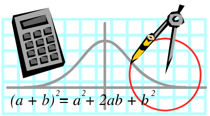


Exemplar 9:
The effects of transformations on the graphs of functions (2)

Objective : Understand the effects of horizontal translation on the graphs of functions and the corresponding changes on their algebraic forms

Key Stage : 4

Learning Unit : Functions and Graphs

Materials required :
1. Spreadsheet software such as *Microsoft Excel* (*graph.xls*), *Graphmatica* and worksheets
2. Graph papers and transparencies with the graphs of functions $y = x^2$ and $y = x^3$ printed on them

Prerequisite Knowledge :
1. Understand the basic properties of the graph of $f(x) = x^2$.
2. Plot graphs

Description of the activity:

1. The teacher revises with students the relation between the graphs of $y = f(x)$ and $y = f(x) + k$ and uses examples such as $y = f(x) + 3$ to assess their understanding of the upward and downward translations of a graph.

2. The teacher asks students to guess the relation between the graphs of $y = f(x)$ and $y = f(x+3)$. The teacher distributes Worksheet 1 to students and let them plot the graph after filling in the values of *x* and *y* in the table.

3. After students complete Worksheet 1, the teacher discusses with students the differences between the graphs of $y = x^2$ and $y = (x+3)^2$ and points out that the graph of $y = (x+3)^2$ is obtained by moving the graph of $y = x^2$ 3 units to the left instead of to the right.

4. If students find difficulties in observing the leftward translation, the teacher may let students move a transparency with the graph of $y = x^2$ and make comparison to the graphs they have drawn.

5. The teacher asks students the relation between the graphs of $y = (x-3)^2$ and $y = x^2$. The students can then verify their conjectures by completing Worksheet 2.

6. After students finish Worksheet 2, the teacher discusses the answers with students and uses the software *graph.xls* to show the relation between the graphs of $y = f(x)$ and $y = f(x+h)$ for various quadratic and cubic functions $f(x)$.
   The teacher uses the software to compare the two functions on:
   (a) the differences between x and y coordinates in the table;
   (b) the differences between x and y coordinates of points in the graphs;
   (c) the differences between the algebraic expressions.

7. The teacher further uses the software *Graphmatica* to sketch the graphs of various functions such as:
   (a) $y = x^4$ and $y = (x+3)^4$,
   (b) $y = \tan x$ and $y = \tan(x + 90°)$.
   Hence, the teacher draws the conclusion that, when $h > 0$,
   (i) moving the graph of $y = f(x)$ h units to the left will give the graph of $y = f(x+h)$;
   (ii) moving the graph of $y = f(x)$ h units to the right will give the graph of $y = f(x-h)$.

8. The teacher asks students to write down the algebraic forms of functions from given geometrical transformations for Questions 1 to 4 in Worksheet 3. The teacher uses Questions 5 and 6 to consolidate:
   (a) write the geometrical changes of functions for the given change of algebraic expressions.
   (b) write the change on algebraic forms of functions for the given geometrical changes,

9. After students finish the worksheets, the teacher may use the following games to consolidate students' concept on the effects of transformation (translation) on the algebraic and graphical representations of functions:

(a) the teacher uses the transparency with the graphs of $y = x^2$ in the Cartesian plane printed on it;

(b) the teacher then uses another transparency with the graph of $y = x^2$ and translates it to different locations and let students find out the algebraic form of the new graphs in groups and write them on the blackboard;

(c) similarly, the teacher writes algebraic forms of different translations on functions $y = x^2$ or $y = x^3$ on the blackboard and let students place the transparency on the coordinates plane to show the graphs of images;

(d) the group getting a correct answer will score two marks. 1 mark will be deducted for a wrong answer. The first group getting 10 marks will be the winner.

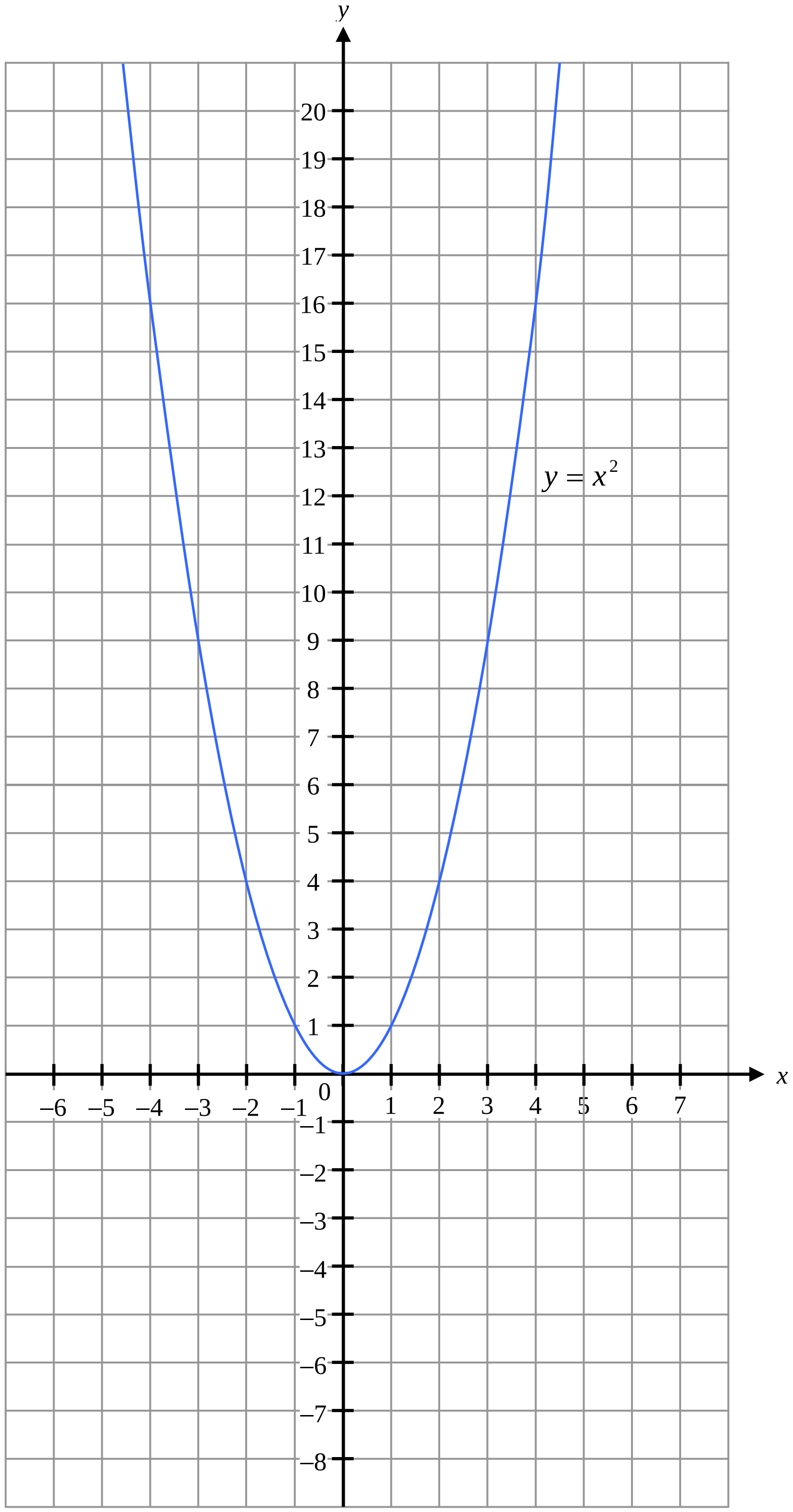
10. When students get familiar with the effects on algebraic forms of horizontal translation on functions, the teacher may integrate vertical translation with horizontal translation in the activities and let students write down the corresponding algebraic forms of functions after two kinds of translations, for example, a horizontal translation after a vertical translation. The procedures in conducting the activities are similar to those in Question 9. The teacher then helps students to consolidate the relation between the changes of the graphs and their algebraic forms through Worksheet 4.

Worksheet 1

1. Complete the following table.

x	C_1 $y = x^2$	C_2 $y = (x + 3)^2$
–4	16	1
–3	9	
–2	4	
–1	1	4
0	0	
1	1	
2	4	25
3	9	36
4	16	49

2. Sketch the graph of C_2 on the graph paper on the right.

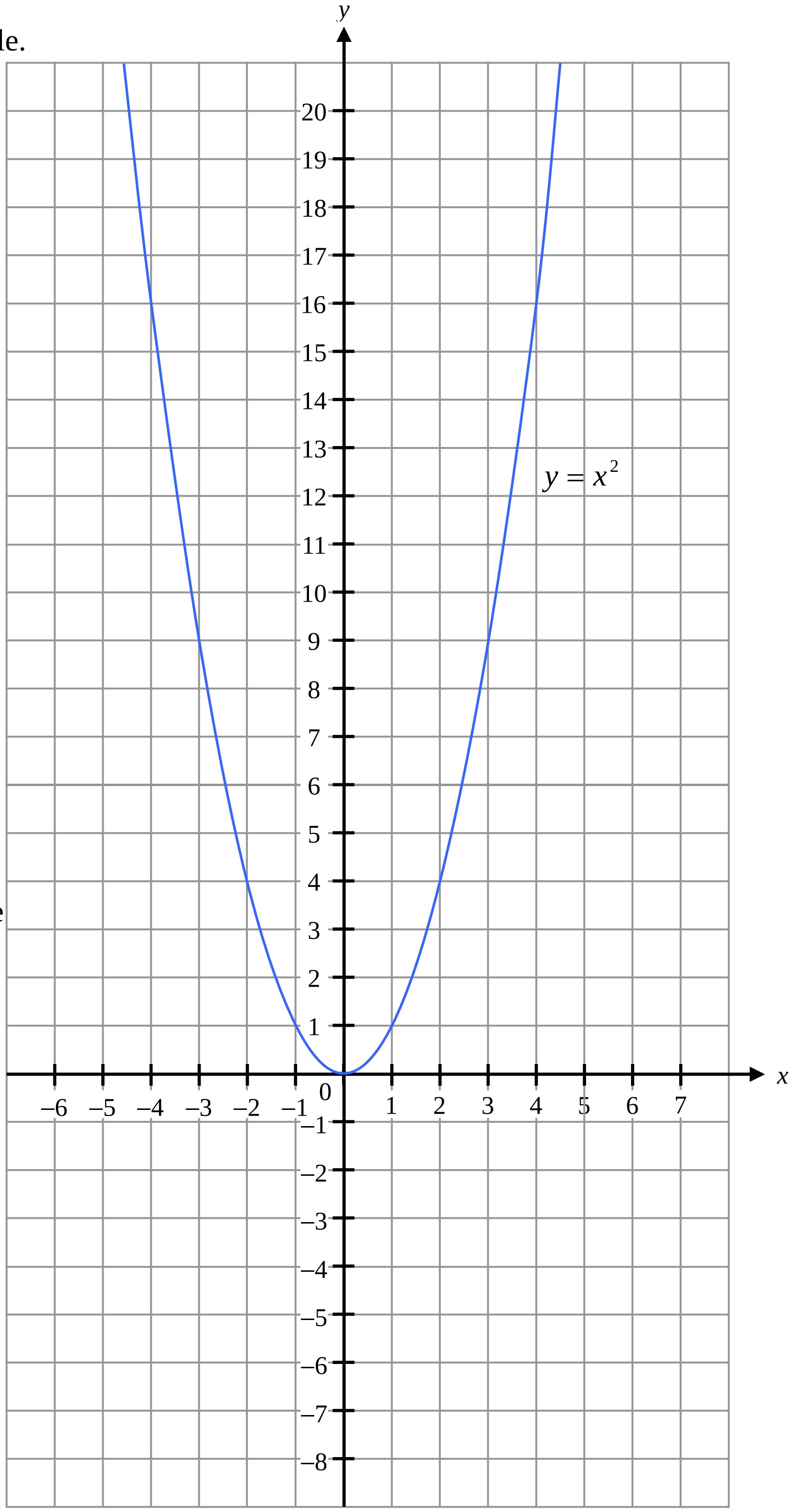


Worksheet 2

1. Complete the following table.

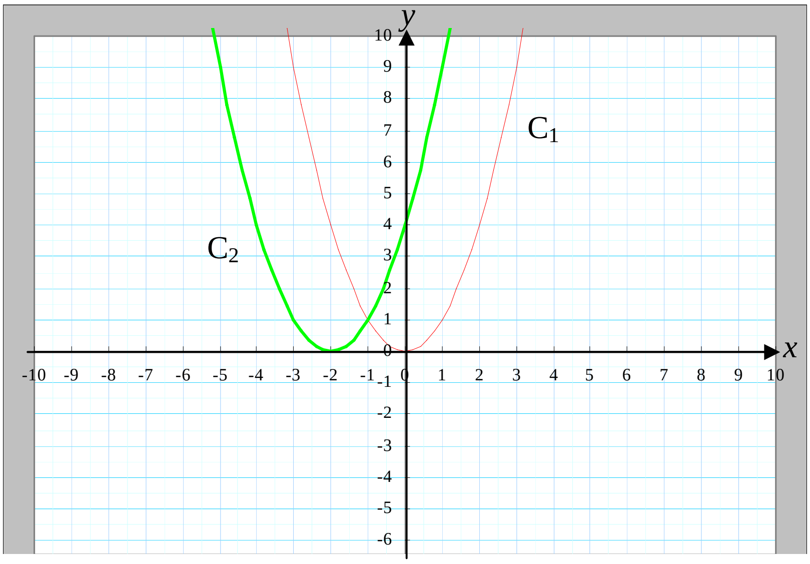
x	C_1 $y = x^2$	C_3 $y = (x - 2)^2$
–4	16	36
–3	9	
–2	4	
–1	1	9
0	0	
1	1	
2	4	0
3	9	
4	16	4

2. Sketch the graph of C_3 on the graph paper on the right.



Worksheet 3

1.

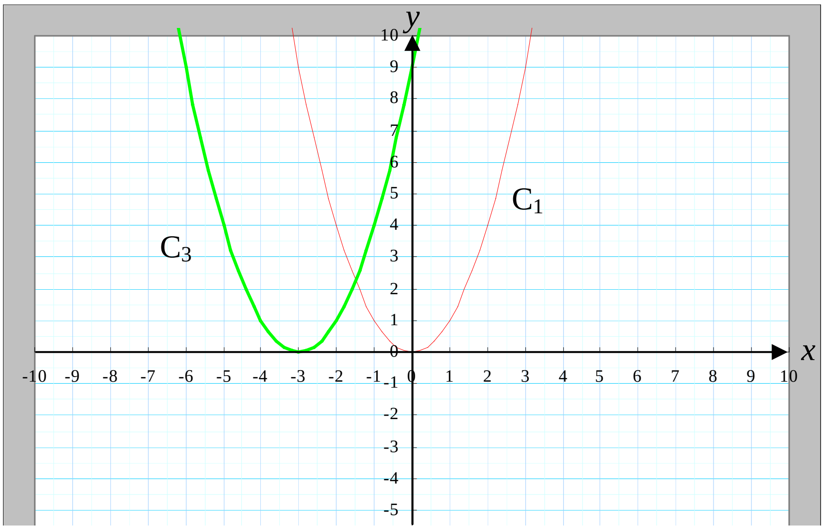


The above figure shows the graph of quadratic functions C_1: $y = x^2$ and C_2.

(a) C_2 is the image of C_1 after moving C_1________ units to the left / right *.

(b) The equation of C_2 is ______________________.

2.



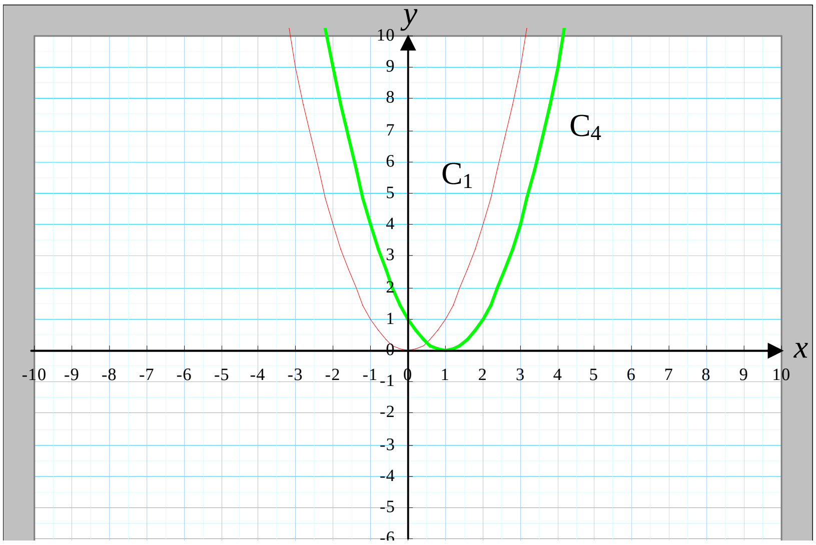
The above figure shows the graph of quadratic functions C_1: $y = x^2$ and C_3.

(a) C_3 is the image of C_1 after moving C_1________ units to the left / right *.

(b) The equation of C_3 is ______________________.

* Circle the appropriate one.

3.

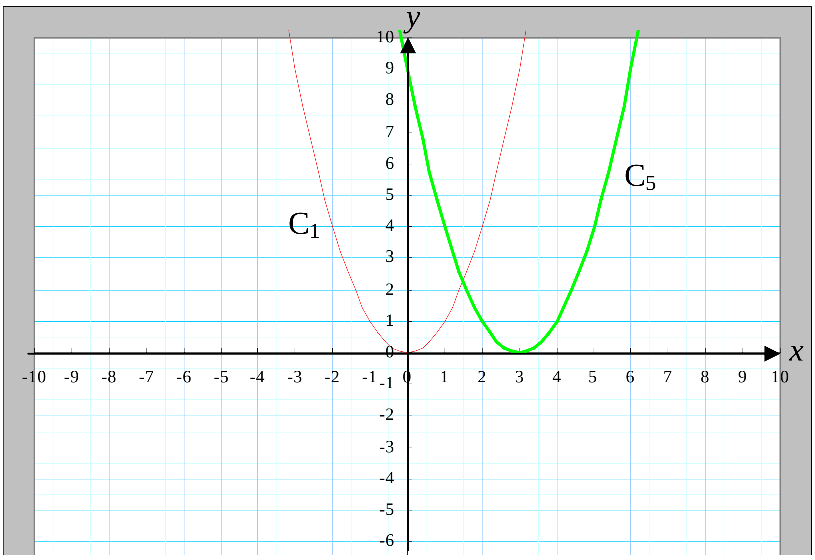


The above figure shows the graph of quadratic functions C_1: $y = x^2$ and C_4.

(a) C_4 is the image of C_1 after moving C_1 ________ units to the left / right *.

(b) The equation of C_4 is ______________________.

4.



The above figure shows the graph of quadratic functions C_1: $y = x^2$ and C_5.

(a) C_5 is the image of C_1 after moving C_1 ________ units to the left / right *.

(b) The equation of C_5 is ______________________.

* Circle the appropriate one.

5. Write down the corresponding changes on the graph after the following transformations.

	Original equations C_1	Horizontal translation	New equations C_2
(a)	$y = x^2$	Move _____ units to the right.	$y = (x-3)^2$
(b)	$y = x^2 + 1$	Move _____ units to the right.	$y = (x-7)^2 + 1$
(c)	$y = x^2 + x$	Move _____ units to the right.	$y = (x-2)^2 + (x-2)$
(d)	$y = x^2 - x$	Move _____ units to the right.	$y = (x-1)^2 - (x-1)$
(e)	$y = x^2 + x + 3$	Move _____ units to the right.	$y = (x-5)^2 + (x-5) + 3$
(f)	$y = x^3$	Move _____ units to the right.	$y = (x-9)^3$
(g)	$y = x^3 + 7$	Move _____ units to the right.	$y = (x-12)^3 + 7$
(h)	$y = x^2$	Move _____ units to the left.	$y = (x+5)^2$
(i)	$y = x^2 + 1$	Move _____ units to the left.	$y = (x+7)^2 + 1$
(j)	$y = x^2 + x$	Move _____ units to the left.	$y = (x+1)^2 + (x+1)$
(k)	$y = 3x^2 + 2x$	Move _____ units to the left.	$y = 3(x+5)^2 + 2(x+5)$
(l)	$y = x^3$	Move _____ units to the left.	$y = (x+2)^3$
(m)	$y = f(x)$	Move _____ units to the left / right*.	$y = f(x-h)$
(n)	$y = f(x)$	Move _____ units to the left / right*.	$y = f(x+h)$
(o)	$y = x^2$	Move _____ units to the left / right*.	$y = (x-12)^2$
(p)	$y = x^2$	Move _____ units to the left / right*.	$y = (x+2)^2$
(q)	$y = x^2 + 3x$	Move _____ units to the left / right*.	$y = (x-5)^2 + 3(x-5)$
(r)	$y = x^2 + 3x$	Move _____ units to the left / right*.	$y = (x+3)^2 + 3(x+3)$
(s)	$y = x^2 - 5x - 7$	Move _____ units to the left / right*.	$y = (x+2)^2 - 5(x+2) - 7$
(t)	$y = x^3 + 4x^2 + 1$	Move _____ units to the left / right*.	$y = (x-7)^3 + 4(x-7)^2 + 1$

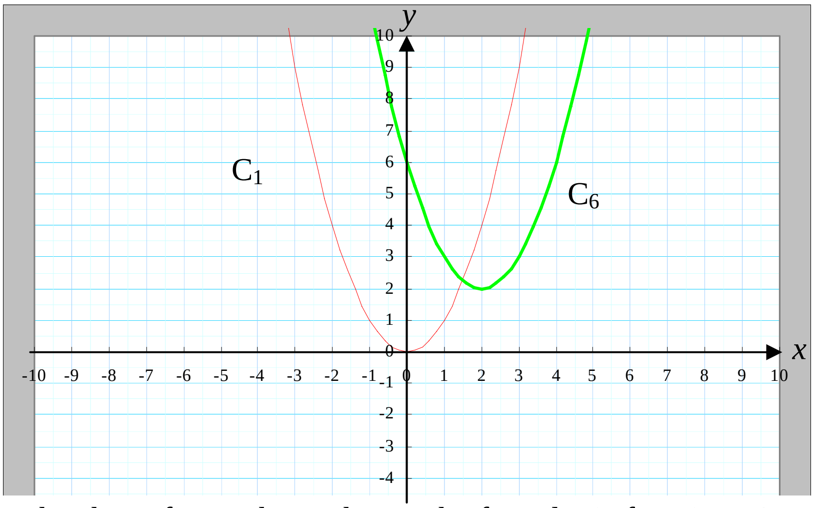
* Circle the appropriate one.

6. Write down the corresponding algebraic forms of the functions after the following transformation.

	Original equations C_1	Horizontal translation	New equations C_2
(a)	$y = x^2$	Move 8 units to the right.	
(b)	$y = x^3$	Move 5 units to the right.	
(c)	$y = x^2 + 1$	Move 5 units to the right.	
(d)	$y = x^2 - x$	Move 3 units to the right.	
(e)	$y = x^2 + 3x + 2$	Move 7 units to the right.	
(f)	$y = x^3 + 3x^2 + 2x - 5$	Move 7 units to the right.	
(g)	$y = x^2$	Move 4 units to the left.	
(h)	$y = x^2 + 1$	Move 2 units to the left.	
(i)	$y = x^3 + x$	Move 4 units to the left.	
(j)	$y = x^2 + 3x + 2$	Move 5 units to the left.	
(k)	$y = x^3 - x^2 + x + 1$	Move 7 units to the left.	
(l)	$y = f(x)$	Move h units to the right.	
(m)	$y = f(x)$	Move h units to the left.	
(n)	$y = x^4$	Move 6 units to the left.	
(o)	$y = x^2 + 5x$	Move 4 units to the right.	
(p)	$y = x^3 + 2x$	Move 3 units to the left.	

Worksheet 4

1.

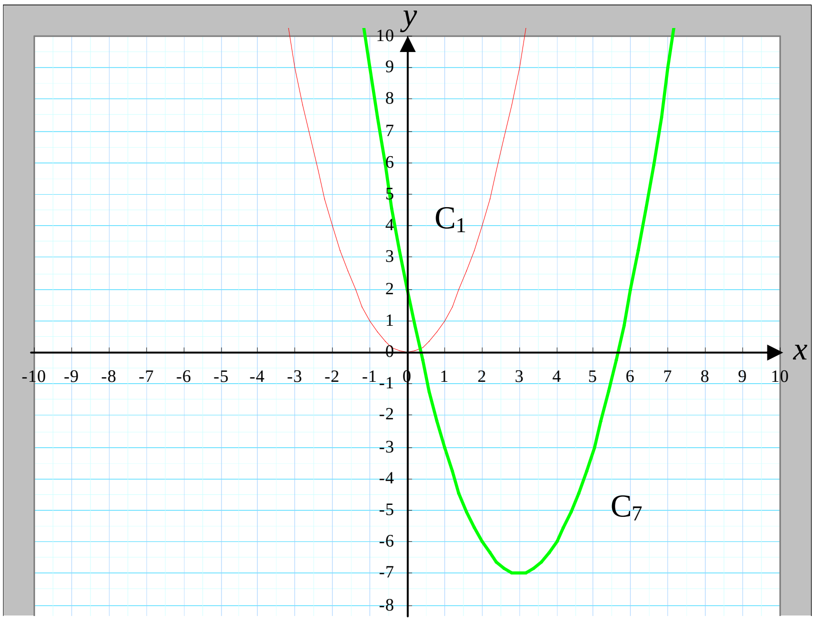


The above figure shows the graph of quadratic functions C_1: $y = x^2$ and C_6.

(a) C_6 is the image of C_1 after moving C_1________ units to the left / right *
and________ units upwards / downwards *.

(b) The equation of C_6 is ______________________.

2.



The above figure shows the graph of quadratic functions C_1: $y = x^2$ and C_7.

(a) C_7 is the image of C_1 after moving C_1________ units to the left / right *
and________ units upwards / downwards *.

(b) The equation of C_7 is ______________________.

* Circle the appropriate one.

3. Write down the corresponding changes after the following transformations.

	Original equations C_1	Horizontal translation	Vertical translation	New equations C_2
(a)	$y = x^2$	Move ______ units to the left / right *.	Move ______ units upwards / downwards *.	$y = (x-3)^2 - 4$
(b)	$y = x^2$	Move ______ units to the left / right *.	Move ______ units upwards / downwards *.	$y = (x+3)^2 + 4$
(c)	$y = x^3$	Move ______ units to the left / right *.	Move ______ units upwards / downwards *.	$y = (x-4)^3 + 3$
(d)	$y = x^3$	Move ______ units to the left / right *.	Move ______ units upwards / downwards *.	$y = (x+6)^3 + 4$
(e)	$y = x^3$	Move ______ units to the left / right *.	Move ______ units upwards / downwards *.	$y = (x-12)^3 - 5$
(f)	$y = f(x)$	Move ______ units to the left / right *.	Move ______ units upwards / downwards *.	$y = f(x+h) + k$ if $h > 0$, $k > 0$
(g)	$y = f(x)$	Move ______ units to the left / right *.	Move ______ units upwards / downwards *.	$y = f(x+h) - k$ if $h > 0$, $k > 0$

* Circle the appropriate one.

4. Write down the corresponding equations after the following transformation.

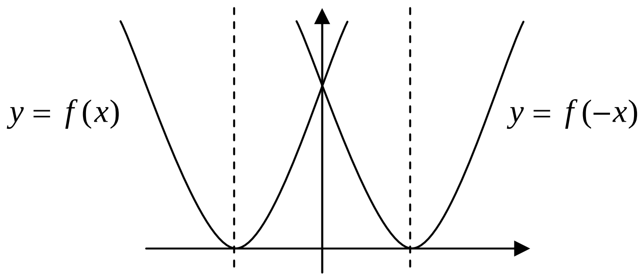
	Original equations C_1	Horizontal translation	Vertical translation	New equations C_2
(a)	$y = x^2$	Move 10 units to the right.	Move 1 unit upwards.	
(b)	$y = x^2$	Move 8 units to the right.	Move 2 units downwards.	
(c)	$y = x^2$	Move 1 unit to the left.	Move 7 units upwards.	
(d)	$y = x^2$	Move 12 units to the left.	Move 1 unit downwards.	
(e)	$y = x^2 + x - 1$	Move 10 units to the right.	Move 20 units downwards.	
(f)	$y = x^3$	Move 1 unit to the left.	Move 3 units upwards.	
(g)	$y = x^3$	Move 3 units to the right.	Move 1 unit downwards.	
(h)	$y = x^3 - 4x + 3$	Move 10 units to the left.	Move 20 units downwards.	
(i)	$y = f(x)$	Move h units to the left.	Move k units upwards.	
(j)	$y = f(x)$	Move h units to the right.	Move k units downwards.	
(k)	$y = f(x)$	Move h units to the right.	Move k units upwards.	

Notes for Teachers:

1. The time required for this activity is about 40–50 minutes.

2. When students study the horizontal translation, they always incorrectly consider the graph of $y = f(x+h)$ as moving the graph $y = f(x)$ to the right if $h > 0$ and to the left if $h < 0$. This misconception is related to the concept of the number line in which numbers on the right are greater than numbers on the left. The teacher may use the tabular values (Worksheet 1) to show the relation between the horizontal translation and the algebraic form of a function. The teacher may explain that, a smaller value of x is required for $y = f(x+h)$ to give the same y value as in $y = f(x)$ to explain the movement of the graph to the left.

3. When the teacher uses some quadratic functions such as translating $y = (x-1)^2$ to $y = (x+1)^2$, students may sometimes consider them as a reflection to each other. In fact, the expansion of $y = [(-x)+1]^2$ is the same as $y = (x-1)^2$.



Therefore, the teacher should choose suitable quadratic functions and the corresponding transformations as examples. It is more appropriate to use cubic functions than quadratic functions to discuss the horizontal translation of functions.

4. The teacher may draw students' attention on the effects of translations on linear functions. It is very difficult to distinguish between graphs of linear functions after horizontal and vertical translations i.e. the difference between $f(x)+k$ and $f(x+k)$ is difficult to be identified. It is also difficult to distinguish between graphs of polynomial functions (including linear and quadratic functions) after horizontal and vertical compression or stretching, i.e. the differences between the graphs of $f(kx)$ and $kf(x)$ are difficult to be identified.

5. The teacher may also use daily life examples (see Exemplar 11) to help students master the concept of horizontal translations.

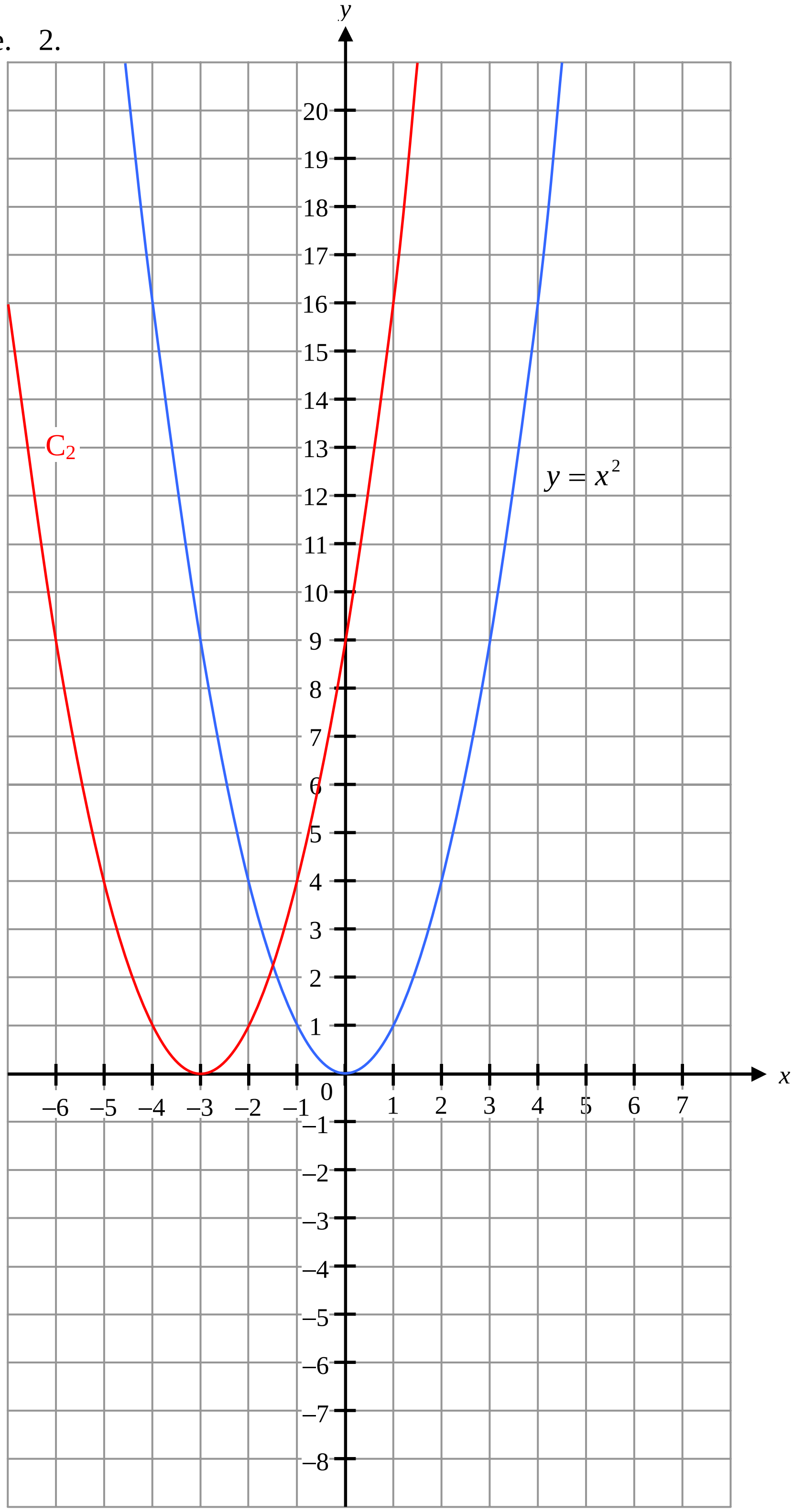
6. Suggested solutions for worksheets.

Worksheet 1

1. Complete the following table.

x	C_1 $y = x^2$	C_2 $y = (x + 3)^2$
–4	16	1
–3	9	0
–2	4	1
–1	1	4
0	0	9
1	1	16
2	4	25
3	9	36
4	16	49

2.



Note:
Some students do not find the corresponding y–coordinates of $x = -5$ and $x = -6$. The corresponding part of the curve will be missing. Thus, students may not visualize the new graph is the translation of the graph $y = x^2$.

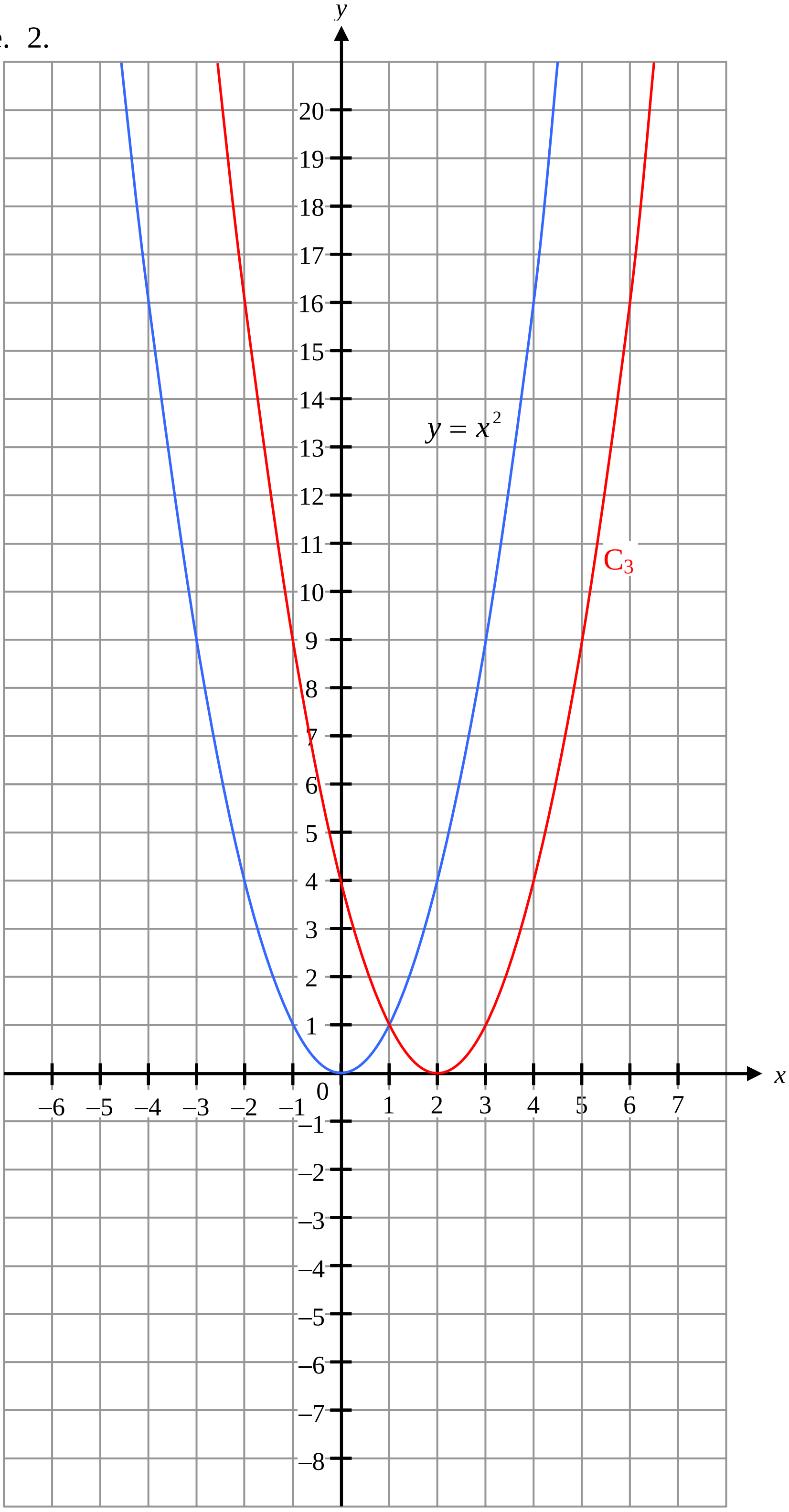
Worksheet 2

1. Complete the following table.

x	C_1 $y = x^2$	C_3 $y = (x-2)^2$
–4	16	36
–3	9	25
–2	4	16
–1	1	9
0	0	4
1	1	1
2	4	0
3	9	1
4	16	4

Note:
Some students do not find the corresponding y–coordinates of $x = -5$ and $x = -6$. The corresponding part of the curve will be missing. Thus, students may not visualize the new graph is the translation of the graph $y = x^2$.

2.



Worksheet 3

1. (a) C_2 is the image of C_1 after moving C_1 2 units to the left / right.

   (b) The equation of C_2 is $y=(x+2)^2$.

2. (a) C_3 is the image of C_1 after moving C_1 3 units to the left / right.

   (b) The equation of C_3 is $y=(x+3)^2$.

3. (a) C_4 is the image of C_1 after moving C_1 1 unit to the left / right.

   (b) The equation of C_4 is $y=(x-1)^2$.

4. (a) C_5 is the image of C_1 after moving C_1 3 unit to the left / right.

   (b) The equation of C_5 is $y=(x-3)^2$.

5. Write down the corresponding changes on the graph after the following transformations.

	Original equations C_1	Horizontal translation	New equations C_2
(a)	$y = x^2$	Move 3 units to the right.	$y = (x-3)^2$
(b)	$y = x^2 + 1$	Move 7 units to the right.	$y = (x-7)^2 + 1$
(c)	$y = x^2 + x$	Move 2 units to the right.	$y = (x-2)^2 + (x-2)$
(d)	$y = x^2 - x$	Move 1 units to the right.	$y = (x-1)^2 - (x-1)$
(e)	$y = x^2 + x + 3$	Move 5 units to the right.	$y = (x-5)^2 + (x-5) + 3$
(f)	$y = x^3$	Move 9 units to the right.	$y = (x-9)^3$
(g)	$y = x^3 + 7$	Move 12 units to the right.	$y = (x-12)^3 + 7$
(h)	$y = x^2$	Move 5 units to the left.	$y = (x+5)^2$
(i)	$y = x^2 + 1$	Move 7 units to the left.	$y = (x+7)^2 + 1$
(j)	$y = x^2 + x$	Move 1 units to the left.	$y = (x+1)^2 + (x+1)$
(k)	$y = 3x^2 + 2x$	Move 5 units to the left.	$y = 3(x+5)^2 + 2(x+5)$
(l)	$y = x^3$	Move 2 units to the left.	$y = (x+2)^3$
(m)	$y = f(x)$	Move h units to the left / right*.	$y = f(x-h)$
(n)	$y = f(x)$	Move h units to the left / right*.	$y = f(x+h)$
(o)	$y = x^2$	Move 12 units to the left / right*.	$y = (x-12)^2$
(p)	$y = x^2$	Move 2 units to the left / right*.	$y = (x+2)^2$
(q)	$y = x^2 + 3x$	Move 5 units to the left / right*.	$y = (x-5)^2 + 3(x-5)$
(r)	$y = x^2 + 3x$	Move 3 units to the left / right*.	$y = (x+3)^2 + 3(x+3)$
(s)	$y = x^2 - 5x - 7$	Move 2 units to the left / right*.	$y = (x+2)^2 - 5(x+2) - 7$
(t)	$y = x^3 + 4x^2 + 1$	Move 7 units to the left / right*.	$y = (x-7)^3 + 4(x-7)^2 + 1$

6. Write down the corresponding algebraic forms of the functions after the following transformations.

	Original equations C_1	Horizontal translation	New equations C_2
(a)	$y = x^2$	Move 8 units to the right.	$y = (x-8)^2$
(b)	$y = x^3$	Move 5 units to the right.	$y = (x-5)^3$
(c)	$y = x^2 + 1$	Move 5 units to the right.	$y = (x-5)^2 + 1$
(d)	$y = x^2 - x$	Move 3 units to the right.	$y = (x-3)^2 - (x-3)$
(e)	$y = x^2 + 3x + 2$	Move 7 units to the right.	$y = (x-7)^2 + 3(x-7) + 2$
(f)	$y = x^3 + 3x^2 + 2x - 5$	Move 7 units to the right.	$y = (x-7)^3 + 3(x-7)^2 + 2(x-7) - 5$
(g)	$y = x^2$	Move 4 units to the left.	$y = (x+4)^2$
(h)	$y = x^2 + 1$	Move 2 units to the left.	$y = (x+2)^2 + 1$
(i)	$y = x^3 + x$	Move 4 units to the left.	$y = (x+4)^3 + (x+4)$
(j)	$y = x^2 + 3x + 2$	Move 5 units to the left.	$y = (x+5)^2 + 3(x+5) + 2$
(k)	$y = x^3 - x^2 + x + 1$	Move 7 units to the left.	$y = (x+7)^3 - (x+7)^2 + (x+7) + 1$
(l)	$y = f(x)$	Move *h* units to the right.	$y = f(x-h)$
(m)	$y = f(x)$	Move *h* units to the left.	$y = f(x+h)$
(n)	$y = x^4$	Move 6 units to the left.	$y = (x+6)^4$
(o)	$y = x^2 + 5x$	Move 4 units to the right.	$y = (x-4)^2 + 5(x-4)$
(p)	$y = x^3 + 2x$	Move 3 units to the left.	$y = (x+3)^3 + 2(x+3)$

Worksheet 4

1. (a) C_6 is the image of C_1 after moving C_1 2 units to the left / right and 2 units upwards / downwards.

(b) The equation of C_6 is $y=(x-2)^2+2$.

2. (a) C_7 is the image of C_1 after moving C_1 3 units to the left / right and 7 units upwards / downwards.

(b) The equation of C_6 is $y=(x-3)^2-7$.

3. Write down the corresponding changes after the following transformations.

	Original equations C_1	Horizontal translation	Vertical translation	New equations C_2
(a)	$y=x^2$	Move 3 units to the left / right*.	Move 4 units upwards / downwards*.	$y=(x-3)^2-4$
(b)	$y=x^2$	Move 3 units to the left / right*.	Move 4 units upwards / downwards*.	$y=(x+3)^2+4$
(c)	$y=x^3$	Move 4 units to the left / right*.	Move 3 units upwards / downwards*.	$y=(x-4)^3+3$
(d)	$y=x^3$	Move 6 units to the left / right*.	Move 4 units upwards / downwards*.	$y=(x+6)^3+4$
(e)	$y=x^3$	Move 12 units to the left / right*.	Move 5 units upwards / downwards*.	$y=(x-12)^3-5$
(f)	$y=f(x)$	Move h units to the left / right*.	Move k units upwards / downwards*.	$y=f(x+h)+k$ if $h>0$, $k>0$
(g)	$y=f(x)$	Move h units to the left / right*.	Move k units upwards / downwards*.	$y=f(x+h)-k$ if $h>0$, $k>0$

4. Write down the corresponding equations after the following transformation.

	Original Equations C_1	Horizontal translation	Vertical translation	New equations C_2
(a)	$y = x^2$	Move 10 units to the right.	Move 1 unit upwards.	$y = (x-10)^2 + 1$
(b)	$y = x^2$	Move 8 units to the right.	Move 2 units downwards.	$y = (x-8)^2 - 2$
(c)	$y = x^2$	Move 1 unit to the left.	Move 7 units upwards.	$y = (x+1)^2 + 7$
(d)	$y = x^2$	Move 12 units to the left.	Move 1 unit downwards.	$y = (x+12)^2 - 1$
(e)	$y = x^2 + x - 1$	Move 10 units to the right.	Move 20 units downwards.	$y = (x-10)^2 + (x-10) - 21$
(f)	$y = x^3$	Move 1 unit to the left.	Move 3 units upwards.	$y = (x+1)^3 + 3$
(g)	$y = x^3$	Move 3 units to the right.	Move 1 unit downwards.	$y = (x-3)^3 - 1$
(h)	$y = x^3 - 4x + 3$	Move 10 units to the left.	Move 20 units downwards.	$y = (x+10)^3 - 4(x+10) - 17$
(i)	$y = f(x)$	Move h units to the left.	Move k units upwards.	$y = f(x+h) + k$
(j)	$y = f(x)$	Move h units to the right.	Move k units downwards.	$y = f(x-h) - k$
(k)	$y = f(x)$	Move h units to the right.	Move k units upwards.	$y = f(x-h) + k$

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