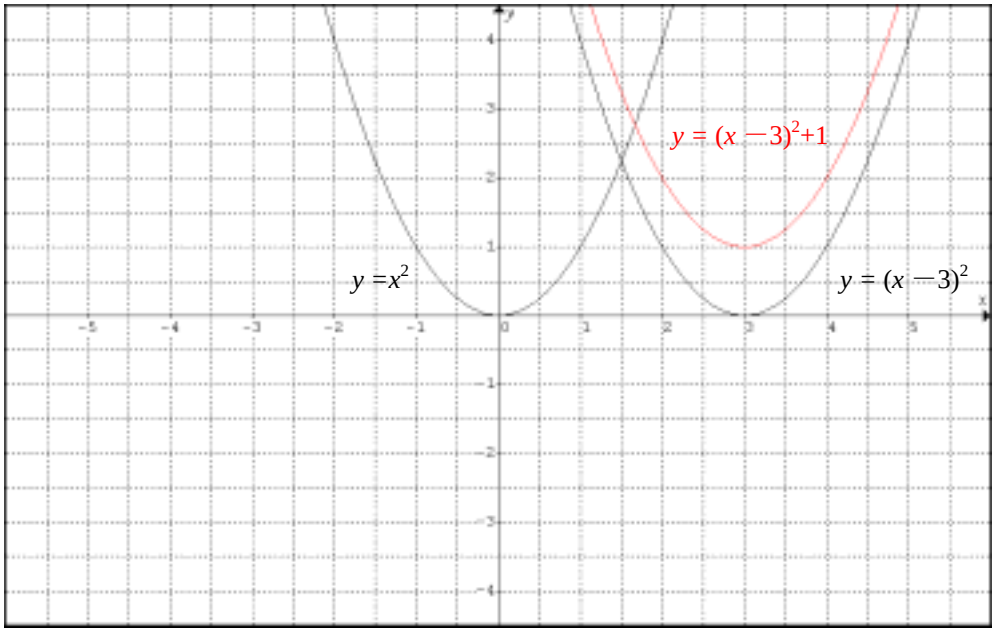
	$y=(x-3)^2$	$y=(x+1)^2$	$y=(x+3)^2$
Axis of symmetry	$x=3$	$x=-1$	$x=-3$
Vertex	$(3,\ 0)$	$(-1,\ 0)$	$(-3,\ 0)$
Maximum/ minimum value of y	0	0	0

15.

When comparing with the graph of $y=x^2$, if $h>0$, the graph of $y=(x-h)^2$ moves h units to the right and the graph of $y=(x+h)^2$ moves h units to the left. For the two functions, the axes of symmetry are $x = h$ and $x = -h$, the vertex is $(h, 0)$. The shape do not change and both functions have minimum values equal to 0.

Activity 2

1. Sketch the graph of the function $y = (x-3)^2 + 1$ in the figure below.



3. Compared to the graph of $y = x^2$, the graph of $y = (x-3)^2 + 2$ moves 3 units to the right and 2 units upward, the axis of symmetry is $x = 3$, the vertex is (3, 2) and the shape does not change.

4.

		Coordinates of Vertex	Axis of Symmetry	Maximum/ minimum value
(a)	$y = (x-3)^2 - 4$	(3, –4)	$x = 3$	(max / min)* value = –4
(b)	$y = (x+3)^2 - 4$	(–3, –4)	$x = -3$	(max / min)* value = –4
(c)	$y = (x-3)^2 + 4$	(3, 4)	$x = 3$	(max / min)* value = 4
(d)	$y = (x+3)^2 + 4$	(–3, 4)	$x = -3$	(max / min)* value = 4
(e)	$y = (x+2)^2 - 5$	(–2, –5)	$x = -2$	(max / min)* value = –5
(f)	$y = (x-12)^2 + 5$	(12, 5)	$x = 12$	(max / min)* value = 5
(g)	$y = (x-7)^2$	(7, 0)	$x = 7$	(max / min)* value = 0

	Coordinates of Vertex	Axis of Symmetry	Maximum/ minimum value
(h) $y = x^2 + 11$	(0, 11)	$x = 0$	(max / min)* value = 11
(i) $y = -x^2 + 3$	(0, 3)	$x = 0$	(max / min)* value = 3
(j) $y = -2x^2$	(0, 0)	$x = 0$	(max / min)* value = 0
(k) $y = 5(x-8)^2 + 1$	(8, 1)	$x = 8$	(max / min)* value = 1
(l) $y = 3(x-8)^2 + 1$	(8, 1)	$x = 8$	(max / min)* value = 1
(m) $y = -3(x-1)^2 - 4$	(1, –4)	$x = 1$	(max / min)* value = –4
(n) $y = -(x+5)^2 + 7$	(–5, 7)	$x = -5$	(max / min)* value = 7
(o) $y = 3(5-x)^2 - 6$	(5, –6)	$x = 5$	(max / min)* value = –6

5. In conclusion, the axis of symmetry and the vertex of $y = a(x + h)^2 + k$ (where a, h and k are constants) are $x = -h$ and $(-h, k)$ respectively.