有效的學與教系列：增強學生學習物理科所需數學能力的有效策略（新辦）

- 推廣物理和數學教師的協作
- 增強STEM教育

April 21, 2016. 2:00 p.m. – 5:00 p.m.
CDI - EDB

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- 推廣物理和數學教師的協作
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當敢，當敢！

攀不上這個口號。

有效的學與教系列:
增強學生學習物理
科所需數學能力的
有效策略（新辦）

❖ 推廣物理和數學教師的協作

❖ 增強STEM教育

數學與物理的和諧優美結合:
實際的課程? 夢想的課程? 能否實現的課程?

蕭文強
香港大學數學系
mathsiu@hku.hk

2015.05.09

傳統的中學課程把數學和物理視為兩個獨立學科，這個教學模式固然有一定的需要和好處，卻不能讓學生深入認識這兩門學科交互編織的關係。物理的問題帶來推動力，由此衍生不少數學的概念和方法，又返回物理上作出多方面的應用。

本講嘗試探討能否在中學課程中結合數學與物理，讓學生了解到數學在認識物理世界的過程中所擔當的角色和它的演化。

不容否認，把這兩門學科融合，在教學上產生不少問題。

例如，是否需要懂很多微積分? 低年級時怎麼辦? 數學與物理的基本分別是什麼？

目前高中課程上的課程安排及選科設置，更為此添加額外的困難。

對於數學與物理的關係，你有何看法？

物理 Sir： 物理與數學當然有關係，數學是學習物理有用的工具。

數學 Miss： 數學與物理當然有關係，物理提供很多應用的例子。

是否僅是如此而已？

數學＝物理？

數學≠物理？

Suppose the temperature on a rectangular slab of metal is given by

$$T(x,y) = k\,(x^2 + y^2)$$

where k is a constant,

What is $T(r,\theta)$?

a physicist's answer:

$$T(r,\theta)=kr^2$$

a mathematician's answer :

$$T(r,\theta)=k\,(r^2 + \theta^2)$$

Vector Calculus Bridge Project

Tevian Dray and Corinne Manogue
Oregon State University
http://math.oregonstate.edu/bridge/

707 (1957)

$\cos(\pi/4)$
$= 0.7071067\ldots$

717

727

737

747

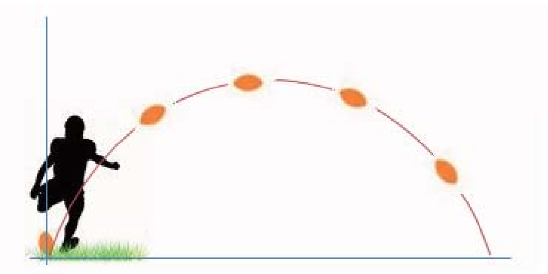
757

767

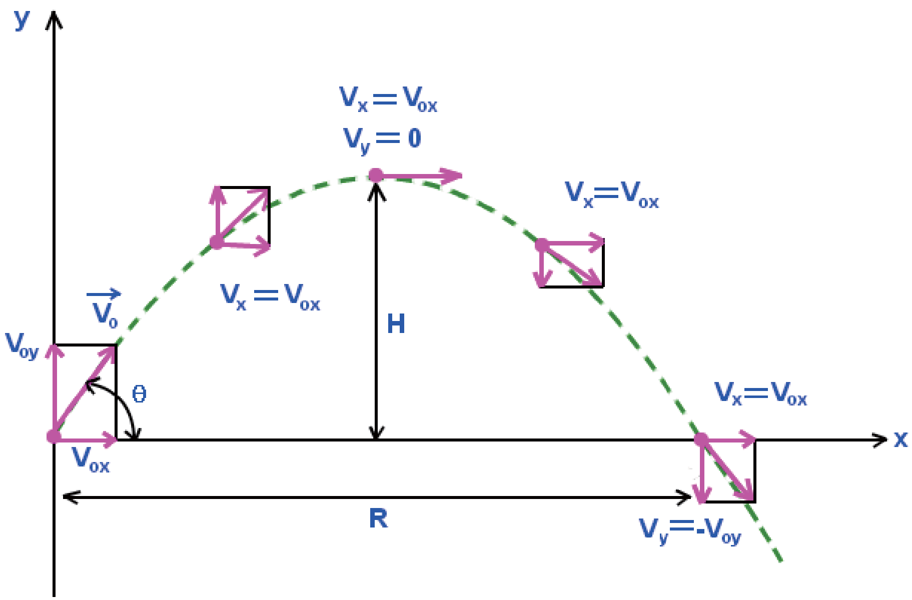
777

787

Boeing Airplanes



Projectile 拋體運動



http://ggbtu.be/m1082291

問題：

你認為以下的座標系統，那一個為較佳？地心系統？抑或日心系統？木(星)心系統又如何？

怎樣判斷那一個座標系統為較佳?

教學就是說故事，說一個能引起好奇、激發想像力的好故事，說一個關於人類如何運用智慧不斷探索及理解周圍客觀世界的故事。……

M. K. Siu, "*Zhi yì xíng nán* (knowing is easy and doing is difficult)" or *vice versa*? ----- A Chinese mathematician's observation on HPM (History and Pedagogy of Mathematics) activities, in *The First Sourcebook on Asian Research in Mathematics Education: China, Korea, Singapore, Japan, Malaysia and India*, (Eds.) B. Sriraman, J. Cai, K. Lee, L. Fan, Y. Shimuzu, C. Lim, & K. Subramaniam, Information Age Publishing, 2015 [Chinese translation by F.K. Siu]

從希臘哲學到現代物理學的整個科學史中，不斷有人力圖把表面上極為複雜的自然現象歸結為幾個簡單的基本觀念和關係，這就是整個自然哲學的基本原理。

Albert Einstein, Leopold Infeld, *The Evolution of Physics : The Growth of Ideas from Early Concepts to Relativity and Quanta* (1938)

Curiosity
Imagination

Disciplined and Critical Thinking

(precision in **mathematics** as well as in **words**)

"Li, Qi And Shu: An Introduction to Science and Civilization in China" by Ho Peng Yoke (1985)

The Concepts of *Yin* 陰 and *Yang* 陽 and *Wuxing* 五行

Qi can exist in two different states. It can be at rest or in motion; and it can contract or expand, giving rise to the two states, *yin* and *yang*. The two words *yin* and *yang* originated from the ideas of darkness and brightness. *Yin* conveys the idea of coldness, clouds, rain, anything feminine, what is inside and dark, the shady part of a mountain or a valley, and so on; while *yang* conveys the opposite idea of warmth, a clear sky, sunshine, anything masculine, what is outside and bright, the sunny part of a mountain or a valley, and so on.[13] *Yijing* (Book of Changes) says: 'One *yin* and one *yang*; that is the *Dao*. (*yi yin yi yang zhi wei dao* 一陰一陽之謂道)'[14], meaning that there are only two components of *qi* operating in nature, one *yin* and one *yang*, each of which dominating over the other successively in a wave-like motion. This can be best illustrated by the *Taijitu* 太極圖 diagram (Fig. 4). Half of the diagram is *yin* and the other half is *yang*. If we imagine the figure of the *Taijitu* rotating about its centre we can see how *yin* and *yang* take over from each other successively in a wave-like action. Hence, *yin* and *yang* are both opposite and complementary to each other.[15]

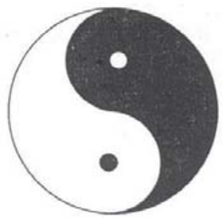


Fig. 4 *Taijitu*.

[13] See Needham, vol. 2, especially section 13 e, p. 273 ff.
[14] See *Yijing xici shang* 繫辭上, p. 3b.
[15] The *taijitu* diagram is not as ancient as its name implies. It is similar to that found in some Taoist texts. See Feng Yulan, pp. 820–4.

很古老的陰陽五行說

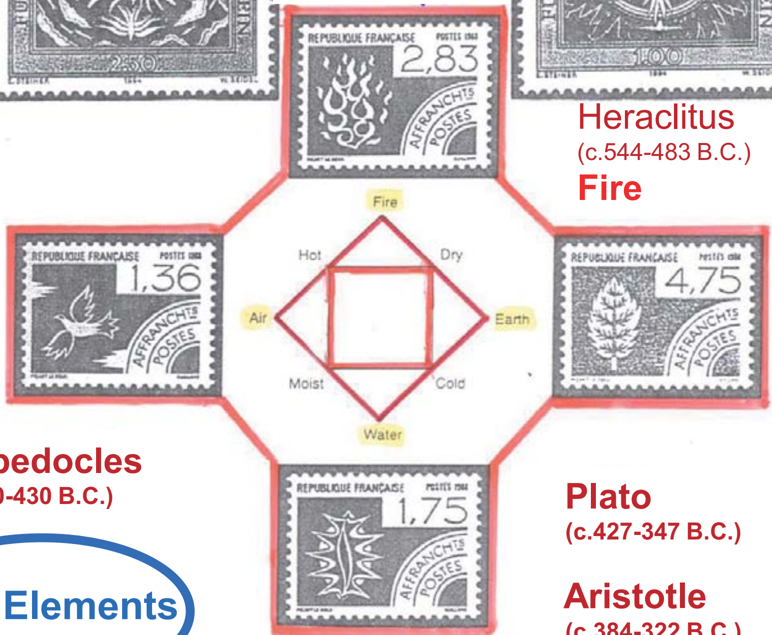
metal wood water fire earth
金 木 水 火 土

Five Phases

Thales
(c.625-547 B.C.)
Water

Anaximenes
(c.585-525 B.C.)
Air

Heraclitus
(c.544-483 B.C.)
Fire

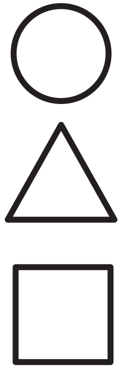


Empedocles
(c.490-430 B.C.)

Plato
(c.427-347 B.C.)

Aristotle
(c.384-322 B,C.)

Four Elements



水 (water)

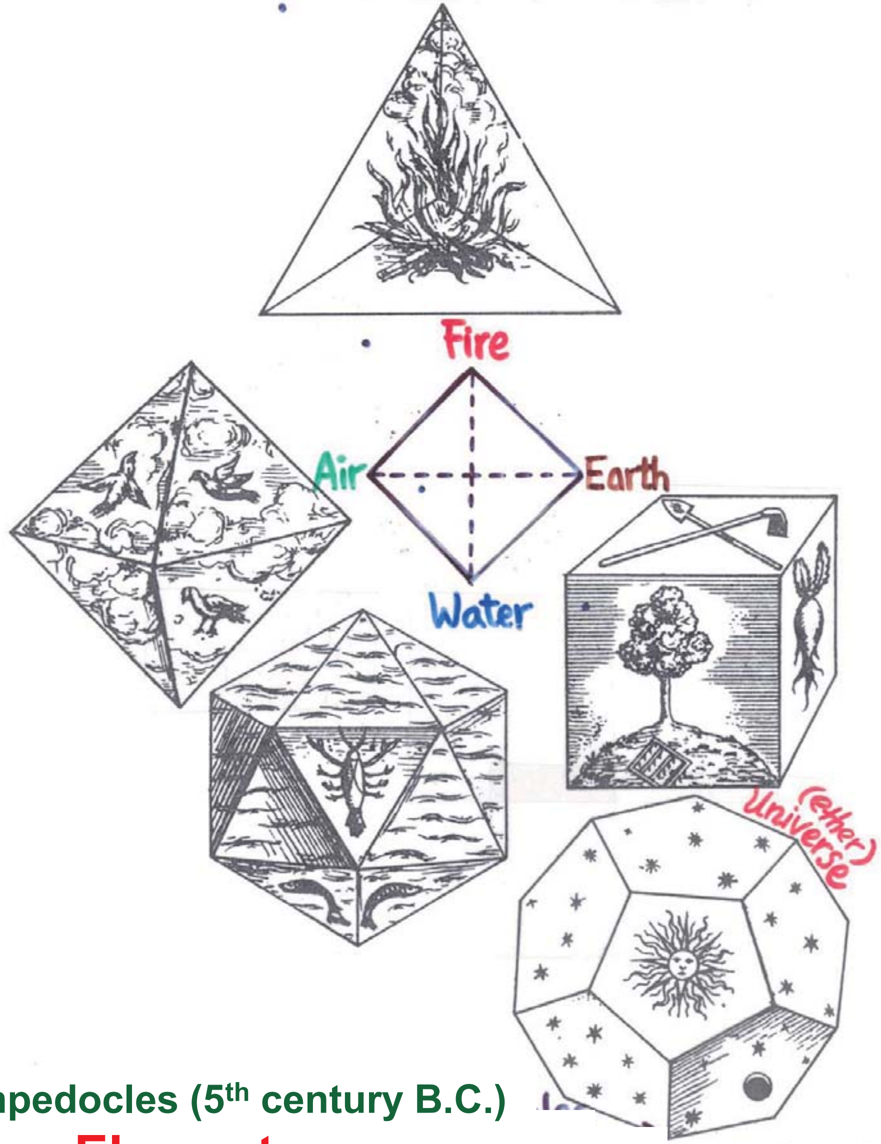
火 (fire)

地 (earth)

風 (air)

禪宗四大思想
Four great thoughts of Chan [Zen]

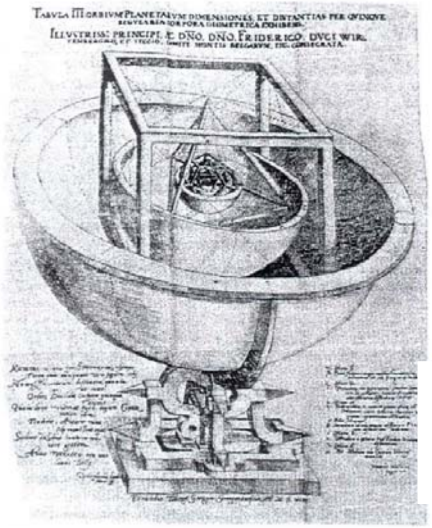
Circle-Triangle-Square Courtyard, Kenninji Temple (建仁寺), Kyoto, Japan .

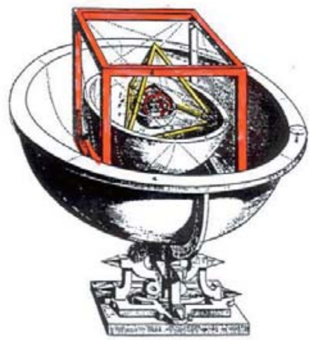


Empedocles (5th century B.C.)

Four Elements

Plato: *Timaeus*

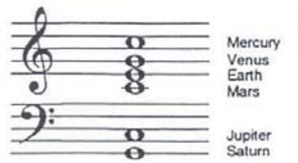




Johannes Kepler,
Mysterium Cosmographicum
(1596)

Polyhedral
Hypotesis
(1596)

Saturn ------- Sphere
Cube
Jupiter ------- Sphere
Tetrahedron
Mars --------- Sphere
Dodecahedron
Earth -------- Sphere
Icosahedron
Venus -------- Sphere
Octahedron
Mercury ------ Sphere



Planetary
Chord (1599)

Johannes Kepler,
Astronomia Nova (1609),
Harmonices Mundi (1619)

Aphorisms [Book One] LXXXII

There remains simple experience which, if taken as it comes, is called accident; if sought for, experiment. ..... But the true method of experience, on the contrary, first lights the candle, and then by means of the candle shows the way; commencing as it does with experience duly ordered and digested, not bungling or erratic, and from it educing axioms, and from established axioms again new experiments; .....

F. Bacon, *Novum Organum*, 1620

Aphorisms [Book One] XCV

Those who have handled sciences have been either men of experiment or men of dogmas. The men of experiment are like the ant, they only collect and use; the reasoners resemble spiders, who make cobwebs out of their own substance. But the bee takes a middle course; it gathers its material from the flowers of the garden and of the field, but transforms and digests it by a power of its own. .....

F. Bacon, *Novum Organum*, 1620

《孫子算經》

Sunzi Suanjing

[Master Sun's Mathematical Manual]

公元四世紀

孫子曰：夫算者，天地之經緯，群生之元首；五常之本末，陰陽之父母；星辰之建號，三光之表裏；五行之準平，四時之終始；萬物之祖宗，六藝之綱紀。稽群倫之聚散，考二氣之降升；推寒暑之迭運，步遠近之殊同；觀天道精微之兆基，察地理從橫之長短；采神祇之所在，極成敗之符驗；窮道德之理，究性命之情。立規矩，準方圓，謹法度，約尺丈，立權衡，平重輕，剖毫釐，析黍絫；歷億載而不朽，施八極而無疆。散之不可勝究，斂之不盈掌握。嚮之者富有餘，背之者貧且窶；心開者幼沖而即悟，意閉者皓首而難精。夫欲學之者必務量能揆己，志在所專。如是則焉有不成者哉。

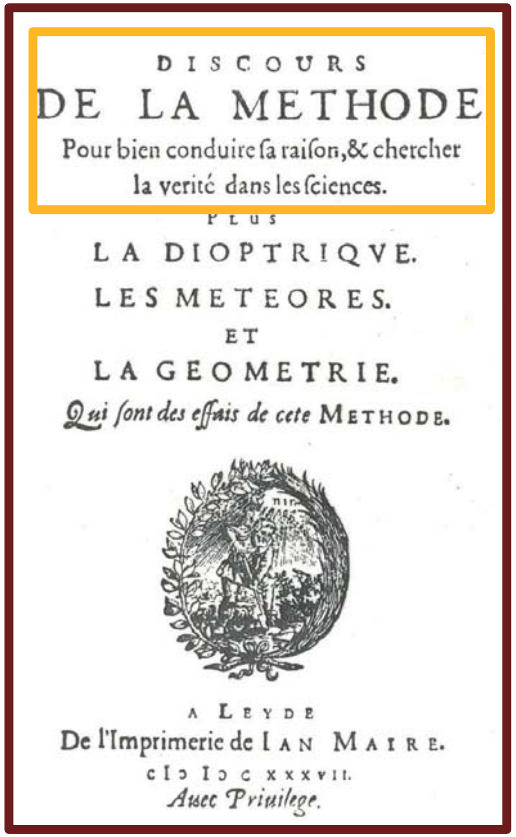
" Philosophy is written in this grand book , the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and reads the letters in which it is composed . It is written in the language of mathematics , and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth."

Il Saggiatore (The Assayer)
Letter to the Illustrious and Very Reverend Don Virginio Cesarini from Galileo Galilei (1623)

Galileo Galilei
(1564-1642)

HOW (MUCH)
rather than **WHY**?

[a quantitative rather than a qualitative description]


DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon,& chercher
la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui ſont des eſſais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
CIↃ IↃ C XXXVII.
Auec Priuilege.

René Descartes
DISCOURSE ON THE METHOD PROPERLY GUIDING THE REASON IN THE SEARCH OF TRUTH IN THE SCIENCES **(1637)**

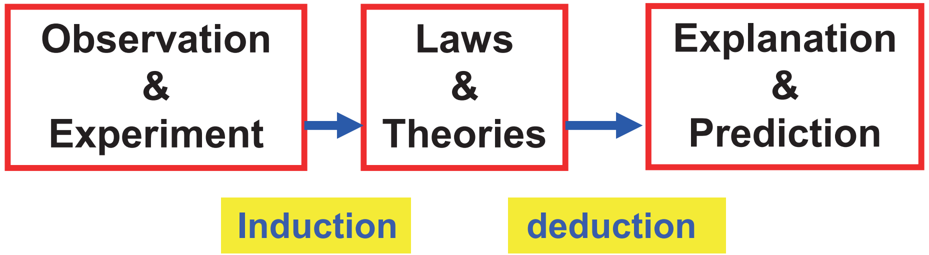
Cogito, ergo sum
(I think, therefore I am)
我思，故我在。

Unification of all sciences by reason

Method: (a) accept only what is so clear in one's mind as to exclude any doubt
(b) divide difficulties into smaller ones
(c) reason from simple to complex
(d) check that nothing is omitted

CARTESIANISM leading to
WORLD MATHEMATIZATION

Aristotle (4th century B.C.)

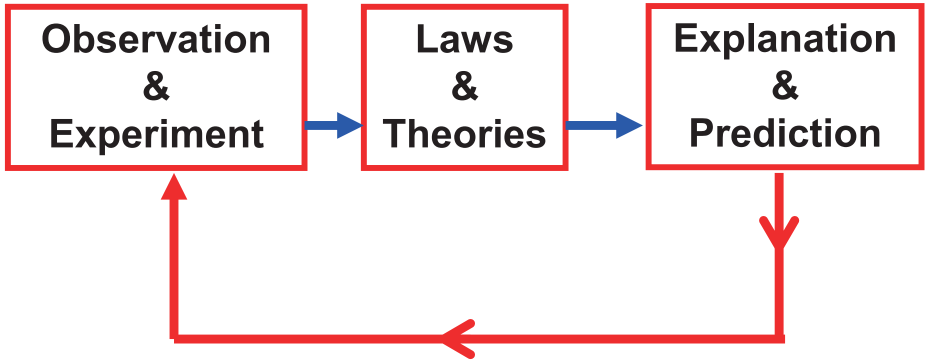


Baconian View (1620)
Novum Organum
(*New Method*)

(17th century)

Medieval Scholasticism

(12th to 15th centuries)



Galileo Galilei
Dialogo dei massimi sistemi del mondo (Dialogue Concerning the Two Chief World Systems), 1632
***Discorsi e dimonstrazioni matematiche intorno a due nuove scienze (Discourse and Mathematical Demonstrations Concerning Two New Sciences*), 1638**

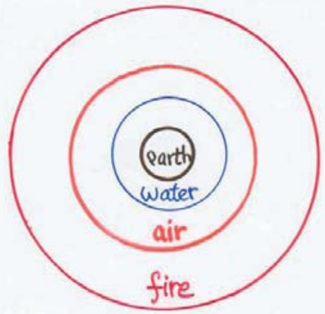
Cosmology in ancient Greece (2-sphere cosmos)

Source: Petrus Apianus, *Cosmographia* (1524)

Aristotle's physical world view (4th century B.C.)

Matter: the four elements
(Empedocles, 5th century B.C.
Plato, 5th/4th centuries B.C.)

Motion: natural motions
violent motions



Phenomena in **heaven** of a distinct nature from phenomena on **earth**.

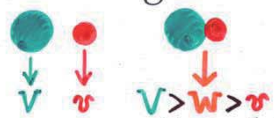
Galileo Galilei, *Discourse and Mathematical Demonstration Concerning Two New Sciences* (1638)

Thought Experiment

Salviati: But, even without further experiment, it is possible to prove clearly, by means of a short and conclusive argument, that a heavier body does not move more rapidly than a lighter one provided both bodies are of the same material and in short such as those mentioned by Aristotle. But tell me, Simplicio, whether you admit that each falling body acquires a definite speed fixed by nature, a velocity which cannot be increased or diminished except by the use of force or resistance.

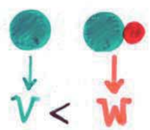
Simplicio: There can be no doubt but that one and the same body moving in a single medium has a fixed velocity which is determined by nature and which cannot be increased except by the addition of momentum or diminished except by some resistance which retards it.

Salviati: If then we take two bodies whose natural speeds are different, it is clear that on uniting the two, the more rapid one will be partly retarded by the slower, and the slower will be somewhat hastened by the swifter. Do you not agree with me in this opinion ?

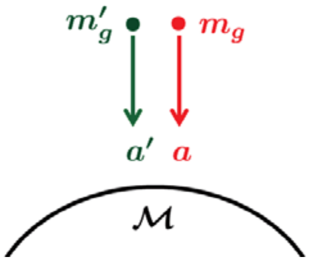


Simplicio: You are unquestionably right.

Salviati: But if this is true, and if a large stone moves with a speed of, say, eight while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. ...



Is this a truly mathematical explanation of a physical phenomenon ?



$F = m_i a$ inertial mass

$F = G m_g M / R^2$ gravitational mass

$$a = \left(\frac{m_g}{m_i}\right)\frac{GM}{R^2}$$

For m and m', the resulting a and a' are the same,

so $\dfrac{m_g}{m_i} = \dfrac{m'_g}{m'_i}$.

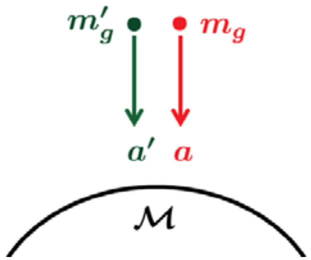
Conclusion: $\dfrac{\text{gravitational mass}}{\text{inertial mass}} = \text{constant}$

Any deeper reason behind?

(Force) = (inertial mass) × (acceleration)

(Force) = (gravitational mass) × (intensity of the gravitational field)

If **gravitational mass** is to be identified with **inertial mass**, then **acceleration** is to be identified with **intensity of the gravitation**. There is a reference system in which the intensity of the gravitation vanishes (locally)!



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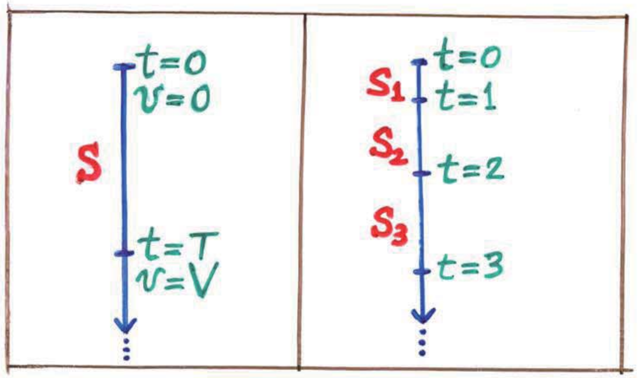
For m and m', the resulting a and a' are the same,

so $\dfrac{m_g}{m_i} = \dfrac{m'_g}{m'_i}$.

This led to the Theory of General Relativity.

Laws of Falling Bodies (in history)



Albert of Saxony (1316-1390) $V \propto S$

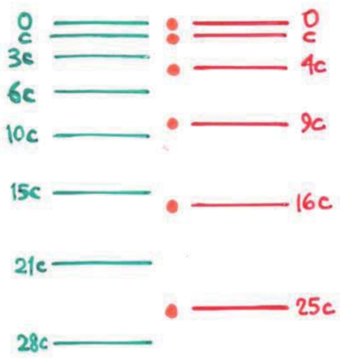
Nicole Oresme (1320-1382) $V \propto T$

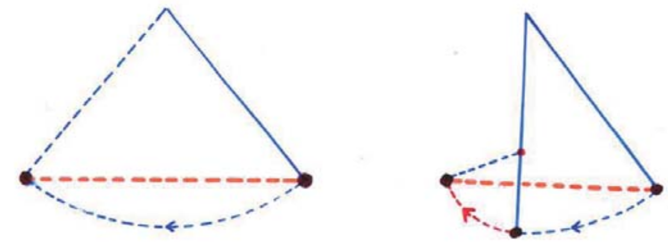
Leonardo da Vinci (1452-1519) $\frac{S_1}{1} = \frac{S_2}{2} = \frac{S_3}{3} = \frac{S_4}{4} = \cdots$

Galileo Galilei (1564-1642) $\frac{S_1}{1} = \frac{S_2}{3} = \frac{S_3}{5} = \frac{S_4}{7} = \cdots$

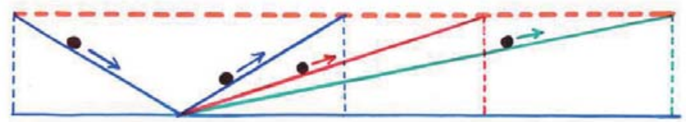
$S \propto T^2$

$V \propto T$



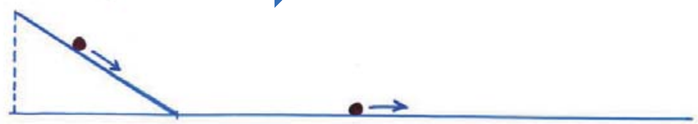


Experiment with a pendulum when the string strikes a nail



Experiment with an inclined plane

thought experiment



"Furthermore we may remark that any velocity once imparted to a moving body will be rigidly maintained as long as the external causes of acceleration or retardation are removed, a condition which is found only on horizontal planes. [...]it follows that motion along a horizontal plane is perpetual [...]" *Galileo Galilei*

Law of Inertia

Newton's First Law of Motion

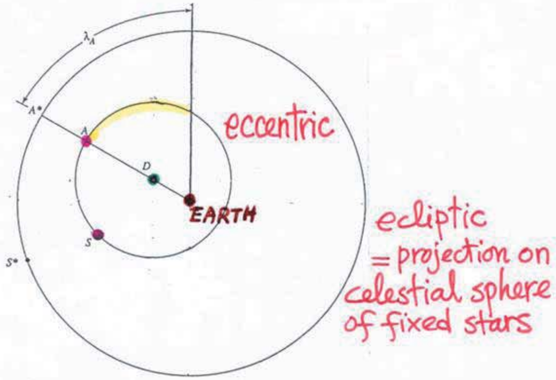
Ancient Greek astronomy

Eudoxus, Plato, Aristotle, Hipparchus (4th century to 2nd century B.C.)
[Heraclides, Aristarchus (4th century to 3rd century B.C.)] heliocentric system
Claudius Ptolemy (c. 100-165)
Almagest

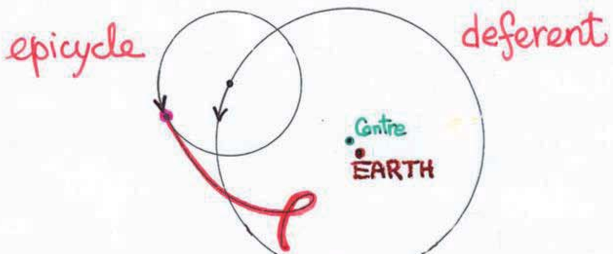
geocentric system

The earth is fixed and is at the centre of the universe. Other [observable] planets [including the sun] move around the earth with constant ~~velocity~~ speed along circular paths.

Problems: (1) different lengths of seasons



(2) retrograde motion of planets




NICOLAI COPERNICI
REVOLVTIONVM LIB. I.
De triplici motu telluris demonstratio. Cap. XI.

Nicolas Copernicus (1473-1543)

De revolutionibus orbium coelestium (1543)

(On the Revolutions of the Heavenly Spheres)

Tycho Brahe
(1546-1601)

Tycho Brahe working in his observatory at Uraniborg, Denmark.

Johannes Kepler
(1571 - 1630)

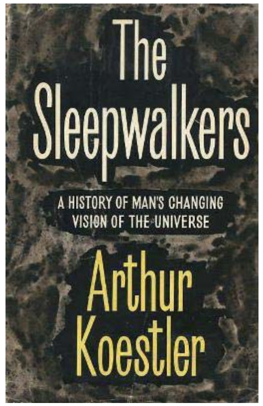
Arthur Koestler, *The Sleepwalkers : A History of Man's Changing Vision of the Universe* (1959)

Introduction by Herbert Butterfield

"No field of thought can be properly laid out by men who are merely measuring with a ruler. Sections of history are liable to be transformed — or, even where not transformed, greatly vivified — by an imagination that comes, sweeping like a searchlight, from outside the historical profession itself."

Herbert Butterfield
(1900-1979)

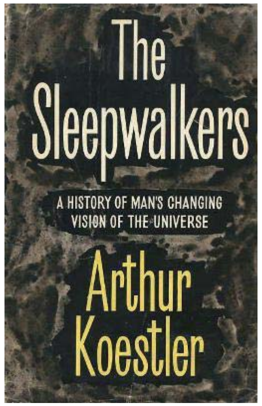
Arthur Koestler
(1905-1983)



Arthur Koestler, *The Sleepwalkers : A History of Man's Changing Vision of the Universe* (1959)

"The progress of Science is generally regarded as a kind of clean, rational advance along a straight ascending line; in fact, it has followed a zig-zag course, at times almost more bewildering than the evolution of political thought."

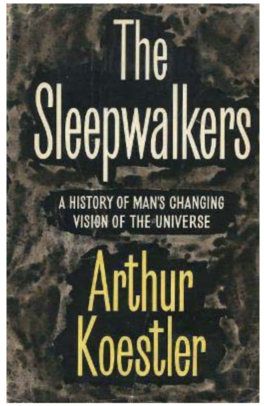
Arthur Koestler
(1905-1983)



Arthur Koestler, *The Sleepwalkers : A History of Man's Changing Vision of the Universe* (1959)

"The history of cosmic theories, in particular, may without exaggeration be called a history of collective obsessions and controlled schizophrenias, and the manner in which some of the most important individual discoveries were arrived at reminds one more of a sleepwalker's performance than an electronic brain's."

Arthur Koestler
(1905-1983)



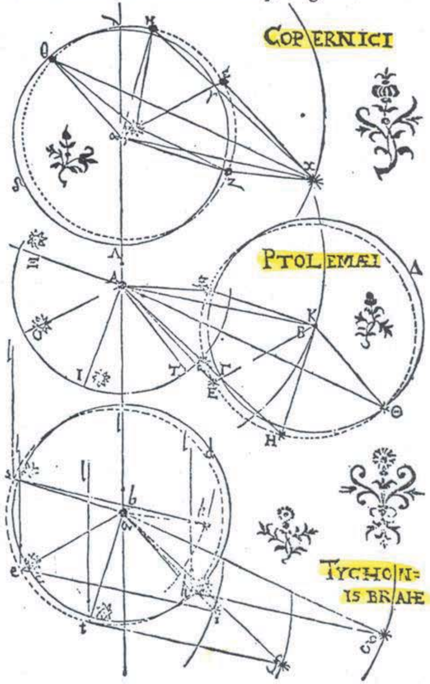
Jam poſtquam ſemel hujus rei periculum fecimus, audacia ſubvecti porro liberiores eſſe in hoc campo incipiemus. Nam conquiram tria vel quotcunque loca viſa MARTIS, Planeta ſemper eodem eccentrici loco verſante: & ex iis lege triangulorum inquiram totidem punctorum epicycli vel orbis annui diſtantias a puncto æqualitatis motus. Ac cum ex tribus punctis circulus deſcribatur, ex trinis igitur hujusmodi obſervationibus ſitum circuli, ejusque augium, quod prius ex præſuppoſito uſurpaveram, & eccentricitatem a puncto æqualitatis inquiram. Quod ſi quarta obſervatio accedet, ea erit loco probationis. CAP. XXIV.

PRIMVM tempus eſto anno MDXCX D. V Martii veſperi H. VII M. X eo quod tunc ♂ latitudine pene caruit, ne quis impertinenti ſuſpicione ob hujus implicationem in percipienda demonſtratione impediatur. Reſpondent momenta hæc, quibus ♂ ad idem fixarum punctum redit: A. MDXCII D. XXI Jan. H. VI M. XLI: A. MDXCIII D. VIII Dec. H. VI. M. XII: A. MDXCV D. XXVI Octob. H. V M. XLIV. Eſtq; longitudo Martis primo tempore ex TYCHONIS reſtitutione 1ˢ. 4°. 38′. 50″: ſequentibus temporib. toties per 1′. 36″ auctior. Hic enim eſt motus præceſſionis congruens tempori periodico unius reſtitutionis MARTIS. Cumq; TYCHO apogæum ponat in 23½ ♌, æquatio ejus erit 11°. 14′. 55″: propterea lõgitudo coæquata anno MDXC 1ˢ. 15°. 53′. 45″.

Eodem vero tempore, & commutatio ſeu differentia medii motus SOLIS a medio Martis colligitur 10ˢ. 18°. 19′. 56″: coæquata ſeu differentia inter medium SOLIS & MARTIS coæquatum eccentricum 10ˢ. 7°. 5′. 1″.

PRIMVM hæc in forma COPERNICANA ut ſimpliciori ad ſenſum proponemus.

Sit α punctum æqualitatis circuitus terræ, qui putetur eſſe circulus δγ ex α deſcriptus: & ſit Sol in partes β, ut αβ linea apogæi:



Johannes Kepler (1571-1630)

***Astronomia Nova*, 1609**

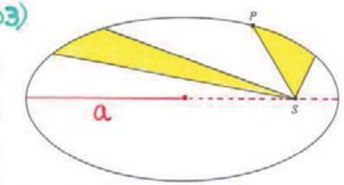
Kepler's "War" with Mars (from 1600 to 1606)

circle to oval: "The conclusion is quite simply that the planet's path is not a circle – it curves inward on both sides and outward again at opposite ends. Such a curve is called an oval. The orbit is not a circle, but an oval figure."

oval to ellipse: "The truth of Nature, which I had rejected and chased away, returned by stealth through the backdoor, disguising itself to be accepted. That is to say, I laid [the original equation] aside, and fell back on ellipses, believing that this was a quite different hypothesis, whereas the two, as I shall prove in the next chapter, are one and the same. .... Ah, what a foolish bird I have been!"

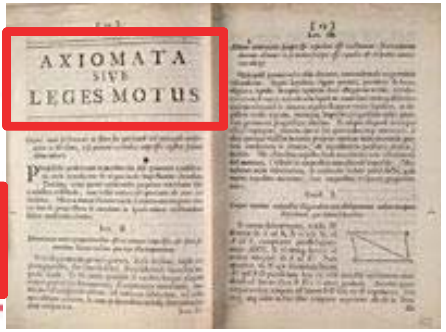
Kepler's Laws of Planetary Motion

1. Each planet moves in an elliptical orbit with the sun at one focus. (1608)
2. A given planet sweeps out equal areas in equal times. (1603)
3. The square of the period of revolution of a given planet is proportional to the cube of its average distance from the sun [the semimajor axis of its elliptical orbit.] (1618)



$$T^2 = ka^3$$

Philosophiae Naturalis Principia Mathematica
(Mathematical Principles of Natural Philosophy)
by Isaac Newton, 1687.



AXIOMS, OR LAWS OF MOTION

LAW I

Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

PROJECTILES continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

LAW II

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

LAW III

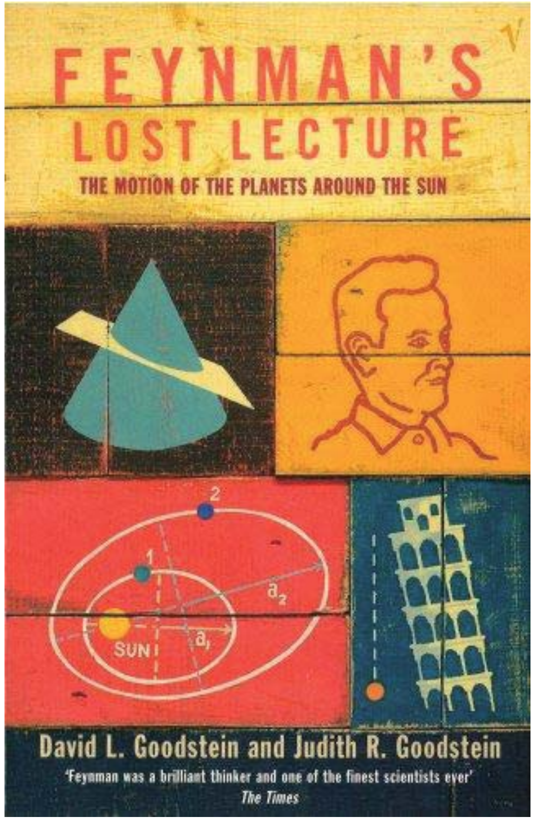
To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the

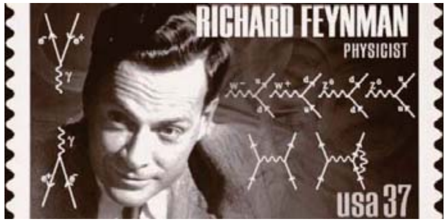
Isaac Newton
(1642-1727)

$$F = \frac{Gm_1 m_2}{r^2}$$

Newton's Law of Universal Gravitation



Richard Phillips Feynman (1918-1988)



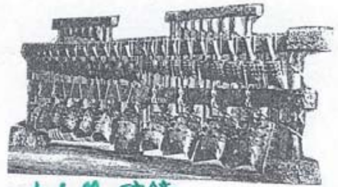
David L. Goodstein, Judith R. Goodstein, *Feynman's Lost Lecture: The Motion of Planets Around the Sun*, Vintage Books, 1997.

(This is a reconstructed account of a lecture given by Richard Feynman on March 13, 1964.)

The Pythagoreans (c. 6th century B.C.) discovered:

- sound caused by a plucked string depends upon the length of the string
- harmonious sounds given off by strings of lengths in ratios of whole numbers to each other

單出曰聲。雜比曰音。
(A singly emerging sound is called *sheng*. An assorted comparative sound is call *yin*.)
《史記》(*Historical Records*)
(1st century B.C.)



set-bells 編鐘

音疾以清。音舒以濁。
(The sound of rapid vibration has a high pitch. The sound of slow vibration has a low pitch.)
《管子》(*Book of Master Guan*)
(7th century B.C.)

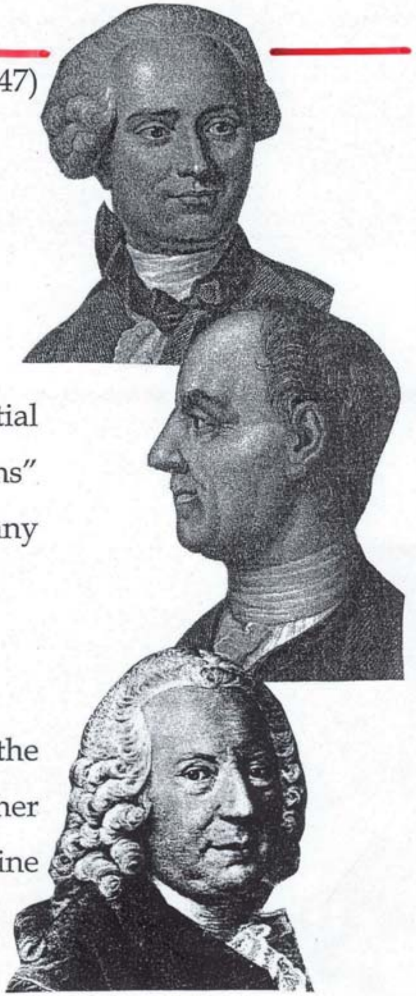
half-tube zither 琴

zither 瑟

From the tomb of Marquis Yi of Zeng (曾侯乙) 5th century B.C.

Vibrating String Problem

Find the motion of a tense string fixed at two ends when it is made to vibrate.



- **Jean le Rond d'Alembert** (1747)

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(L,t) = 0$$

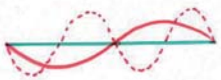
$$u = f(x - at) + g(x + at)$$

- **Leonhard Euler** (1748)

Calculation on initial conditions involve "functions" which depend on x in any manner

- **Daniel Bernoulli** (1753)

$u(x,t)$ is a (infinite) sum of the fundamental and higher harmonics (expressed as sine and cosine functions)



Fourier series

1807 *Sur la propagation de la chaleur*

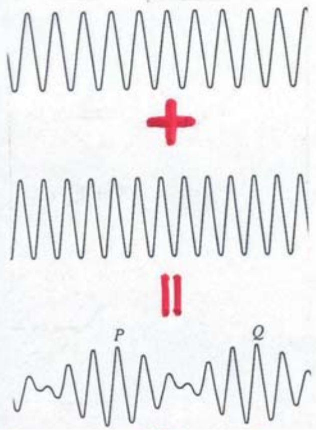
1822 *Théorie analytique de la chaleur*

Study on heat conduction

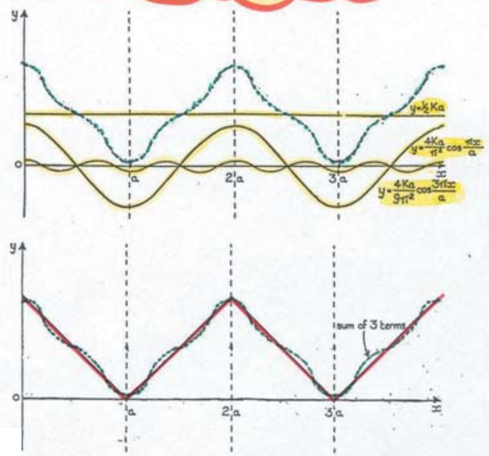
Jean Baptiste Joseph Fourier (1768-1830)

Under certain conditions a **periodic** function can be represented as a (infinite) sum of **sine** and **cosine** functions.

$$F(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots$$



Superposition of two waves



- J. C. Maxwell, A dynamical theory of the electromagnetic field, *Philosophical Transactions of the Royal Society of London*, 155, 1865, 459-512.

- J. C. Maxwell, On a method of making a direct comparison of electrostatic with electromagnetic force; with a note on the electromagnetic theory of light, *Philosophical Transactions of the Royal Society of London*, 158, 1868, 643-657.

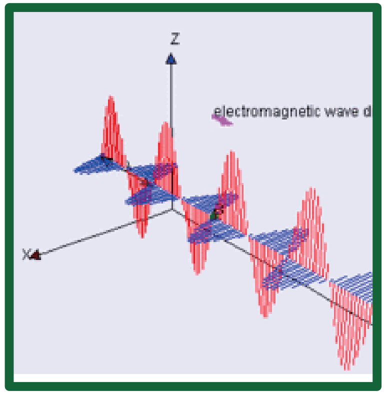
- J. C. Maxwell, *A Treatise on Electricity and Magnetism* , Oxford University Press, 1873.

James Clerk Maxwell
(1831-1879)

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}
$$

Maxwell's Equations

Electromagnetic Field

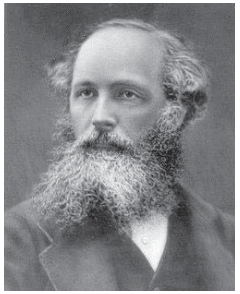


S1 「自然科學」
Basic Science

S2 「技術科學」
Research & Development

T 「技術」
Technology

中國傳統文化和思維對科學起阻礙作用嗎？
王綬琯院士訪談錄，孫小淳、儲姍姍 整理,
《科學文化評論》第 8 卷第 2 期 (2011), 97-116頁
Would traditional Chinese culture and thinking inhibit the development of science?
An interview with Academician Wang Shou-guan, compiled by Sun Xiao-chun and Chu Shan-shan,
Science and Culture Review, 8 (2) (2011), 97-116.

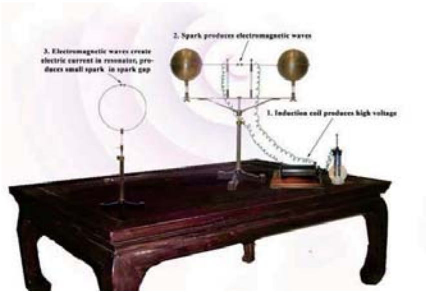


$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

S1

James Clerk Maxwell
(1831-1879)

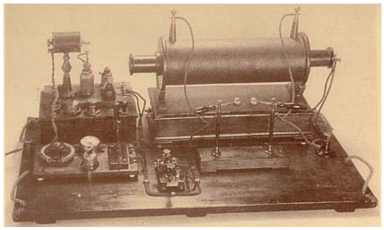
1865



S2

Heinrich Rudolf Hertz
(1875-1894)

1886

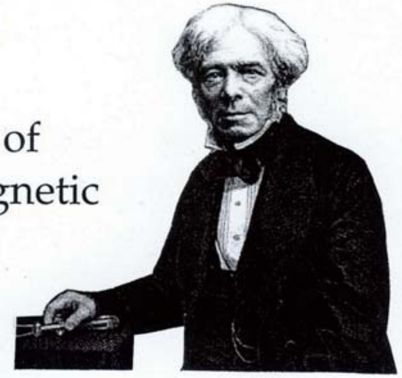


T

Guglielmo Marconi
(1874-1937)

1896

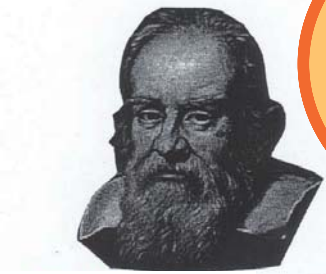
Michael Faraday
(1791-1867)
1831 Discovery of electromagnetic induction



James Clerk Maxwell
(1831-1879)

Albert Einstein
(1879-1955)
1905 "On the electrodynamics of moving bodies" (Special Theory of Relativity)

Story of Relativity



Galileo Galilei
(1564-1642)

Isaac Newton
(1642-1727)

Albert Einstein
(1879-1955)

Stephen Hawking's visit to Hong Kong [*Ming Pao*, 16.06.06]

霍金訪港 講座全文

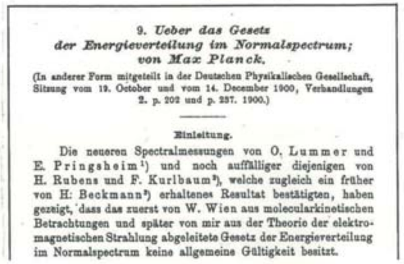
Stephen Hawking

宇宙的起源

史蒂芬・霍金 劍橋大學盧卡斯數學教授

Stephen Hawking
(1942-)

Black body radiation
Max Planck (1901)



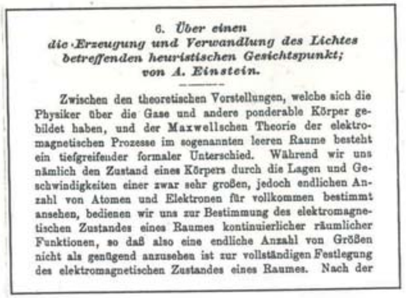
9. *Ueber das Gesetz der Energieverteilung im Normalspectrum;*
von Max Planck.

(In anderer Form mitgeteilt in der Deutschen Physikalischen Gesellschaft, Sitzung vom 19. October und vom 14. December 1900, Verhandlungen 2. p. 202 und p. 237. 1900.)

Einleitung.

Die neueren Spectralmessungen von O. Lummer und E. Pringsheim[1]) und noch auffälliger diejenigen von H. Rubens und F. Kurlbaum[2]), welche zugleich ein früher von H. Beckmann[3]) erhaltenes Resultat bestätigten, haben gezeigt, dass das zuerst von W. Wien aus molecularkinetischen Betrachtungen und später von mir aus der Theorie der elektromagnetischen Strahlung abgeleitete Gesetz der Energieverteilung im Normalspectrum keine allgemeine Gültigkeit besitzt.

Photoelectric effect
Albert Einstein (1905)



6. *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt;*
von A. Einstein.

Zwischen den theoretischen Vorstellungen, welche sich die Physiker über die Gase und andere ponderable Körper gebildet haben, und der Maxwellschen Theorie der elektromagnetischen Prozesse im sogenannten leeren Raume besteht ein tiefgreifender formaler Unterschied. Während wir uns nämlich den Zustand eines Körpers durch die Lagen und Geschwindigkeiten einer zwar sehr großen, jedoch endlichen Anzahl von Atomen und Elektronen für vollkommen bestimmt ansehen, bedienen wir uns zur Bestimmung des elektromagnetischen Zustandes eines Raumes kontinuierlicher räumlicher Funktionen, so daß also eine endliche Anzahl von Größen nicht als genügend anzusehen ist zur vollständigen Festlegung des elektromagnetischen Zustandes eines Raumes. Nach der

Electron interference
Louis de Broglie (1924)
Clinton Joseph Davisson and Lester Halbert Germer (1927)
George Paget Thomson (1927)

Quantum Mechanics

"I am now exclusively occupied with the problem of gravitation, and hope, with the help of a local mathematician friend, to overcome all the difficulties. One thing is certain, however, that never in my life have I been quite so tormented. A great respect for mathematicians has been instilled within me, the subtler aspects of which, in my stupidity, I regarded until now as pure luxury. Against this problem, the original problem of the theory of relativity is child's play."

Letter from Albert Einstein to a colleague in 1912.

C.F. Gauss (1777-1855)

G.F.B. Riemann (1826-1866)

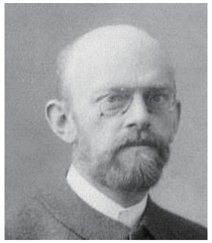
"..... It remains now to examine the question how, in what degree and to what extent these assumptions are guaranteed by experience. ... Either then the actual things forming the groundwork of a space must constitute a discrete manifold, or else the basis of metric relation must be sought for outside that actuality, in colligating forces that operate upon it.

... This path leads out into the domain of another science, into **the realm of physics**, into which the nature of this present occasion forbids us to penetrate."

Georg Friedrich Bernhard Riemann
Űber die Hypothesen, welche der Geometrie zu Grunde liegen (On the Hypotheses Which Lie at the Foundations of Geometry), 1854

Hermann Minkowski (1864-1909)

"[…] through a peculiar, **pre-established harmony**, it has been shown that by trying logically to elaborate the existing edifice of **mathematics**, one is directed on exactly the same path as by having responded to questions arising from the facts of **physics and astronomy**."



David Hilbert (1862-1943)

Hilbert's Sixth Problem:
"To treat in the same manner, by means of axioms, those **physical sciences** in which **mathematics** plays an important part; […]"

D. Hilbert, *Grundlagen der Geometrie* (1899)
D. Hilbert, *Grundlagen der Physik* (1915)

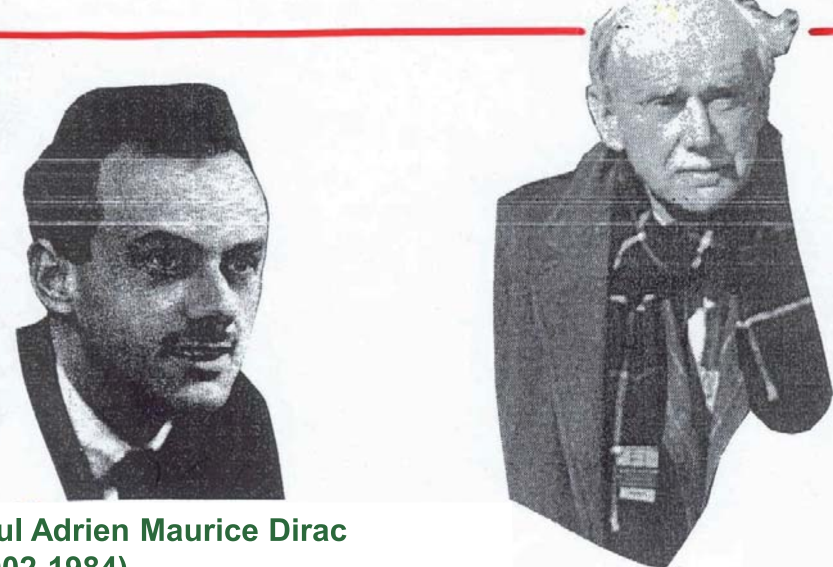
Georg-August-Universität Göttingen, founded in 1734.

"A physical law must possess mathematical beauty."

Paul Adrien Maurice Dirac
Inscription on a blackboard
at University of Moscow in 1956

Dirac Equation (1928)

$$\left[\gamma^{\mu}\left(i\frac{\partial}{\partial x^{\mu}} - eA_{\mu}(x)\right) + m\right]\psi(x) = 0$$



Paul Adrien Maurice Dirac
(1902-1984)
Nobel Laureate in Physics 1933

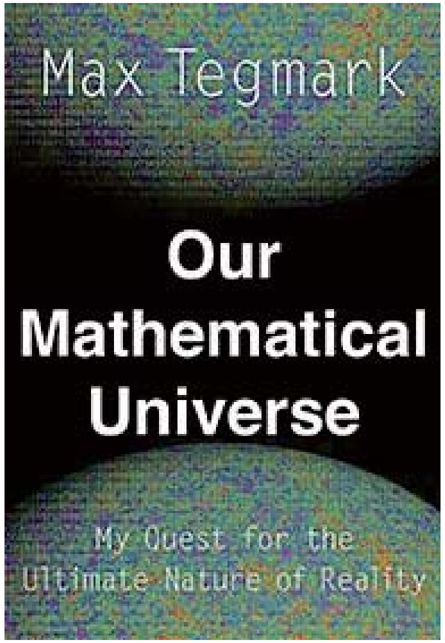
The unreasonable effectiveness of mathematics in the natural sciences

Richard Courant Lecture in Mathematical Sciences delivered by Eugene Wigner at New York University on May 11, 1959, published in *Communications in Pure and applied Mathematics*, 13 (1) (1960), 1-14.

" The first point is that mathematical concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections. "

" Secondly, just because of this circumstance, and because we do not understand the reasons of their usefulness, we cannot know whether a theory formulated in terms of mathematical concepts is uniquely appropriate . "

Eugene Paul Wigner
(1902-1995)
Nobel Laureate
in Physics 1963



Physical reality is a mathematical structure !

Max Tegmark, *Our Mathematical Universe: My Quest for the Ultimate Nature of Reality*, 2014.

You can't hope to understand the [physics / math] until you've understood the [math / physics].

Robert Mills (1927-1999)
Space, Time and Quanta: An Introduction to Contemporary Physics (1994), p.337.

C.N. Yang and Robert Mills at Stony Brook in 1999

End of Part I

接着，讓我從個人在學生時代學習物理的某些片段拿出來，作為事例與大家共同探討一下以上提及的想法。

MECHANICS

Quantity	Definition	Formula	Units
Velocity (v)	$\frac{\text{distance}}{\text{time}}$	$\frac{s}{t}$	ft. per sec.; cm. per sec. (60 m.p.h = 88 ft. per sec.)
Acceleration (f)	$\frac{\text{velocity change}}{\text{time}}$	$\frac{v-u}{t}$	ft. per sec.^{2}; cm. per sec.^{2} (g = 32 ft. per sec.^{2}, or 980 cm. per sec.^{2})
Force (P)	Mass x acceleration	mf	poundal, lb. wt. (32 poundals = 1 lb. wt.) dyne, gm. wt. (980 dynes = 1 gm. wt.)
Work (Energy)	Force x distance	$P \times s$	ft. poundal, ft. lb. wt.; erg, joule
Kinetic energy	Energy by virtue of motion	$\frac{1}{2}mu^2$	(Same units as "Work")
Potential energy	Energy by virtue of level or position	mgh	(Same units as "Work")
Power	Energy (Work) per second	$\frac{\text{Work}}{\text{time}}$	ft. lb. wt. per sec.; erg per sec.; watt
Momentum	Mass x velocity	mv	poundal-sec.; dyne sec.
Resolved component of force	Force x cosine of angle concerned	$P \cos\theta$	(Same as "Force")
Moment of Force	Force x perpendic. distance	—	lb. wt. ft.; gm. wt. cm.
Mechanical Advantage ($M.A$)	$\frac{\text{Load}}{\text{Effort}}$	$\frac{W}{P}$	—
Velocity Ratio ($V.R.$)	$\frac{\text{Distance per sec. moved by effort}}{\text{Distance per sec. moved by load}}$	—	—
Efficiency	$\frac{\text{Work (Energy) obtained}}{\text{Work (Energy) supplied}} \times 100\%$	$\frac{M.A.}{V.R.} \times 100\%$	—
Density (d)	$\frac{\text{Mass}}{\text{Volume}}$	$\frac{M}{V}$	lb. per cu. ft. gm. per c.c.
Specific gravity (s)	$\frac{\text{Weight of substance}}{\text{Weight of equal vol. of water}}$	—	—
Pressure (p)	Force per unit area	hd	lb. wt. per sq. ft.; dynes per sq. cm.

HEAT

Quantity	Definition	Formula	Units
Coefficient of linear expansion	Increase in length of 1 cm. for 1 degree rise in temperature	—	per °C; per °F
Coefficient of volume expansion	Increase in volume of 1 c.c. for 1 degree rise in temperature	—	per °C; per °F
Calorie	Heat to raise temperature of 1 gm. water by 1°C.	—	—
British Thermal Unit	Heat to raise temperature of 1 lb. water by 1°F.	—	—
Therm	100,000 B. Th. U.	—	—
Specific heat (s)	Heat to raise temperature of 1 gm. of substance by 1 degree	—	cal. per gm. per °C. B. Th. U. per lb. per °F.
Water equivalent	Mass X specific heat	—	grams (of water)
Thermal capacity	Mass X specific heat	—	calories per °C. B. Th. U. per °F.
Latent Heat (L)	Heat to change 1 gram of substance from solid to liquid, or from liquid to vapour, state without change of temperature	—	calories per gm.; B. Th. U. per lb.
Joule's equivalent (J)	$\frac{\text{Work done}}{\text{Heat produced}}$	—	ergs or joules per cal. ft. lb. wt. per B. Th. U.
Volume coefficient of gas	Increase in volume of 1 c.c. of gas at 0°C. when temperature rises 1°C., pressure being constant	—	per °C.; per °F.
Pressure coefficient of gas	Increase in pressure per unit pressure at 0°C. when temperature rises 1°C., volume being constant	—	per °C.; per °F.
Absolute temperature (T)	Temperature measured from absolute zero	$273 + t$ (°C)	°K.
Relative humidity	$\frac{\text{S.V.P. at dew point}}{\text{S.V.P. at air temp.}} \times 100\%$	—	—

LIGHT

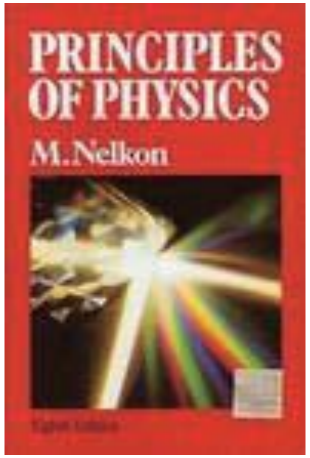
Quantity	Definition	Formula	Units
Focal length (f)	Distance from focus to mirror or lens	—	cm.; inches
Mirror or lens formula	—	$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$	—
Magnification (m)	$\frac{\text{Image length}}{\text{Object length}}$	$\frac{v}{u}$	—
Refractive index (μ)	$\frac{\text{Sine of angle of incidence}}{\text{Sine of angle of refraction}}$	$\frac{\sin i}{\sin r}$	—
Critical angle (c)	Angle at which total internal reflection just begins	$\sin c = \frac{1}{\mu}$	degrees
Candle power (I)	$\frac{\text{Luminous intensity of lamp}}{\text{Luminous intensity of lamp of 1 c.p.}}$	—	c. p.
Intensity of illumination (E)	Light energy per unit area per second	—	foot-candle; cm.-candle

SOUND

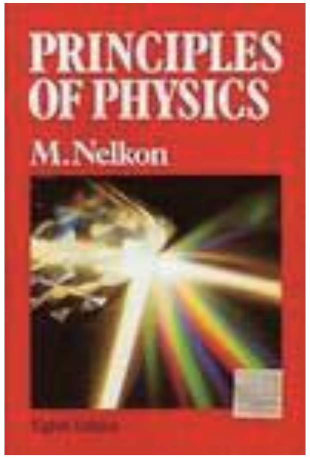
Quantity	Definition	Formula	Units
Frequency (f)	Number of vibrations per second	—	c. p. s.
Wavelength (λ)	Distance between successive crests or troughs	—	feet; metres or cm.
Velocity of wave (v)	Distance travelled per sec.	$v = f\lambda$	feet per sec.; metres per sec.
Frequency of stretched string	—	$\frac{1}{2l}\sqrt{\frac{T}{m}}$	c. p. s.
Fundamental frequency of closed pipe	Lowest frequency obtainable from pipe	$\frac{v}{4l}$	c. p. s.
Fundamental frequency of open pipe	(As above)	$\frac{v}{2l}$	c. p. s.

ELECTRICITY

Quantity	Definition	Formula	Units
Current (I)	Quantity of electricity per second	$I = \frac{V}{R}$	amperes (A)
Potential Difference (V)	Energy per coulomb	$V = IR$	volts (V)
Resistance (R)	$\frac{\text{Potential Difference}}{\text{Current}}$	$R = \frac{V}{I}$	ohms (Ω)
Quantity	—	$Q = It$	coulombs
Resistance in Series	—	$R = R_1 + R_2$	—
Resistance in Parallel	—	$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$	—
Electrical Energy (W)	—	$W = QV = IVt$	joules
Electrical Power (P)	—	$P = IV$	watts
Heat in Resistor (H)	—	$H = \frac{I^2Rt}{4.2}$ or $\frac{V^2t}{4.2R}$ or $\frac{IVt}{4.2}$	calories
Electrochemical equivalent (z)	Weight of element deposited by 1 amp. in 1 sec.	$w = zIt$	gms. per coulomb
Electro-Motive Force (E)	P.D. at terminals when no current flows	$I = \frac{E}{R+r}$	volts
Resistivity (Specific resistance) (ρ)	Resistance of 1 cm. or 1 in. cube of material	$R = \frac{\rho l}{a}$	ohm-cm.; ohm-in.
Unit pole-strength (m)	Strength of that pole which repels a similar pole 1 cm. away in a vacuum with a force of 1 dyne	$F = \frac{m_1 m_2}{d^2}$	c.g.s. unit
Magnetic Moment (M)	Pole-strength x magnetic length	$m \times 2l$	c.g.s. unit
Angle of Dip (θ)	Angle between the horizontal and the earth's resultant field	$\tan\theta = \frac{V}{H_0}$	degrees
Intensity of Field (H)	Force in dynes acting on a pole of unit strength	$F = Hm$	oersteds
Electrostatic unit of quantity	That charge which repels a similar charge 1 cm. away in a vacuum with a force of 1 dyne	—	e. s. u.
Capacitance (C)	$\frac{\text{Charge on Condenser}}{\text{P.D. between plates}}$	$C = \frac{Q}{V}$	farads



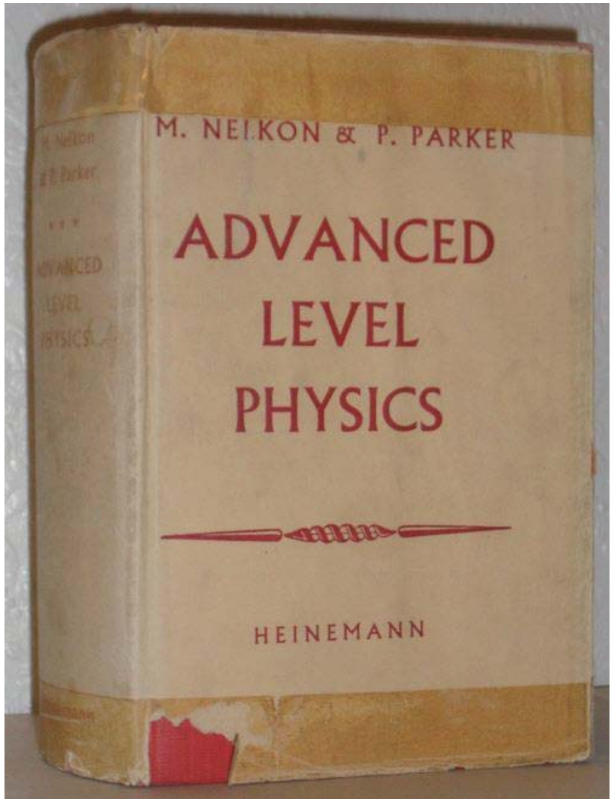
Michael Nelkon, *Principles of Physics*, 8th Edition, 1990; first published, 1951.



Michael Nelkon, *Principles of Physics*, 8th Edition, 1990; first published, 1951.

A personal anecdote :

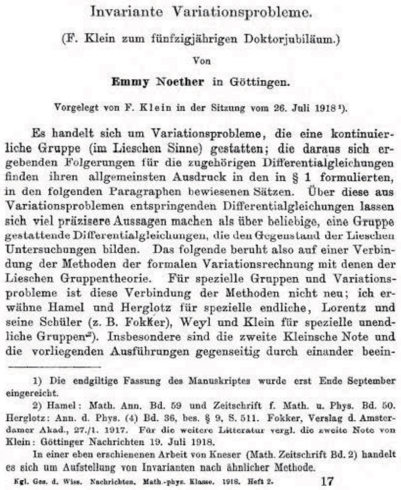
What made me choose mathematics instead of physics in graduate school ?



Michael Nelkon and Philip Parker, *Advanced Level Physics*, Heinemann, London, 1958.

One day I asked myself the question: "What is energy?" I could not give a satisfying answer for myself. I doubted whether I had in me a "sense of physics" or not. I wondered whether I liked physics only because I liked its mathematics.

Many years later, after teaching for decades, ...



Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

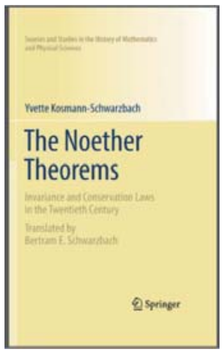
In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17

Emmy Noether
(1882-1935)

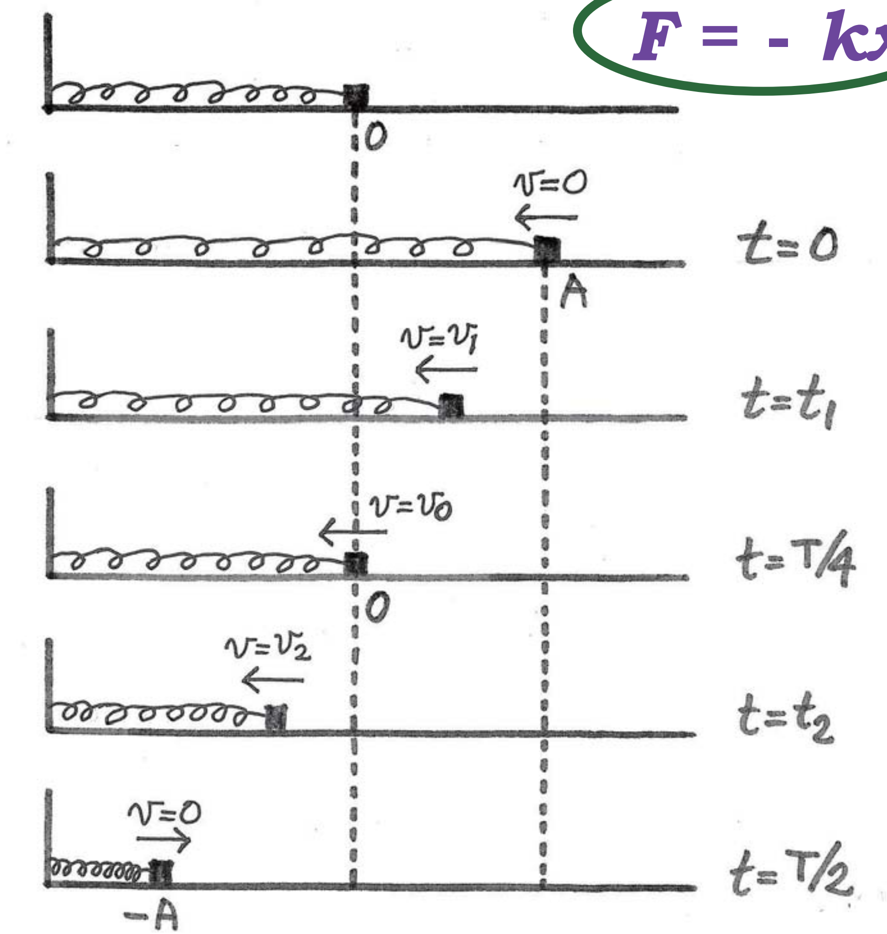
E. Noether, Invariante Variationsprobleme, *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse* (1918), 235-257.

Symmetry of a physical system ⟺ Conservation Law

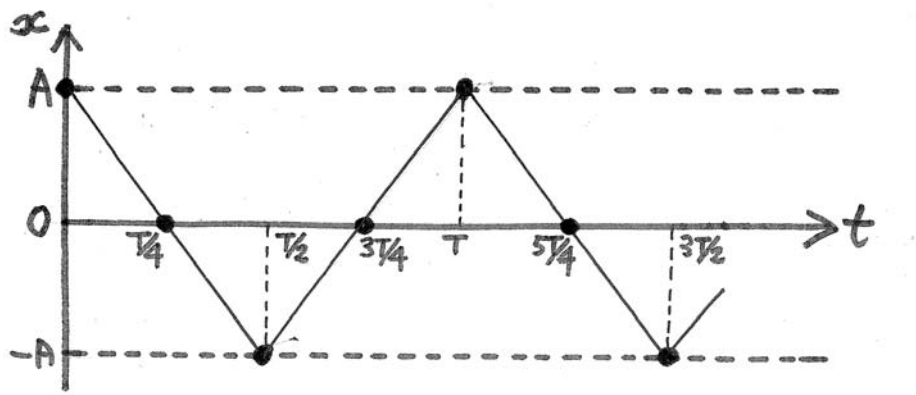


Yvette Kosmann-Schwarzbach, *The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century*, 2011; original edition in French, 2006.

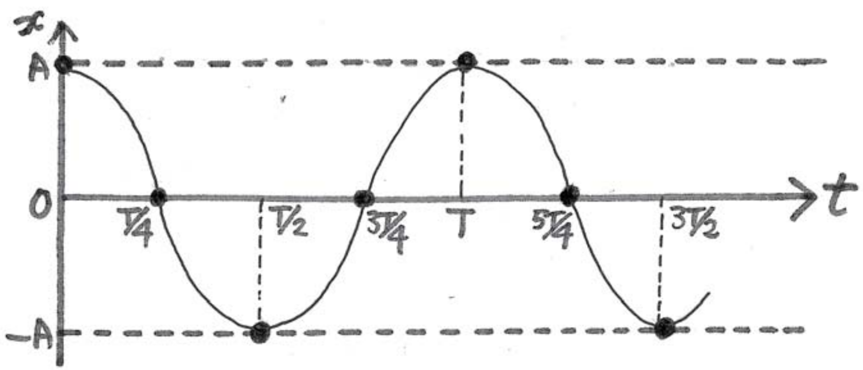
$F = -kx$



簡諧運動 (Simple Harmonic Motion)



This is not possible!
(Why?)



Does this curve ring a bell?
How does x varies with t ?

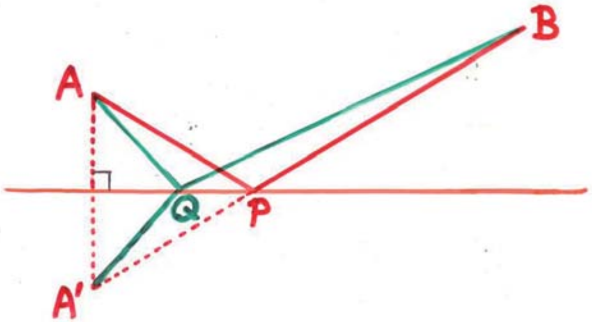
A lesson from optics

- **Euclid** in his *Optics* (*c.* 300 B.C.E.) reduced the study of optics to geometry by observing that light propagates in a straight line.
- **Heron** (*c.* 100 C.E.) introduced the ***Shortest Path Principle***.
- **Fermat** (17th century) introduced the ***Quickest Path Principle***, with which he explained the phenomenon of both reflection and refraction at one stroke as a problem on finding minimum. (In solving this problem, Fermat invented differential calculus.)

Principle of Least Action

(Gottfried Leibniz, Leonhard Euler, Pierre Louis Maupertuis, 18th century)

Fermat's Principle: Light travels between two given points along the path of shortest time.



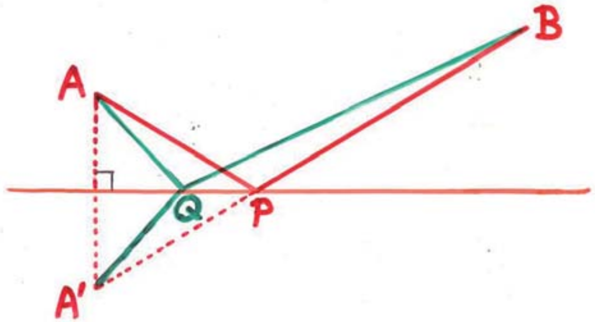
$$\begin{aligned} AQ + BQ &= A'Q + BQ \\ &> A'B = A'P + BP \\ &= AP + BP. \end{aligned}$$

Or, you can locate the point P that yields a minimum by using the standard method of calculus, which is not as fast.

Principle of Least Action

(Gottfried Leibniz, Leonhard Euler, Pierre Louis Maupertuis, 18th century)

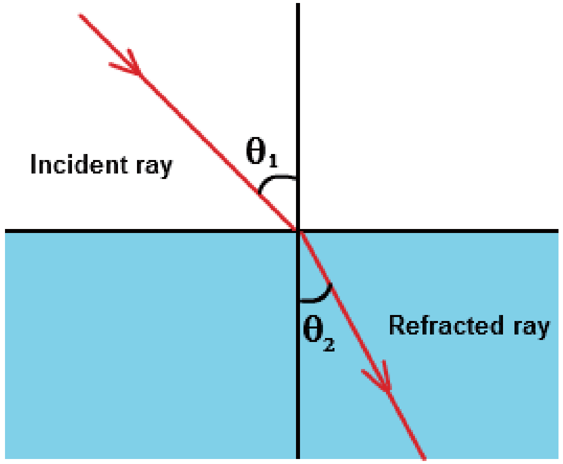
Fermat's Principle: Light travels between two given points along the path of shortest time.



$$\begin{aligned} AQ + BQ &= A'Q + BQ \\ &> A'B = A'P + BP \\ &= AP + BP. \end{aligned}$$

A similar reasoning goes for the phenomenon of refraction (Snell's Law of refraction).

Willebrod Snellius [known in the English-speaking world as Snell] (1580-1626)



Snell's Law of the refraction of light

Snell's Law of the refraction of light

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2}=\frac{v_1}{v_2}=\frac{n_2}{n_1}}$$

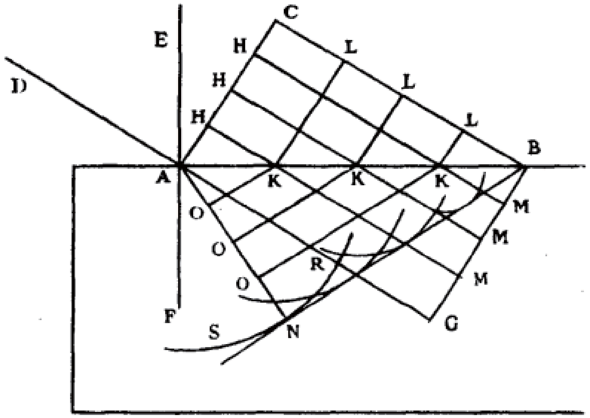
where $v_1 = \frac{c}{n_1}$, $v_2 = \frac{c}{n_2}$ and c is the speed of light in vacuum.

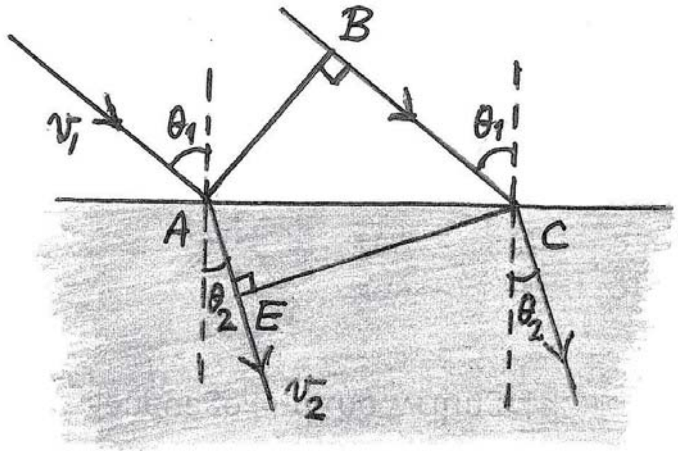
$$\boxed{\frac{\sin\theta_1}{\sin\theta_2}=\frac{v_1}{v_2}=\frac{n_2}{n_1}}$$

where v_1, v_2 are respectively the velocity of light in the first and second medium;

n_1, n_2 are respectively the refractive index of the first and second medium.

Christiaan Huygens
(1629-1695)





Christiaan Huygens,
***Traité de la Lumière* (1678)**

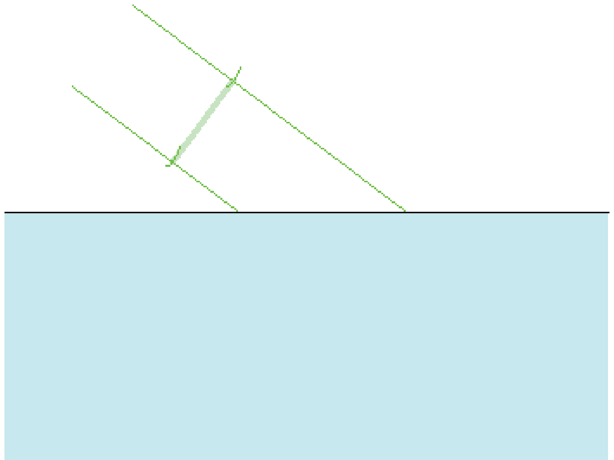
$BC/v_1 = AE/v_2$,

that is,

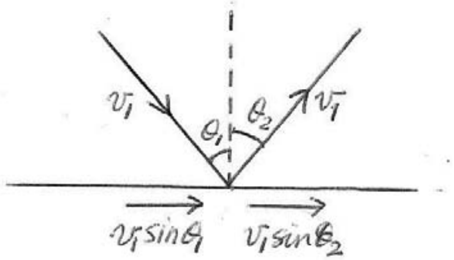
$AC\sin\theta_1/v_1 = AC\sin\theta_2/v_2$.

Hence,

$$\sin\theta_1/\sin\theta_2 = v_1/v_2.$$

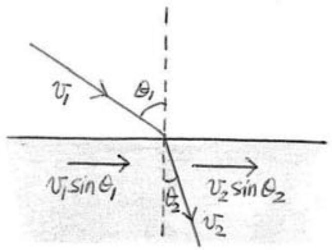


Newton's Corpuscular Model (around 1660)



Reflection of light

$v_1 \ \sin\theta_1 = v_1 \ \sin\theta_2$,
so $\sin\theta_1 = \sin\theta_2$, or $\theta_1 = \theta_2$.



Refraction of light

$v_1 \ \sin\theta_1 = v_2 \ \sin\theta_2$,
so $\sin\theta_1 / \sin\theta_2 = v_2 / v_1$.

This implies that $v_1 < v_2$, that is, light travels faster in water than in air!
In 1862 Léon Foucault verified by his experiment that the opposite is true.

Christiaan Huygens
(1629-1695)

Wave Theory of Light

Isaac Newton
(1642-1727)

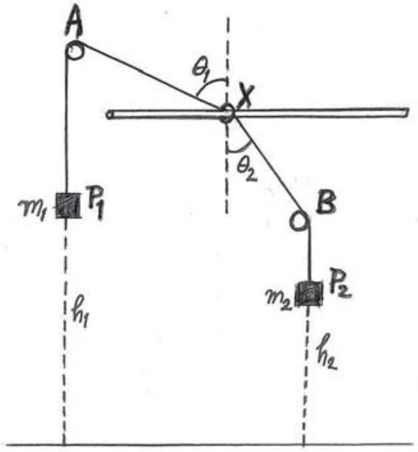
Particle Theory of Light

Thomas Young
(1773-1829)

Augustin Jean Fresnel
(1788-1827)

Interference and Diffraction of Light as a Wave

Wave-Particle Duality



A mechanical model (à la Pólya) to see when $AX/v_1+XB/v_2$ is a minimum?

At equilibrium the potential energy is a minimum, so $m_1h_1+m_2h_2$ is a minimum, so $m_1\ AP_1+m_2\ BP_2$ is a maximum, so $m_1\ AX+m_2\ XB$ is a minimum.

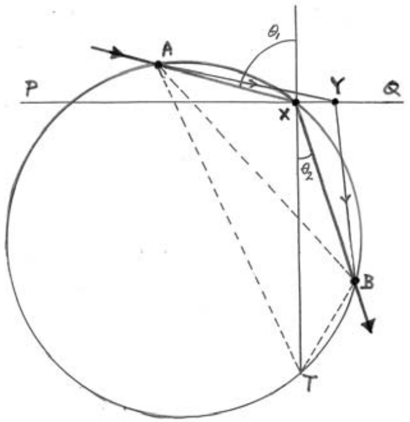
Put $m_1 = 1/v_1$ and $m_2 = 1/v_2$, we have $AX/v_1 + XB/v_2$ is a minimum.

At equilibrium we also have $m_1 \sin\theta_1 = m_2 \sin\theta_2$, that is,

$$\boxed{\sin\theta_1/\sin\theta_2 = v_1/v_2}\ .$$

The vertical line intersects the circle AXB at T. X is located on the horizontal line PQ such that $\sin\theta_1/\sin\theta_2 = v_1/v_2$.
We want to show that

$$AX/v_1 + XB/v_2 < AY/v_1 + YB/v_2 .$$



By Ptolemy's Theorem we have

$$AX \cdot BT + AT \cdot XB = AB \cdot XT,$$

and $AY \cdot BT + AT \cdot YB > AB \cdot YT$.

Hence, $AX \cdot BT + AT \cdot XB < AY \cdot BT + AT \cdot YB$ because $XT < YT$ (#).

But $BT = 2R\sin\theta_2$ and $AT = 2R\sin\theta_1$ where R is the radius of the circle AXB (Why?)

Hence, $AT/BT = \sin\theta_1/\sin\theta_2 = v_1/v_2$.

Substitute into (#), we have

$$AX + XB \cdot v_1/v_2 < AY + YB \cdot v_1/v_2 ,$$

or $$AX/v_1 + XB/v_2 < AY/v_1 + YB/v_2 .$$

A short enrichment course/workshop in ten three-hour sessions was conducted each year from 2006 to 2011 at HKU for youngsters about to embark on their undergraduate study. It tried to integrate the two subjects mathematics and physics with a historical perspective, to show how the two subjects are intimately interwoven.

The underlying theme would be the role and evolution of mathematics (mainly calculus, with related topics in linear algebra and geometry) in understanding the physical world, from the era of Isaac Newton's mechanics to that of James Clerk Maxwell's electromagnetism and possibly beyond, to that of Albert Einstein's relativity. In other words it tries to tell the story of triumph in mathematics and physics over the past four centuries. The physics would provide both the sources of motivation and the applications.

4th century B.C.	Physical view of Aristotle	Euclidean geometry
Many centuries in between		Geometry (area/volume) Algebra (equations)
17th century	Physical view of Copernicus, Kepler, Galileo, Newton	Calculus (functions — polynomial, rational, trigonometric, logarithmic and exponential)

18th century	Wave and particle	Differential equations Fourier analysis
19th century	Theory of electromagnetism (Maxwell's equations)	Stokes' Theorem (Fundamental Theorem of Calculus)
20th century	Special and general theory of relativity, quantum mechanics	Non-Euclidean geometries of spacetime, probability theory

Problem 2 in Tutorial 1

A ball is dropped at a point of height *H* from the ground. Suppose every time the ball rebounds its velocity is 3/4 of that with which it hits the ground.

Discuss the subsequent motion of the ball. (Will the ball bounce forever? What is the total distance the ball will travel?)

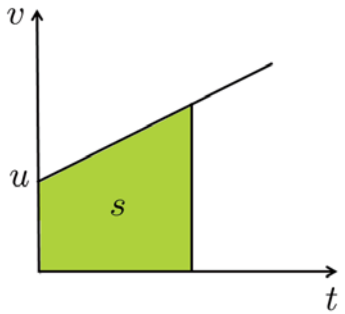
Motion in a straight line with uniform acceleration

$$\frac{v-u}{t} = a, a \text{ is a constant.}$$

$$\therefore v = u + at$$

What is the distance *s* covered in time *t* ?

$$s = \frac{1}{2}(u+v)t = \frac{1}{2}(2u+at)t = ut + \frac{1}{2}at^2$$

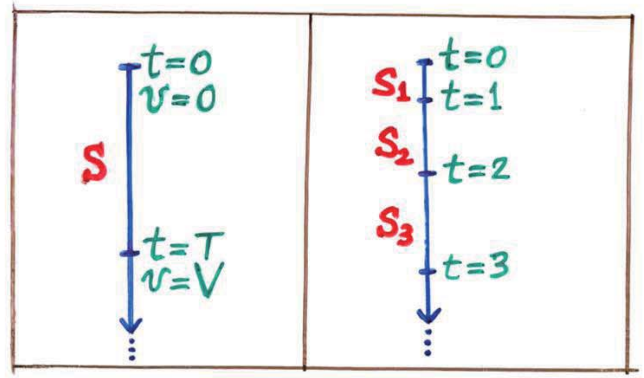


$$\boxed{\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ v &= u + at \end{aligned}}$$

Eliminating *t* we obtain

$$\boxed{v^2 = u^2 + 2as}$$

Laws of Falling Bodies (in history)

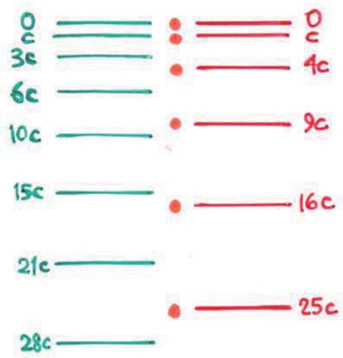


Albert of Saxony (1316-1390) $V \propto S$

Nicole Oresme (1320-1382) $V \propto T$

Leonardo da Vinci (1452-1519) $\frac{S_1}{1} = \frac{S_2}{2} = \frac{S_3}{3} = \frac{S_4}{4} = \cdots$

Galileo Galilei (1564-1642) $\frac{S_1}{1} = \frac{S_2}{3} = \frac{S_3}{5} = \frac{S_4}{7} = \cdots$



$S \propto T^2$

$V \propto T$

Which case tells us that $S \propto T^2$?
If $S \propto T^2$, why is $V \propto T$, and conversely?

Free fall

t (sec)	s (cm)	Average velocity $\frac{s(1)-s(t)}{1-t}$ (cm./sec.)	
0	0	(490 – 0)/1	= 490
0.1	49	(490 – 49)/0.9	= 490
0.5	122.5	(490 – 122.5)/0.5	= 735
⋮	⋮	⋮	
0.9	396.9	(490 – 396.9)/0.1	= 931
0.95	442.23	(490 – 442.23)/0.05	= 955.4
⋮	⋮	⋮	
0.99	480.25	(490 – 480.25)/0.01	= 975
0.995	485.11	(490 – 485.11)/0.005	= 978
0.999	489.02	(490 – 489.02)/0.001	= 980
⋮	⋮		
1	490		

Instantaneous velocity at $t = 1$ is 980 cm./sec.

If $S \propto T^2$, why is $V \propto T$?

$S = S(t) = ct^2$

$V = V(t) = ?$ (instantaneous velocity)

$$V = \lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{c(t+\Delta t)^2 - ct^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{c[2t\Delta t + (\Delta t)^2]}{\Delta t} = \lim_{\Delta t \to 0} c[2t + \Delta t] = 2ct.$$

In general, for a function $y = f(x)$, the **derived function** $\frac{dy}{dx}$ (also written as $f'(x)$, called the **derivative** of $y = f(x)$) is given by $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (if it exists).

Geometric interpretation: Slope of tangent to a curve.

In general, for a function $y=f(x)$, the **derived function** $\frac{dy}{dx}$ (also written as $f'(x)$, called the **derivative** of $y=f(x)$) is given by $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (if it exists).

Geometric interpretation: Slope of tangent to a curve.

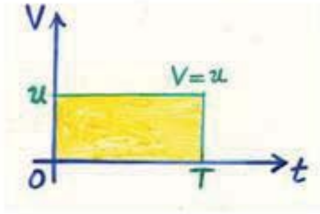
Consult any standard textbook on calculus to learn (i) some basic properties about $f'(x)$, (ii) some formulas of $f'(x)$ of certain elementary functions (rules of differentiation).

A simple example:

$$f(x)=x^5,\ \frac{df}{dx}=f'(x)=5x^4$$
$$\frac{d^2f}{dx^2}=f''(x)=20x^3, \text{ etc.}$$

Given $V=\frac{dS}{dt}=u$ (constant velocity), what is $S=S(T)$?

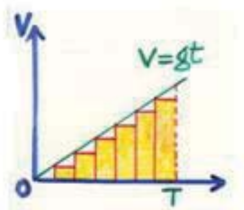
If $V \propto T$, why is $S \propto T^2$?



$S=uT$

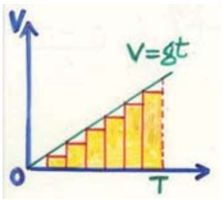
= area under the curve on the interval $[0,T]$

Given $V=\frac{dS}{dt}=gt$ (constant acceleration), what is $S=S(T)$?



$$S=\lim_{N\to\infty}\left[V\left(\frac{T}{N}\right)+V\left(\frac{2T}{N}\right)+V\left(\frac{3T}{N}\right)+\cdots+V\left(\frac{(N-1)T}{N}\right)\right]\left(\frac{T}{N}\right)$$
$$=\lim_{N\to\infty}\left[g\left(\frac{T}{N}\right)^2\frac{N(N-1)}{2}\right]$$
$$=\lim_{N\to\infty}\frac{g}{2}T^2\left[1-\frac{1}{N}\right]=\frac{1}{2}gT^2.$$

Given $V = \dfrac{dS}{dt} = gt$ (constant acceleration), what is $S = S(T)$?



$$S = \lim_{N\to\infty}\left[V\left(\frac{T}{N}\right)+V\left(\frac{2T}{N}\right)+V\left(\frac{3T}{N}\right)+\cdots+V\left(\frac{(N-1)T}{N}\right)\right]\left(\frac{T}{N}\right)$$

$$= \lim_{N\to\infty}\left[g\left(\frac{T}{N}\right)^2\frac{N(N-1)}{2}\right]$$

$$= \lim_{N\to\infty}\frac{g}{2}T^2\left[1-\frac{1}{N}\right]=\frac{1}{2}gT^2.$$

In general, for a function $y = f(x)$, a working definition of the (**definite**) **integral** $\int_a^b f(x)dx$ is given by

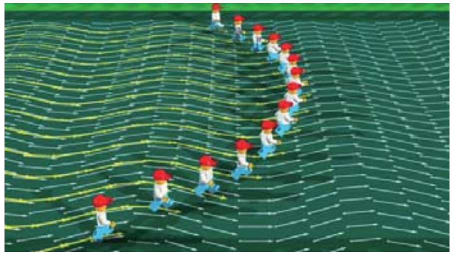
$$\lim_{N\to\infty}\left[f(a)+f\left(a+\frac{b-a}{N}\right)+f\left(a+\frac{2(b-a)}{N}\right)+\cdots+f\left(a+\frac{(N-1)(b-a)}{N}\right)\right]\left(\frac{b-a}{N}\right)$$

(if it exists).

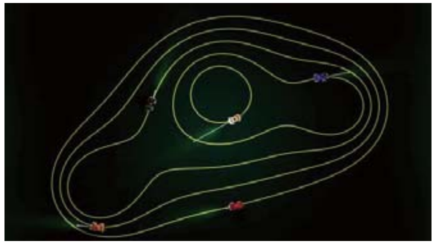
Geometric interpretation: area under a curve on the interval $[a,b]$.

Michael Spivak, *The Hitchhiker's Guide to Calculus* (1995)

Otto Toeplitz, *The Calculus: A Genetic Approach* (1963)



Calculus, differential equation and vector field — Chapter Two in the video CHAOS

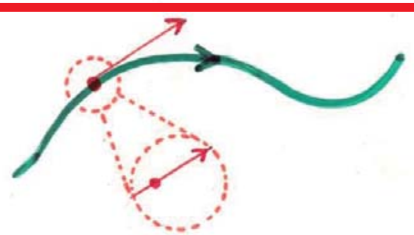


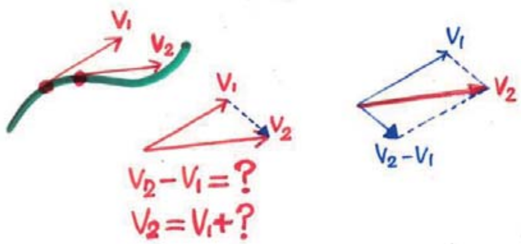
produced by Étienne Ghys, Jos Leys and Aurélien Alvarez

http://www.chaos-math.org/en/film

Motion on a curve

Q. What is meant by velocity and acceleration in this case?

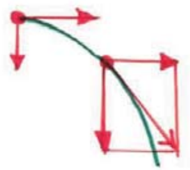




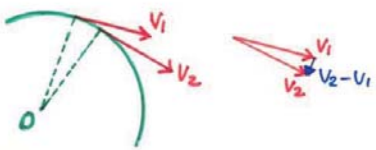
Vector (direction + magnitude)

scalar (only magnitude)

Projectile

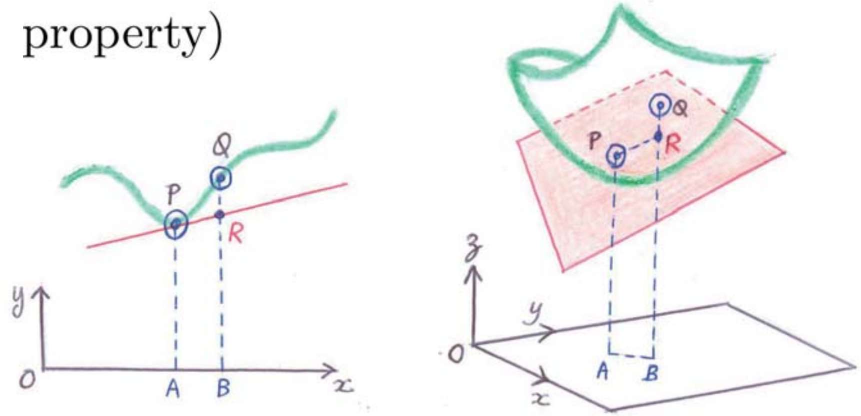


Circular Motion



DIFFERENTIATION

"best" **linear** approximation (a local property)



QR vanishes "faster" than AB,

i.e. $\dfrac{QR}{AB}$ becomes arbitrarily small as AB is made sufficiently small.

1-dimensional case

$$f(x_0+h) = f(x_0) + L(h) + \varepsilon(h)|h|,$$

where L is **linear** and $\lim_{h\to 0} \varepsilon(h) = 0.$

2-dimensional case

$$f(x_0+h, y_0+k) = f(x_0,y_0) + L(h,k) + \varepsilon(h,k)|(h,k)|,$$

where L is **linear** and $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0.$

1-dimensional case

$$f(x_0+h) = f(x_0) + L(h) + \varepsilon(h)|h|,$$

where L is **linear** and $\lim_{h\to 0} \varepsilon(h) = 0.$

$L(h) = ah$, where

$$a = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \frac{df}{dx}(x_0),$$

derivative of f at x_0.

2-dimensional case

$$f(x_0+h, y_0+k) = f(x_0, y_0) + L(h,k) + \varepsilon(h,k)|(h,k)|,$$

where L is **linear** and $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0.$

$L(h,k) = ah + bk$, where

$$a = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial x}(x_0, y_0),$$

$$b = \lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k} = \frac{\partial f}{\partial y}(x_0, y_0),$$

partial derivatives of f at (x_0, y_0).

RATE OF CHANGE

DIFFERENTIAL EQUATION

= an equation involving derivatives (partial derivatives) of various orders

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots\right) = 0, \quad \text{where}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right), \text{ etc.}$$

$$G\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \cdots\right) = 0, \text{ where}$$

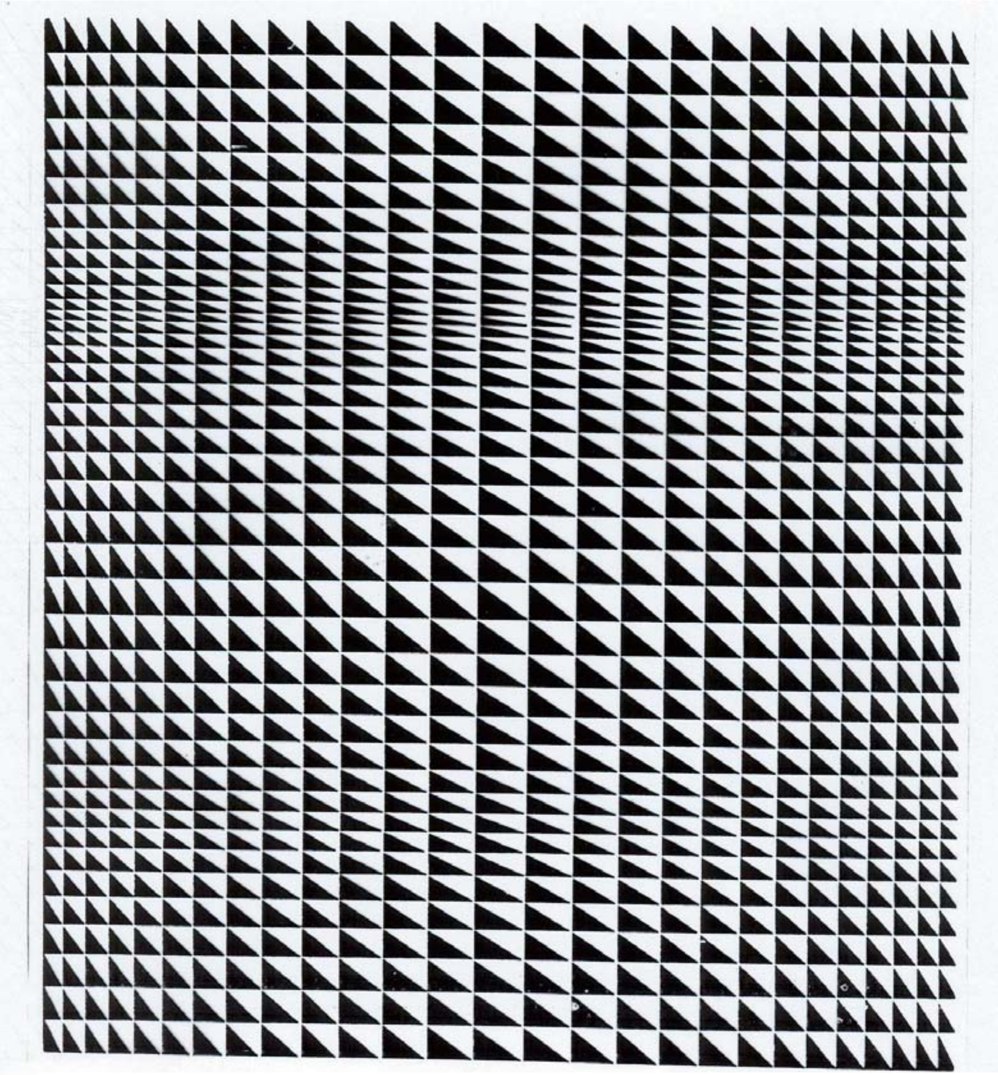
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right), \text{ etc.}$$

The simplest type of differential equation is

$$\frac{dy}{dx} = f(x) \cdots\cdots (*).$$

A solution of $(*)$ is called a **primitive** (or an **anti-derivative**, or an **indefinite integral** of f), in symbol $\int f(x)dx$.



Bridget Riley : Straight Curves (1963)

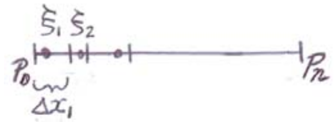
Compare with lines of force of Faraday, notion of field of Maxwell in physics, integral curves in mathematics.

INTEGRATION

"summation" by going to the limit (a global property)

One example: Knowing the shape and the density distribution *f* of an object, calculate its mass.

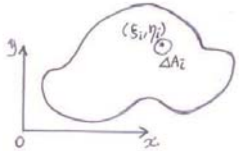
Case 1: Straight line



$$\text{mass} = \lim_{\Delta\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

idea of definite integral

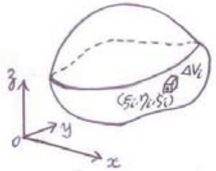
Case 2: Plane lamina



$$\text{mass} = \lim_{\Delta\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta A_i$$

idea of double integral

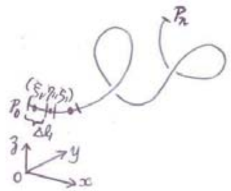
Case 3: Solid



$$\text{mass} = \lim_{\Delta\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta V_i$$

idea of volume integral

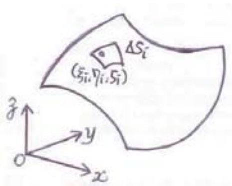
Case 4: Curling wire



$$\text{mass} = \lim_{\Delta\to 0}\sum f(\xi_i,\eta_i,\zeta_i)\Delta \ell_i$$

idea of line integral

Case 5: Curved lamina



$$\text{mass} = \lim_{\Delta\to 0}\textstyle\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta S_i$$

idea of surface integral

This particular
differential equation

$$\boxed{\frac{d^2y}{dx^2} = -kx}$$

comes up in
numerous instances,
whenever some sort
of oscillation occurs.

Simple Harmonic Motion

This particular
differential equation

$$\boxed{\frac{dy}{dx} = ky}$$

comes up in
numerous instances,
whenever the rate of
change of a quantity
is proportional to the
quantity itself.

Exponential growth

謝謝香港教育局課程發展處與香港數理教育學會的邀請，讓我有此機會與大家談談數學和物理。

謝謝蔡偉峰老師應允作回應嘉賓，與大家分享他的高明識見。

柯志明先生協助製作 *GeoGebra* 顯示，以輔助講解，香港大學數學系呂美美女士協助製作圖片，為講座添色，謹此一併致謝。

HARMONIES IN NATURE: A DIALOGUE BETWEEN MATHEMATICS AND PHYSICS

Man-Keung SIU
Department of Mathematics
The University of Hong Kong
Pokfulam, Hong Kong
mathsiu@hkucc.hku.hk

ABSTRACT

The customary practice in school to teach mathematics and physics as two separate subjects has its grounds. However, such a practice deprives students of the opportunity to see how the two subjects are intimately interwoven. This paper discusses the design and implementation of an enrichment course for school pupils in senior secondary school who are about to embark on their undergraduate study. The course tries to integrate the two subjects with a historical perspective.

1 Why is an enrichment course on mathematics-physics designed?

In school it is a customary practice to teach mathematics and physics as two separate subjects. In fact, mathematics is taught throughout the school years from primary school to secondary school, while physics, as a full subject on its own, usually starts in senior secondary school. This usual practice of teaching mathematics and physics as two separate subjects has its grounds. To go deep into either subject one needs to spend at least a certain amount of class hours, and to really understand physics one needs to have a sufficiently prepared background in mathematics. However, such a practice deprives students of the opportunity to see how the two subjects are intimately interwoven. Indeed, in past history there was no clear-cut distinction between a scientist, not to mention so specific as a physicist, and a mathematician.

Guided by this thought we try to design an enrichment course for school pupils in senior secondary school, who are about to embark on their undergraduate study in two to three years' time, that tries to integrate the two subjects with a historical perspective. Conducting it as an enrichment course, we are free from an examination-oriented teaching-learning environment and have much more flexibility with the content. Admittedly, this is not exactly the same as the normal classroom situation with the constraint imposed by an official syllabus and the pressure exerted by a public examination. However, just like building a mathematical model, we like to explore what happens if we can have a bit more freedom to do things in a way we feel is nearer to our ideal.

Albert Einstein and Leopold Infeld sum up the situation succinctly, "In the whole history of science from Greek philosophy to modern physics there have been constant attempts to reduce the apparent complexity of natural phenomena to some simple fundamental ideas and relations. This is the underlying principle of all natural philosophy." [Einstein & Infeld, 1938]. Such a process makes demand on one's curiosity and imagination, but at the same time requires disciplined and critical thinking. Precision in mathematics as well as in words is called for. Galileo Galilei already referred to mathematics as the language of science in his *Il Saggiatore* (*The Assayer*) of 1623, "Philosophy

is written in this grand book — I mean the universe — which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth."

By promoting this view Galileo made a significant step forward in switching the focus from trying to answer "why" to trying to answer "how (much)", that is, from a qualitative aspect to a quantitative aspect. In the Eastern world a similar sentiment was expressed by many authors of ancient classics that may sound like bordering on the mystical side. One such typical example is found in the preface of *Sun Zi Suan Jing* (*Master Sun's Mathematical Manual*) in the 4th century, "Master Sun says: Mathematics governs the length and breadth of the heavens and the earth; affects the lives of all creatures; forms the alpha and omega of the five constant virtues; acts as the parents for yin and yang; establishes the symbols for the stars and the constellations; manifests the dimensions of the three luminous bodies; maintains the balance of the five phases; regulates the beginning and the end of the four seasons; formulates the origin of myriad things; and determines the principles of the six arts."

The conviction in seeing beauty and order in Nature was long-standing. Plato's association of the five regular polyhedra to the theory of four elements in *Timaeus* (c.4th century B.C.) is an illustrative example. Over a millennium later, Johannes Kepler tried to fit in the motion of the six known planets (Saturn, Jupiter, Mars, Earth, Venus, Mercury) in his days with the five regular polyhedra in *Mysterium Cosmographicum* of 1596. By calculating the radii of inscribed and circumscribed spheres of the five regular polyhedra nestled in the order of a cube, a tetrahedron, a dodecahedron, an icosahedron and an octahedron, he obtained results that agreed with observed data to within 5% accuracy! He also thought that he had explained why there were six planets and not more! Now we realize the lack of physical ground in his theory, beautiful as it may seem. Still, it is a remarkable attempt to associate mathematics with physics, and indeed it led to something fruitful in the subsequent work of Kepler.

Well into the modern era the explanatory power of mathematics on Nature is still seen by many to be mystical but fortunate. Eugene Paul Wigner, 1963 Nobel Laureate in physics, refers to it as "the unreasonable effectiveness of mathematics in the natural sciences". Heinrich Rudolf Hertz even said (referring to the Maxwell's equations which predicted the presence of electromagnetic wave that he detected in the laboratory in 1888.), "One cannot escape the feeling that these mathematical formulas have an independent existence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them." Robert Mills, an eminent physicists of the Yang-Mills gauge theory fame, says, "You can't hope to understand the [physics / math] until you've understood the [math / physics]." [Mills, 1994]. This dictum that emphasizes a two-way relationship between mathematics and physics furnishes the guideline for our enrichment course.

2 How is such a course run?

The enrichment course, with its title same as that of this paper, ran for ten sessions each taking up three hours on a weekend (outside of the normal school hours). It had been run four times, in the spring of 2006 to 2009, in collaboration with a colleague at the Department of Physics in my university. Much as we wish to offer a truly integrated course, other constraints and factors (individual expertise, affordable time of preparation, inadequacy on our part, lack of experience in this new venture) force some sort of division of labour so that each one of us took up about half of the course. However, we still tried to maintain a spirit of integration in having a balanced emphasis on the mathematics and the physics in a suitable manner. In this paper I will naturally tell more about the part I took up, which involved the first two sessions, two intermittent sessions and the final session.

The underlying theme of the course is the role and evolution of mathematics, mainly geometry and calculus, with related topics in linear algebra, in an attempt to understand the physical world, from the era of Isaac Newton to that of James Clerk Maxwell and beyond it to that of Albert Einstein. In other words, it tries to tell the story of triumph in mathematics and physics in the past four centuries. The physics provides both the source of motivation and the applications of a number of important topics in mathematics. Along the way both ideas and methods are stressed, to be learnt in an interactive manner through discussion in tutorials and group work on homework assignments. A rough sketch of the content of the course is summarized in Table 1. Considering the level of the course, it is to be expected that topics near to the end are treated only after a fashion, mainly for broadening the vista of the students rather than for teaching them the technical details.

Table 1

3 A sketch of the content of the course

Each session of the enrichment course consists of a lecture in the first hour followed by a tutorial. The lecture serves to highlight some keypoints and outline the development of the topic. What is covered is selective in the sense that the material illustrates some theme rather than provides a comprehensive account. Interested students are advised to read up on their own relevant references suggested in each session. [A selected sample of such books can be found in the list of references, some of which are more suitable for the teacher than the student (Barnett, 1949; Boyer, 1968; Einstein & Infeld, 1938; Feynman, 1995; Hewitt, 2006; Lines, 1994; Longair, 1984; Mills, 1994; Olenik, Apostol & Goldstein, 1985/1986; Pólya, 1963; Siu, 1993).] The course is seen as a means to arouse, to foster and to maintain the enthusiasm of students in mathematics and physics more than as a means to equip them with a load of knowledge.

To keep within the prescribed length of the paper I would not give a full account of the content but select certain parts, particularly the beginning part that sets the tone of the course, with supplementary commentary, to illustrate the intent of the enrichment course. The intent is to highlight the beautiful (some would say uncanny!) and intimate relationship between mathematics and physics, in many cases even mathematical ideas that have lain quietly in waiting for many years (sometimes more than a thousand years!) that enhance theoretical understanding of physical phenomena. In fact the relationship is two-way so that the two subjects benefit mutually from each other in their development. In section 4 some sample problems in tutorials are appended in the hope of better illustrating this intention.

The course begins with a discussion on the Aristotelian view of the physical world that came to be known since the 4th century B.C.. All terrestrial matters, which are

held to be different from heavenly matters, are believed to contain a mixture of the four elements in various compositions. Each of the four elements is believed to occupy a natural place in the terrestrial region, in the order of earth (lowest), water, air, fire (uppermost). Left to itself, the natural motion of an object is to go towards its natural position, depending on the composition and the initial position. Hence, a stone (earth) falls to the ground but a flame (fire) goes up in the air. A natural motion has a cause. It is believed that the weight of a stone is the cause for its free falling motion. According to the Aristotelian view, a heavier stone will fall faster than a lighter one. Any motion that is not a natural motion is called a violent motion, believed to be caused by a force.

We next bring in the physical world view that Galileo propounded in the first part of the 17th century. In particular, he demolished the theory that a heavier object falls faster by mathematical reasoning (thought-experiment) in *Discorsi e dimonstrazioni matematiche intorno a due nuove scienze* (*Discourses and Mathematical Demonstrations Concerning Two New Sciences*) of 1638. Suppose object A_1 has a larger weight W_1 than the weight W_2 of object A_2. Tie the objects A_1 and A_2 together to form an object of weight $W_1 + W_2$. The more rapid one will be partly retarded by the slower; the slower one will be somewhat hastened by the swifter. Hence, the united object will fall slower than A_1 alone but faster than A_2 alone. However, the united object, being heavier than A_1, should fall faster than A_1 alone. This is a contradiction! [Hawking, 2002, p.446]. A commonly told story says that Galileo dropped two balls of different weights from the top of the Tower of Pisa to arrive at his conclusion. There is no historical evidence that he actually did that. The significant point does not lie so much in whether Galileo actually carried out the experiment but in his arrival at the conclusion by pure reasoning. Together with pure reasoning, Galileo was known for his emphasis on observations and experiments as well, notably his experiments with an inclined plane. By observing that a ball rolling down an inclined plane will travel up another inclined plane joined to the first one at the bottom until it reaches the same height, he saw that the ball will travel a greater distance if the second inclined plane is placed less steep than the first one, the greater if the second inclined plane is less steep. From thence a thought-experiment comes in again. If the second inclined plane is actually placed in a horizontal position, the ball will travel forever without stopping. "Furthermore we may remark that any velocity once imparted to a moving body will be rigidly maintained as long as the external causes of acceleration or retardation are removed, a condition which is found only on horizontal planes. ... it follows that motion along a horizontal plane is perpetual ..." [Hawking, 2002, p.564]. This motivated him to announce his famous law of inertia, which becomes the first law of motion in Newton's *Philosophiae naturalis principia mathematicas* (*Mathematical Principles of Natural Philosophy*) of 1687: "Every body persevers in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon." [Hawking, 2002, p.743]. This fundamental modification on the Aristotelian view (in a sense actually more natural according to daily experience!) that a force acting on an object is exemplified not by the speed of its motion but by the change in speed (acceleration), led to a quantitative description of this relationship in Newton's second law of motion (which yields the famous formula $F = ma$). It turned a new page in the development of physics. We follow with a discussion on the work of Johannes Kepler in calculating the orbit of Mars based on the meticulously kept observed data of Tycho Brahe [Koestler, 1959]. On the one hand the story displays a beautiful interplay between theory and experiment. On the other hand Kepler's laws on planetary motion provide a nice lead to a discussion on Newton's law of universal gravitation.

We next discuss the theory of wave motion along with the mathematics, culminating in the theory of electromagnetism and Maxwell's equations. Mathematics owed to physics a great debt in that a large part of mathematical analysis that was developed in the 18th and 19th centuries have to do with the Vibrating String Problem. We talk about the all-important notions of function and of equation. Together with the discussion on vector calculus and the generalized Fundamental Theorem of Calculus, there is much more material than enough to take up the second part of the course. The unification of electricity, magnetism and light through the electromagnetic wave is a natural lead into the final third of the course, which is spent on a sketch of the theory of relativity and on quantum mechanics. Some probability theory is introduced to let students appreciate the stochastic aspect that is not usually encountered in the usual school curriculum. The close relationship between geometry and physics is stressed in the final episode on the theory of general relativity. In a letter to Arnold Sommerfeld dated October 29, 1912 (collected in A. Hermann, *Einstein/Sommerfeld Briefwechsel*, Schwabe Verlag, Stuttgart, 1968, p.26) Albert Einstein wrote, "I am now exclusively occupied with the problem of gravitation, and hope, with the help of a local mathematician friend, to overcome all the difficulties. One thing is certain, however, that never in my life have I been quite so tormented. A great respect for mathematicians has been instilled within me, the subtler aspects of which, in my stupidity, I regarded until now as pure luxury. Against this problem, the original problem of the theory of relativity is child's play." The 'mathematician friend' refers to Einstein's school friend Marcel Grossmann, and the mathematics refers to Riemannian geometry and tensor calculus. The story on the work of Carl Friedrich Gauss and Georg Friedrich Bernhard Riemann in revealing the essence of curvature which lies at the root of the controversy over the Fifth Postulate in Euclid's *Elements* (but which had been masked for more than two thousand years when the attention of mathematicians was directed into a different direction) and its relation to Einstein's idea on gravitation theory is fascinating for both mathematics and physics. No wonder Riemann concluded his famous 1854 lecture titled *Über die Hypothesen welche der Geometrie zu Grunde liegen* (*On the hypotheses which lie at the foundation of geometry* (an English translation can be found in David Eugene Smith (ed.), *A Source Book in Mathematics*, McGraw-Hill, New York, 1929, pp.411-425) with: "This path leads out into the domain of another science, into the realm of physics, into which the nature of this present occasion forbids us to penetrate."

4 Some sample problems in tutorials

In this course more than half of the time in each session is spent as a tutorial, which is regarded as an integral part of the learning experience. Students work in small groups with guidance or hint provided on the side by the teacher and a team of (four) teaching assistants. At the end of each session there is a guided discussion with presentations by students. A more detailed record of the solution is put on the web afterward for those who are interested to probe further. Some sample problems in the tutorials are given below to convey a flavour of the workshop.

Question 1. A, B, C, D move on straight lines on a plane with constant speeds. (The speed of each chap may be different from that of another.) It is known that each of A and B meets the other three chaps at **distinct** points. Must C and D meet?

Under what condition will the answer be 'yes' (or 'no')? [The question was once given in an examination at Oxford University.]

Discussion: C and D will (respectively will not) meet if they do not move (respectively move) in the same or opposite directions. The catch is a commonly mistaken first reaction to draw a picture with two straight lines emanating from a common point M_{AB} (the point where A and B meet) and two more straight lines, one intersecting the first line at M_{AC} and the second line at M_{BC}, the other one intersecting the first line at M_{AD} and the second line at M_{BD}. It seems that the answer comes out obviously from the picture until one realizes that a geometric intersecting point needs not be a physical intersecting point! This problem is set as the first problem in the first tutorial to lead the class onto the important notion of spacetime, which will feature prominently in the theory of relativity. Viewed in this context, no calculation is needed at all!

Question 2. Suppose you only know how to calculate the area of a rectangle — our ancestors started with that. Explain how you would calculate the area of a triangle by approximating it with many many rectangles of very small width. This answer, by itself, does not sound too exciting. You can obtain it by other means, for instance by dissection — our ancestors did just that! However, what is exciting is the underlying principle that can be adapted to calculate the area of regions of other shapes. Try to carry out a similar procedure for a parabolic segment. (Find the area under the curve given by $y = kx^2$ from $x = 0$ to $x = a$. What happens if you are asked to find the area under the curve $y = kx^3$? $y = kx^4$? $\cdots$? Later you will see how a result enables us to solve this kind of problem in a uniform manner.)

Discussion: This problem is set at the beginning of the course to introduce some ideas and methods devised by ancient Greeks and ancient Chinese on problems in quadrature, to be contrasted with the power of calculus developed during the 17th and 18th centuries, culminating in the Fundamental Theorem of Calculus with its generalized form (Stokes' Theorem) established through the development of the theory of electromagnetism in the 19th century. For this particular problem some clever formulae on the sum of consecutive rth power of integers $1^r + 2^r + 3^r + \cdots + N^r$ are needed. That kind of calculation is not totally foreign to the experience of school pupils and yet offers some challenge beyond what they are accustomed to, which is therefore of the level of difficulty the workshop is gauged at. After struggling with specific but seemingly ad hoc 'tricks' of this sort, students would appreciate better the power afforded by the Fundamental Theorem of Calculus when they learn it later.

Question 3. (a) By computing the sum

$$1 + z + z^2 + \cdots + z^n$$

where $z = e^{i\theta}$, and using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

find a simple expression for

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$$

and

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta.$$

(b) Apply the result in **(a)** to calculate the area under the curve $y = \sin x$ on $[0, \pi]$ from scratch in the way you did for $y = x^2$ in the first tutorial. Do the same for $y = \cos x$ on $[0, \pi]$. (How do you normally calculate this area in your class at school?)

Discussion: Besides introducing a most beautiful formula in mathematics, this problem further strengthens students' appreciation of the Fundamental Theorem of Calculus. In the course of explaining Euler's formula students are led into the realm of complex numbers, to the 'twin' functions of logarithm and exponentiation.

Question 4. (a) Pierre Simon Laplace (1749-1827) once said, "By shortening the labors, the invention of logarithms doubled the life of the astronomer." To appreciate this quotation, let us work on an multiplication problem (81276×96343) like people did before the invention of logarithm. The method, known as "prosthaphaeresis", is based on the addition formula of trigonometric functions.

(i) If $2\cos A = 0.81276$ and $\cos B = 0.96343$, find A and B.

(ii) Calculate $A + B$, $A - B$, and hence calculate $\cos(A + B)$, $\cos(A - B)$.

(iii) Calculate $\cos(A+B)+\cos(A-B)$, which is equal to $2\cos A\cos B$, and hence find out what 81276×96343 is.

(b) Compare Napier's logarithm with the natural logarithm you learn in school.

(c) Making use of the idea Leonhard Euler (1707-1783) explained in Chapter XXII of his *Vollständige Anleitung zur Algebra* (1770), compute the common logarithm of 5, $\log 5$, in the following steps:

(i) As 5 lies between 1 and 10, so $\log\ 5$ lies between 0 and 1. Take the average of 0 and 1, which is $1/2$. Compute $10^{1/2}$, which is the square root of 10, say a_1.

(ii) Decide whether 5 falls into $[1, a_1]$ or $[a_1, 10]$. Hence decide whether $\log\ 5$ falls into $[0, 1/2]$ or $[1/2, 1]$. It turns out $\log\ 5$ falls into $[1/2, 1]$. Take the average of $1/2$ and 1, which is $3/4$. Compute $10^{3/4}$, which is the square root of 10 multiplied by the square root of $10^{1/2}$, say a_2.

(iii) Decide whether 5 falls into $[a_1, a_2]$ or $[a_2, 10]$. Hence decide whether $\log\ 5$ falls into $[1/2, 3/4]$ or $[3/4, 1]$. It turns out $\log\ 5$ falls into $[1/2, 3/4]$. Take the average of $1/2$ and $3/4$, which is $5/8$. Compute $10^{5/8}$, which is the square root of $10^{1/2}$ multiplied by the square root of $10^{3/4}$, say a_3.

(iv) Continue with the algorithm until you reach a value of $\log 5$ accurate to three decimal places.

Discussion: Note the similar underlying idea of converting multiplication to addition in "prosthapharesis" and in logarithm. That allows the class to see how John Napier and later Henry Briggs devised their logarithm in the early 17th century. The bisection algorithm explained in **(c)**, though seemingly cumbersome from a modern viewpoint, is nonetheless very natural and simple, reducing the calculation to only finding square root. It provides an opportunity to go into the computation of square root by the ancients, first propounded in detail in the ancient Chinese classics *Jiu Zhang Suan Shu* (*Nine Chapters on the Mathematical Art*) compiled between 100 B.C. and 100 A.D. For the generation of youngsters who grow up with calculators and computers, this kind of 'old' techniques may add a bit of amazement as well as deeper comprehension.

Question 5. In an $x - t$ spacetime diagram drawn by an observer S who regards himself as stationary, draw the world-line for S and the world-line for an observer S'

moving with uniform velocity v (relative to S). At $t = 0$ both S and S' are at the origin O. Both S and S' observe a light signal sent out from O at $t = 0$, reflected back by a mirror at a point P, then received by S' at Q. Which point on the world-line for S' will S' regard as an event **simultaneous** with the reflection of the light signal at P? Call this point P'. Show that the slope of the line $P'P$ is equal to v/c^2, where c is the speed of light (units omitted). [The physical interpretation is as follows. S regards two events, perceived as simultaneous by S', as separated by a time Δt given by $\Delta t = (v/c^2)\Delta x$, where Δx is the distance between the events measured by S and v is the velocity of S' relative to S.]

Discussion: We pay attention to the physical interpretation of a mathematical calculation and vice versa. This problem focuses on the key notion of simultaneity in the theory of special relativity. There is a note of caution for this problem. The picture of the spacetime diagram (according to the observer S) is to be seen in two ways: (i) the picture as it is, just like a picture one is accustomed to see in school geometry, (ii) the coordinate system of S with coordinates assigned to each event. In the lecture we take good care in denoting points in (i) by letters P, Q, P', O, etc., and events in (ii) by $(x(P), t(P)), (x(Q), t(Q)), (x(P'), t(P')), (x(O), t(O))$, etc. One can read the same in the shoes of the other observer S', in which case events in (ii) will be denoted by $(x'(P), t'(P)), (x'(Q), t'(Q)), (x'(P'), t'(P')), (x'(O), t'(O))$, etc. In the lecture we also explain how $x(P), t(P)$ are related to $x'(P), t'(P)$ and vice versa (by the Lorentz transformation).

5 Conclusion

The triumph of Maxwell's theory on electromagnetism resolved many problems and yet introduced new difficulties that were resolved by Einstein's theory of special relativity. The triumph of Einstein's theory of special relativity resolved many problems and yet introduced new difficulties that were resolved by Einstein's theory of general relativity. But then the theory of general relativity introduces a more difficult problem on incompatibility with quantum mechanics, which is not revealed until one comes up with a situation where both the mass involved is very large and the size involved is very small, for instance, a black hole [Greene, 1999; Penrose, 2004]. Physics will march on to solve further problems, and so will mathematics, hand-in-hand with physics, in a harmonious way.

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Time period	**Physics**	**Mathematics (mainly)**
4th century B.C.	Physical view of Aristotle	Euclidean geometry
many centuries in between		geometry (area / volume) algebra (equations)
17th century	physical view of Copernicus, Kepler, Galileo, Newton, ...	vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, calculus in one variable (functions, including polynomial, rational, trigonometric, logarithmic and exponential)
18th century	wave and particle	differential equations, Fourier analysis, complex numbers
19th century	theory of electromagnetism (Maxwell's equations)	vector calculus, Stokes' Theorem (Fundamental Theorem of Calculus)
20th century	theory of special and general relativity, quantum mechanics	probability theory, non-Euclidean geometries of spacetime

Table 1

A dialogue between Mathematics & Physics: Examples in Secondary School Curriculum

Choi Wai Fung, Brian (St. Paul's Co-educational College) 21st April, 2016

I. Mechanics

1. Kinematics & Dynamics

(a)Work done by an applied force $\vec{F}$ with displacement $\vec{s}$ $\qquad W = \vec{F}\cdot\vec{s} = Fs\cos\theta$

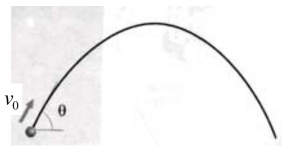
(b) Power by an applied force $\vec{F}$ with velocity $\vec{v}$ $\qquad P = \frac{dW}{dt} = \vec{F}\cdot\vec{v} = Fv\cos\theta$

(c) Moment (or torque) by an applied force $\vec{F}$ with distance $\vec{r}$ from a fixed point

$\vec{\mathrm{T}} = \vec{r}\times\vec{F} = rF\sin\theta\,\hat{\mathbf{u}}$

2. Projectile motion

When an object is thrown on the ground with an angle θ from the horizontal, the maximum range of an object is obtained when $\theta = \frac{\pi}{4}$, which is independent of the initial velocity v_0.



	Horizontal Component	Vertical Component (Taking upward as positive)
Initial velocity	$u_x = v_0\cos\theta$	$u_y = v_0\sin\theta$
Final velocity	$v_x = v_0\cos\theta$	$v_y = -v_0\sin\theta$
Acceleration	$a_x = 0$	$a_y = -g$
Time	t	

Vertical Acceleration: $\qquad a_y = \frac{v_y - u_y}{t}$

$$-gt = (-v_0\sin\theta) - v_o\sin\theta$$

$$t = \frac{2v_0\sin\theta}{g}$$

Horizontal Displacement: $\qquad x = v_x t = (v_0\cos\theta)\left(\frac{2v_0\sin\theta}{g}\right) = \frac{v_0{}^2(2\sin\theta\cos\theta)}{g} = \frac{v_0{}^2\sin 2\theta}{g}$

x is max. when $\sin 2\theta = 1$, i.e. $2\theta = \frac{\pi}{2}$, i.e. $\theta = \frac{\pi}{4}$.

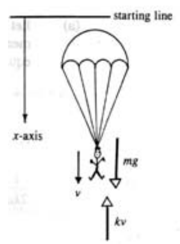
The use of double angle formula in trigonometry helps to determine the exact value of the angle for max. range.

Extension: If the object is projected on an inclined plane with angle θ and the angle between the initial velocity and the plane is α, then for max. range, $\alpha = \frac{\pi}{4} - \frac{\theta}{2}$.

Web link: http://ggbtu.be/m1082291

3. Motion under gravity with air resistance

Consider an object with mass m falling under gravity with air resistance and initial velocity v_0.



Suppose air resistance $\propto$ velocity v. Let the air resistance $= kv$, where k is a positive constant.

Net force acting on the object $F = mg - kv$.

If $k = \dfrac{mg}{v_0}$, net force is 0, so the object will fall under constant velocity v_0.

If $k \neq \dfrac{mg}{v_0}$, by Newton's 2nd Law, $ma = mg - kv$

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

$$\int_{v_0}^{v} \frac{dv}{g - \frac{k}{m}v} = \int_0^t dt$$

$$\int_{v_0}^{v} \frac{d\left(g - \frac{k}{m}v\right)}{g - \frac{k}{m}v} = -\frac{k}{m}t$$

$$\left[\ln\left(g - \frac{k}{m}v\right)\right]_{v_0}^{v} = -\frac{k}{m}t$$

$$\ln\left(g - \frac{k}{m}v\right) - \ln\left(g - \frac{k}{m}v_0\right) = -\frac{k}{m}t$$

$$\ln\left(\frac{mg - kv}{mg - kv_0}\right) = -\frac{k}{m}t$$

$$\frac{mg - kv}{mg - kv_0} = e^{-\frac{k}{m}t}$$

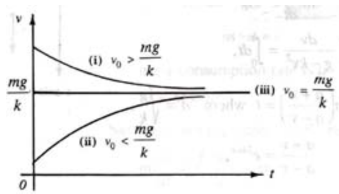
$$mg - kv = e^{-\frac{k}{m}t}(mg - kv_0)$$

$$kv = mg + e^{-\frac{k}{m}t}(kv_0 - mg)$$

$$v = \frac{mg}{k} + e^{-\frac{k}{m}t}\left(v_0 - \frac{mg}{k}\right)$$

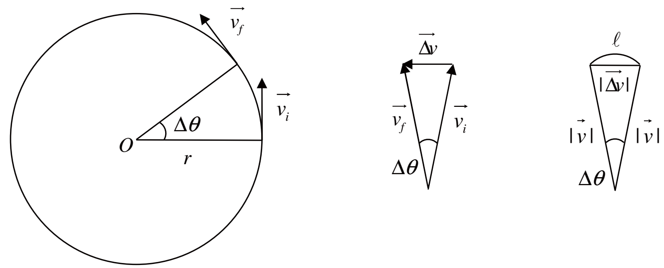
In the long run, $\lim_{t \to \infty} v = \lim_{t \to \infty}\left[\frac{mg}{k} + e^{-\frac{k}{m}t}\left(v_0 - \frac{mg}{k}\right)\right] = \frac{mg}{k}$

The limit $\dfrac{mg}{k}$ is called the terminal velocity.



4. Circular Motion

To obtain the expression of centripetal acceleration, we have to use vectors operation and limits.



In the first figure, the velocity vectors $\vec{v_i}$ and $\vec{v_f}$ are tangential to the circular path. Thus they are perpendicular to the radius.

Consider the direction of acceleration, i.e. the direction of $\overrightarrow{\Delta v}$. When $\Delta\theta \to 0$, $\overrightarrow{\Delta v}$ will tend to a direction perpendicular to $\vec{v_i}$ and $\vec{v_f}$, i.e. $\overrightarrow{\Delta v}$ is along radial direction towards the centre. Hence the acceleration is called centripetal acceleration.

Consider the magnitude of $\overrightarrow{\Delta v}$, when $\Delta\theta \to 0$, $|\overrightarrow{\Delta v}| \approx \ell = |\vec{v}|\Delta\theta$.

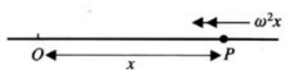
$\therefore$ centripetal acceleration $|\vec{a}| = \lim_{\Delta t\to 0}\frac{|\overrightarrow{\Delta v}|}{\Delta t} = \lim_{\Delta t\to 0}\frac{|\vec{v}|\Delta\theta}{\Delta t} = \lim_{\Delta t\to 0}|\vec{v}|\frac{\Delta\theta}{\Delta t} = \lim_{\Delta t\to 0}\frac{|\vec{v}|}{r}\frac{r\Delta\theta}{\Delta t}$

$= \frac{|\vec{v}|}{r}\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \frac{|\vec{v}|^2}{r}$

In scalar notation, $a = \frac{v^2}{r}$.

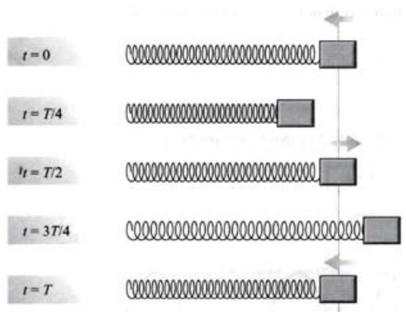
5. Simple Harmonic Motion (S.H.M.)

(a) Definition: Relation between acceleration and displacement is given by $a = -kx$, where k is a positive constant.



(i) k is usually rewritten as ω^2, where $\omega > 0$ is called the angular frequency.

(ii) Consider $a = \frac{d^2x}{dt^2}$. The relation is changed to a 2nd order ordinary differential equation $\frac{d^2x}{dt^2} + \omega^2 x = 0$. The general solution is given by $x = A\cos\omega t + B\sin\omega t$, where A and B are constants.



(iii) Initial condition:

If the object starts from the equilibrium position, then $x = B\sin\omega t$, where $B \neq 0$.

If the object starts from the extreme position, then $x = A\cos\omega t$, where $A \neq 0$.

(iv) Simple harmonic motion is isochronous, i.e. the period of S.H.M. is independent of its amplitude, as shown in the general solutions.

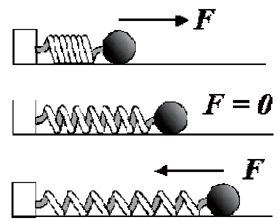
(b) Example 1: Spring Mass System

(i) Horizontal System

Consider an object with mass m which is tied to a spring of force constant k along a horizontal ground.

When the mass is displaced about its equilibrium position with displacement x,

by Hooke's Law, force $F = -kx$



By Newton's 2nd Law, $ma = -kx$

$a = -\frac{k}{m}x$

$\therefore \omega^2 = \frac{k}{m}$ angular velocity $\omega = \sqrt{\frac{k}{m}}$

Period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$, which is independent of the amplitude of oscillation.

If we consider the system starting from extreme position A,

$x = A\cos\omega t$

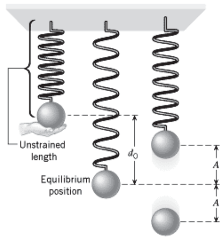
$v = \frac{dx}{dt} = -\omega A\sin\omega t$

Elastic Potential Energy $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2\omega t$

Kinetic Energy $K=\frac{1}{2}mv^2=\frac{1}{2}m\omega^2A^2\sin^2\omega t=\frac{1}{2}kA^2\sin^2\omega t$

Total Energy $=U+K=\frac{1}{2}kA^2\cos^2\omega t+\frac{1}{2}kA^2\sin^2\omega t=\frac{1}{2}kA^2$, which is a constant.

(ii) Vertical System



Consider an object with mass m which is tied to a spring of force constant k vertically.

At equilibrium position, there is an extension e of the spring. $\because$ net force = 0, $\therefore mg=ke$.

When the mass is displaced downwards about its equilibrium position with displacement x,

Net force: $ma=mg-k(e+x)$ (taking downwards as positive)

$ma=mg-ke-kx$

$ma=-kx$

$a=-\frac{k}{m}x$, which is the same as the case of horizontal system.

(iii) Inclined System

Consider an object with mass m which is tied to a spring of force constant k along an inclined plane with angle θ.

At equilibrium position, there is an extension e of the spring. $\because$ net force = 0,

$\therefore mg\sin\theta=ke$

When the mass is displaced downwards about its equilibrium position with displacement x,

Net force: $ma=mg\sin\theta-k(e+x)$ (taking downwards as positive)

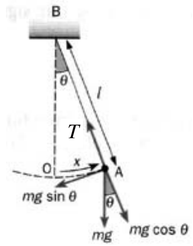
$ma=mg\sin\theta-ke-kx$

$ma=-kx$

$a=-\frac{k}{m}x$, which is still the same as the case of horizontal system.

(c) Example 2: Simple Pendulum

The 2 forces acting on an object in simple pendulum are the gravitational force mg and the tension T. Let θ be the angle between the vertical and the string and l be the length of the string.



Consider radial component, $mg\cos\theta=T$.

Net force along tangential direction: $ma=-mg\sin\theta$

$a=-g\sin\theta$

Arc length $x=l\theta$

If $\theta\to 0$, then $\sin\theta\approx\theta$, equivalently $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$.

$\therefore a\approx -g\theta=-g\left(\frac{x}{l}\right)=-\left(\frac{g}{l}\right)x$

6. Damped Harmonic Motion (D.H.M.)

Consider a mass–spring system with mass m and force constant of the spring k. If there is a damping force $F\propto v$, i.e. $F=-b\frac{dx}{dt}$, where b is a constant,

then net force $=-kx-b\frac{dx}{dt}$

By Newton's 2^{nd} Law, $m\dfrac{d^2x}{dt^2} = -kx - b\dfrac{dx}{dt}$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 ......(*)$$

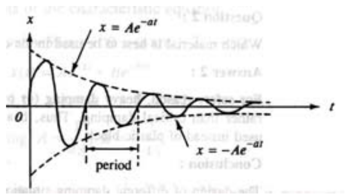
Let $2a = \dfrac{b}{m}$, $\omega^2 = \dfrac{k}{m}$, so the characteristic equation of (*) is

$$\alpha^2 + 2a\alpha + \omega^2 = 0$$

$$\alpha^2 + 2a\alpha + a^2 = a^2 - \omega^2$$

$$(\alpha + a)^2 = a^2 - \omega^2$$

$$\alpha = -a \pm \sqrt{a^2 - \omega^2}$$



We consider the case of light damping, i.e. b is a small number such that $\omega^2 > a^2$. Then the roots of the characteristic equation are both complex.

The general solution of (*) is $x = Ae^{-at}\cos(t\sqrt{\omega^2 - a^2} + \phi)$, where A and ϕ are constants determined by initial conditions.

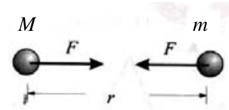
For simplicity, consider the object at the extreme position initially, $\therefore$ take $\phi = 0$.

$$\therefore x = Ae^{-at}\cos(t\sqrt{\omega^2 - a^2})$$

Period of the oscillation $T = \dfrac{2\pi}{\sqrt{\omega^2 - a^2}}$, which is larger than S.H.M. without damping.

The amplitude of oscillation is affected by the damping factor e^{-at}. Indeed the 2 graphs $x = \pm Ae^{-at}$ form the envelopes of the graph of the D.H.M.

<u>7. Gravitational Field</u>



(a) Newton's Law of Gravitation: gravitational force $F = \dfrac{GMm}{r^2}$

(b) Gravitational field strength g is the gravitational force per unit mass, i.e. $g = \dfrac{F}{m} = \dfrac{GM}{r^2}$.

Equivalently, $F = mg$.

(c) Gravitational potential energy U is the work done by the gravitational force in moving a mass m from infinity to r.

Work done on a mass by an infinitesimal displacement $= \vec{F}\cdot\vec{dr} = Fdr\cos 0° = Fdr$

$$U = \int_\infty^r Fdr = \int_\infty^r \frac{GMm}{r^2}dr = \left[\frac{-GMm}{r}\right]_\infty^r = \frac{-GMm}{r}$$

Equivalently, $F = \dfrac{dU}{dr}$.

(d) Gravitational potential is the gravitational potential energy per unit mass, i.e. $V = \dfrac{U}{m} = -\dfrac{GM}{r}$, or equivalently $U = mV$.

Moreover, $\because F = \dfrac{dU}{dr}$, $mg = \dfrac{d(mV)}{dr}$, $g = \dfrac{dV}{dr}$, or equivalently $V = \displaystyle\int_\infty^r gdr$.

The following figure shows the relationship between F, g, U and V.

$$\begin{tikzcd}
F = \dfrac{GMm}{r^2} \arrow[rr, leftrightarrow, "F = \dfrac{dU}{dr}"] \arrow[d, leftrightarrow, "F = mg"'] & & U = \dfrac{-GMm}{r} \arrow[d, leftrightarrow, "U = mV"] \\
g = \dfrac{GM}{r^2} \arrow[rr, leftrightarrow, "g = \dfrac{dV}{dr}"'] & & V = -\dfrac{GM}{r}
\end{tikzcd}$$

(e) The approximation of gravitational potential energy near Earth surface can be obtained by the following:

Let R be the radius of the Earth. Consider a mass m which is of height h above the Earth surface, and $h << R$.

$$U = -\frac{GMm}{R+h} = -\frac{GMm}{R\left(1+\frac{h}{R}\right)} = -\frac{GMm}{R}\left(1+\frac{h}{R}\right)^{-1}$$

By first order approximation of Binomial Expansion, $U \approx -\frac{GMm}{R}\left(1-\frac{h}{R}\right)$

$U \approx -\frac{GMm}{R} + \frac{GMm}{R^2}h$, where $-\frac{GMm}{R}$ is the gravitational potential energy on the Earth surface, denoted it by U_0.

$$U \approx U_0 + \frac{GM}{R^2}mh$$

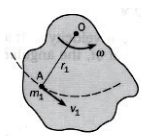
Change in gravitational potential energy $\Delta U = U - U_0 \approx \frac{GM}{R^2}mh = mgh$

8. Rotational Motion of a Rigid Body

(a) Moment of Inertia

Consider a rigid body rotating in a constant angular velocity ω. The body is divided into small mass m_i, each with a distance r_i from the axis of rotation.

K.E. of a small mass $= \frac{1}{2}m_i v_i^{\ 2} = \frac{1}{2}m_i(r\omega_i)^2 = \frac{1}{2}\omega^2(m_i r_i^{\ 2})$



K.E. of the rigid body $= \int \frac{1}{2}\omega^2 r^2 dm = \frac{1}{2}\omega^2 \int r^2 dm$

Define moment of inertia $I = \int r^2 dm$

$\therefore$ K.E. of the rigid body $= \frac{1}{2}I\omega^2$, compared with a rigid body in linear motion K.E. $= \frac{1}{2}mv^2$

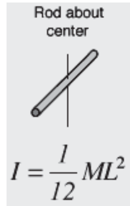
Hence moment of inertia takes the role of mass in rotational motion.

(b) Example of moment of inertia

Consider a uniform rod of mass m, length L and linear density $\rho = \frac{m}{L}$

infinitesimal mass with infinitesimal length: $dm = \rho dr$

moment of inertia $I = \int r^2 dm = \int r^2 \rho dr = \int r^2 \left(\frac{m}{L}\right) dr = \frac{m}{L}\int r^2 dr$

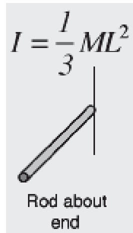


(i) Rotating along an axis passing through its mid–point

$$I = \frac{m}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 dr = \frac{2m}{L}\int_0^{\frac{L}{2}} r^2 dr = \frac{2m}{L}\left[\frac{r^3}{3}\right]_0^{\frac{L}{2}} = \frac{2m}{3L}\left(\frac{L^3}{8}\right) = \frac{mL^2}{12}$$

(ii) Rotating along one end of the rod

$$I = \frac{m}{L}\int_0^L r^2 dr = \frac{m}{L}\left[\frac{r^3}{3}\right]_0^L = \frac{m}{3L}(L^3) = \frac{mL^2}{3}$$



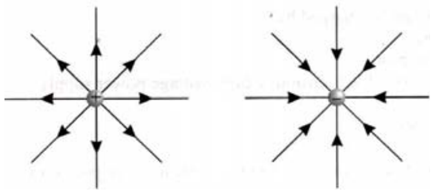
II. Electricity and Magnetism

1. Electric Field

(a) Electric force on a charge q by a fixed charge Q is given by

$F = \dfrac{Qq}{4\pi\varepsilon_0 r^2}$. (Coulomb's Law)

(b) Electric field strength E by a fixed charge Q is electric force per unit charge i.e. $E = \dfrac{F}{q} = \dfrac{Q}{4\pi\varepsilon_0 r^2}$. Equivalently $F = qE$.



(c) Electric potential energy U is the work done by the electric force F in moving a charge q from infinity to r.

If charges Q and q are of the same sign, the external force required to put the charge q from infinity is opposite to the electric force. Mathematically, by scalar product

work done on a charge q by an infinitesimal displacement $= \vec{F} \cdot \vec{dr} = Fdr\cos 180° = -Fdr$

Work done to put the charge q from infinity to a distance r from charge Q is given by

$$U = -\int_\infty^r Fdr = -\int_\infty^r \frac{Qq}{4\pi\varepsilon_0 r^2}dr = \left(\frac{-Qq}{4\pi\varepsilon_0}\right)[-r^{-1}]_\infty^r = \frac{Qq}{4\pi\varepsilon_0 r}$$

Equivalently, $F = -\dfrac{dU}{dr}$.

(d) Electric potential is the electric potential energy per unit test charge, i.e. $V = \dfrac{U}{q} = \dfrac{Q}{4\pi\varepsilon_0 r}$.

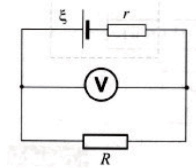
Equivalently $U = qV$.

Moreover, $\because F = -\dfrac{dU}{dr}$, $qE = -\dfrac{d(qV)}{dr}$, $\therefore E = -\dfrac{dV}{dr}$, or equivalently $V = -\int_\infty^r Edr$.

The following figure shows the relationship between F, E, U and V.

$$\begin{tikzcd}
F = \dfrac{Qq}{4\pi\varepsilon_0 r^2} \arrow[rr, leftrightarrow, "F = -\frac{dU}{dr}"] \arrow[d, leftrightarrow, "F = qE"'] & & U = \dfrac{Qq}{4\pi\varepsilon_0 r} \arrow[d, leftrightarrow, "U = qV"] \\
E = \dfrac{Q}{4\pi\varepsilon_0 r^2} \arrow[rr, leftrightarrow, "E = -\frac{dV}{dr}"'] & & V = \dfrac{Q}{4\pi\varepsilon_0 r}
\end{tikzcd}$$

2. Circuit with internal resistance in battery



Let the electromotive force (e.m.f.) of a battery be V_0, with internal resistance r and equivalent resistance in the circuit R.

By Ohm's Law, current $I = \dfrac{V_0}{r+R}$

Power consumed by the internal resistance $P = I^2 r = \left(\dfrac{V_0}{r+R}\right)^2 r = \dfrac{V_0^2 r}{(r+R)^2}$

Consider the extremum of this power,

$$\frac{dP}{dr} = V_0^2 \times \frac{(r+R)^2(1) - r(2)(r+R)(1)}{(r+R)^4} = \frac{V_0^2}{(r+R)^3}[(r+R) - 2r] = \frac{V_0^2(R-r)}{(r+R)^3}$$

When $\dfrac{dP}{dr} = 0$, $r = R$.

By first derivative test, P is max. when $r = R$.

Max. power consumed by the internal resistance $P = \dfrac{V_0{}^2 R}{(R+R)^2} = \dfrac{V_0{}^2 R}{(2R)^2} = \dfrac{V_0{}^2}{4R}$

On the other hand, the total power consumed $P_0 = \dfrac{V_0{}^2}{2R} = 2P$

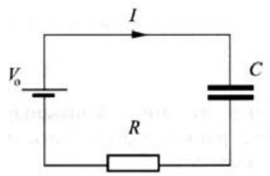
3. Capacitance

Definition: Capacitance is the charge stored per unit voltage. $C = \dfrac{Q}{V}$

Charging a capacitor

Consider a circuit with equivalent capacitance C and equivalent resistance R. The capacitor contains zero charge initially.

Potential difference:



$$V_0 = V_C + V_R = \frac{Q}{C} + IR = \frac{Q}{C} + R\frac{dQ}{dt}$$

$$\frac{dQ}{dt} = \frac{1}{R}\left(V_0 - \frac{Q}{C}\right) = \frac{CV_0 - Q}{RC}$$

$$\int_0^Q \frac{dQ}{CV_0 - Q} = \int_0^t \frac{dt}{RC}$$

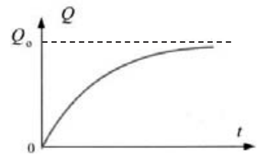
$$-\int_0^Q \frac{d(Q_0 - Q)}{Q_0 - Q} = \left[\frac{t}{RC}\right]_0^t$$

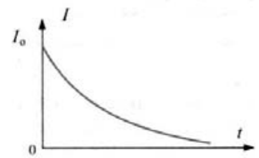
$$[\ln(Q_0 - Q)]_0^Q = \frac{-t}{RC}$$

$$\ln(Q_0 - Q) - \ln Q_0 = \frac{-t}{RC}$$

$$\ln\left(\frac{Q_0 - Q}{Q_0}\right) = \frac{-t}{RC}$$

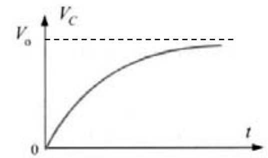
$$1 - \frac{Q}{Q_0} = e^{\frac{-t}{RC}}$$





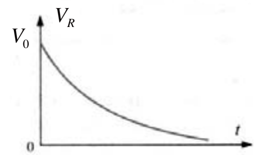
$$\frac{Q}{Q_0} = 1 - e^{\frac{-t}{RC}}$$

$$Q = Q_0\left(1 - e^{\frac{-t}{RC}}\right)$$



$$I = \frac{dQ}{dt} = Q_0\left(-e^{\frac{-t}{RC}}\right)\left(\frac{-1}{RC}\right) = \frac{CV_0}{CR}e^{\frac{-t}{RC}} = \frac{V_0}{R}e^{\frac{-t}{RC}} = I_0 e^{\frac{-t}{RC}}$$

$$V_C = \frac{Q}{C} = \frac{Q_0}{C}\left(1 - e^{\frac{-t}{RC}}\right) = V_0\left(1 - e^{\frac{-t}{RC}}\right)$$



$$V_R = IR = I_0 R(e^{\frac{-t}{RC}}) = V_0 e^{\frac{-t}{RC}}$$

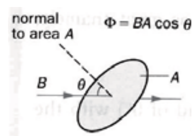
Define the time constant be RC.

When $t = 5RC$, $Q = Q_0(1 - e^{-5}) \approx 0.993Q_0$, the capacitor is regarded as fully charged.

4. Electromagnetism

(a) Magnetic Flux ϕ and Magnetic Field Strength $\vec{B}$

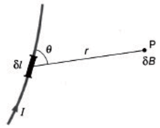
(i) $\phi = \vec{B} \cdot \vec{A} = BA\cos\theta$, where A is the area that the magnetic flux passing through, θ is the angle between $\vec{B}$ and $\vec{A}$.



(ii) magnetic flux linkage $\Phi = N\phi$, where N is the number of turns of a coil

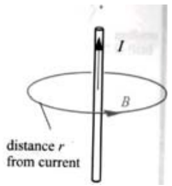
(iii) magnetic field strength B by Biot–Savart Law: $B = \int dB = \int \frac{\mu_0 I \sin\theta}{4\pi r^2} dl$

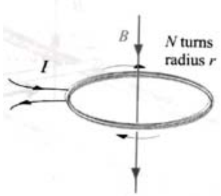
(I) by a current carrying wire with distance r: $B = \frac{\mu_0 I}{2\pi r}$

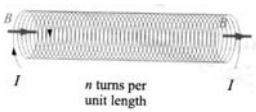


(II) by N circular coils with radius R: $B = \frac{\mu_0 NI}{2R}$

(III) by a solenoid with n turns per unit length: $B = \mu_0 nI$



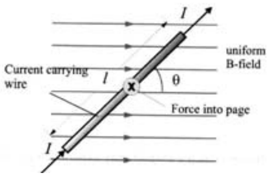


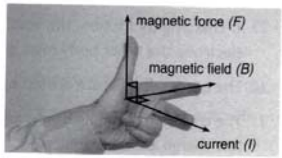


(b) Magnetic Force

(i) on current carrying wire

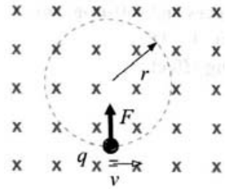
The magnitude of magnetic force $\vec{F}$ acting on a current carrying wire with current $\vec{I}$ and length ℓ by a magnetic field $\vec{B}$ is given by $F = BI\ell\sin\theta$. The direction is given by Fleming's Left Hand Rule.





In vector notation, $\vec{F} = \ell \vec{I} \times \vec{B}$.

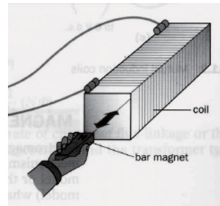
(ii) on a point charge



The magnitude of magnetic force $\vec{F}$ acting on a charge q with velocity $\vec{v}$ by a magnetic field $\vec{B}$ is given by $F = qvB\sin\theta$. The direction is also given by Fleming's Left Hand Rule.

In vector notation, $\vec{F} = q\vec{v} \times \vec{B}$.

(c) Faraday's Law of Electromagnetic Induction



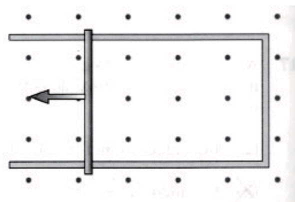
When there is a change in magnetic field, an induced e.m.f. will be produced.

Induced e.m.f. $\varepsilon = -N\frac{d\phi}{dt}$, where N is the number of turns in a coil and ϕ is the magnetic flux.

(d) Application of Faraday's Law:

(i) Induced e.m.f. of a moving conductor

Suppose a conductor moves with a constant velocity $\vec{v}$, perpendicular to a uniform magnetic field $\vec{B}$.

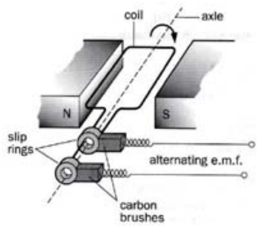


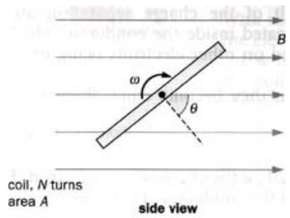
$$\varepsilon = -N\frac{d\phi}{dt} = -1\frac{d}{dt}(\vec{B}\cdot\vec{A}) = -\frac{d}{dt}(BA)$$

$$= -B\frac{d}{dt}(ls) = -Bl\frac{ds}{dt} = -Blv$$

(ii) Generator

Consider a coil with N turns and area A rotating with a constant angular velocity ω in a uniform magnetic field B.





$$\varepsilon = -N\frac{d\phi}{dt} = -N\frac{d}{dt}(\vec{B}\cdot\vec{A}) = -N\frac{d}{dt}(BA\cos\theta) = -NBA\frac{d}{dt}(\cos\omega t) = NBA\omega\sin\omega t$$

(iii) Search Coil

Search coil is used to measure a changing magnetic field at a certain point. Suppose the changing magnetic field is produced by an a.c. signal, i.e. $B = B_0\sin\omega t$

By rotating the search coil to obtain the maximum signal so that B is perpendicular to the coil, we have

$$\varepsilon = -N\frac{d\phi}{dt} = -N\frac{d}{dt}(\vec{B}\cdot\vec{A}) = -N\frac{d}{dt}(BA) = -NA\frac{d}{dt}(B_0\sin\omega t) = -NAB_0\omega\cos\omega t$$

Max. induced e.m.f. $= NAB_0\omega$.

By measuring the max. induced e.m.f., we get the magnetic field strength B.

(e) Inductance

Definition: Inductance is the magnetic flux linkage per unit current. $L = \dfrac{N\phi}{I}$

(compared with capacitance)

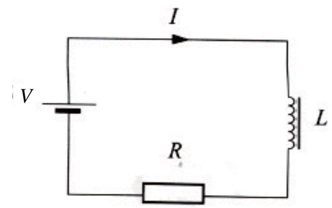
By Faraday's Law, $\varepsilon = -N\dfrac{d\phi}{dt} = -\dfrac{d(LI)}{dt}$

When a current I passing through a solenoid is changed, there is an induced e.m.f. in the solenoid.

$\because$ for solenoid, $\phi = \vec{B}\cdot\vec{A} = BA\cos 0° = (\mu_0 nI)A$,

Inductance of a solenoid $L = \dfrac{N\phi}{I} = \dfrac{N(\mu_0 nIA)}{I} = \mu_0 nNA = \dfrac{\mu_0 N^2 A}{l}$

In a LR circuit, when the switch is turned on, there is an increase in current across the solenoid, hence a change in magnetic flux linkage. Therefore there will be an induced e.m.f. to oppose the change, which is the voltage across the solenoid (the inductor).



$$V_0 = V_L + V_R$$

$$I_0 R = L\frac{dI}{dt} + IR$$

$$\frac{(I_0 - I)R}{L} = \frac{dI}{dt}$$

$$\int_0^I \frac{dI}{I_0 - I} = \int_0^t \frac{R}{L}dt$$

$$\int_0^I \frac{d(I_0 - I)}{I_0 - I} = -\frac{R}{L}t$$

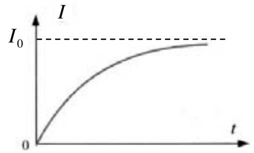
$$[\ln(I_0 - I)]_0^I = -\frac{R}{L}t$$

$$\ln(I_0 - I) - \ln I_0 = -\frac{R}{L}t$$

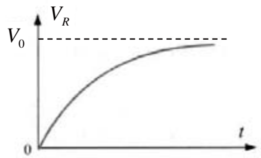
$$\ln\left(\frac{I_0 - I}{I_0}\right) = -\frac{R}{L}t$$

$$1-\frac{I}{I_0}=e^{\frac{-Rt}{L}}$$

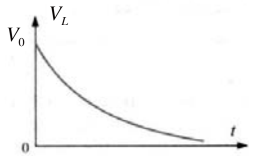
$$\frac{I}{I_0}=1-e^{\frac{-Rt}{L}}$$



$$I=I_0\left(1-e^{\frac{-t}{\left(\frac{L}{R}\right)}}\right)$$



$$V_R=IR=I_0R\left(1-e^{\frac{-t}{\left(\frac{L}{R}\right)}}\right)=V_0\left(1-e^{\frac{-t}{\left(\frac{L}{R}\right)}}\right)$$



$$V_L=V_0-V_R=V_0-V_0\left(1-e^{\frac{-t}{\left(\frac{L}{R}\right)}}\right)=V_0e^{\frac{-t}{\left(\frac{L}{R}\right)}}$$

<u>5. Alternating Current</u>

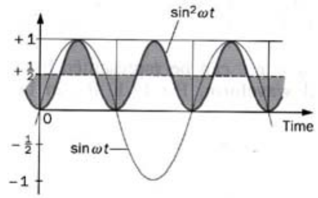
(a) Root–Mean–Square value of current

Consider an a.c. $I=I_0\sin\omega t$, where angular frequency $\omega=\frac{2\pi}{T}$ & T is the period

$$I_{rms}{}^2=\frac{1}{T}\int_0^T I^2dt=\frac{1}{T}\int_0^T I_0{}^2\sin^2\omega t dt$$

$$=\frac{I_0{}^2}{2T}\int_0^T(1-\cos 2\omega t)dt$$

$$=\frac{I_0{}^2}{2T}\left[t-\frac{\sin 2\omega t}{2\omega}\right]_0^T$$



$$=\frac{I_0{}^2}{2T}\left(T-\frac{\sin 2\omega T}{2\omega}\right)$$

$$=\frac{I_0{}^2}{2T}\left(T-\frac{\sin 4\pi}{2\omega}\right)$$

$$=\frac{I_0{}^2}{2}$$

$$\therefore I_{rms}=\frac{I_0}{\sqrt{2}}$$

(b) Capacitive and Inductive Reactance

Consider an a.c. circuit with a capacitor,

$$V=V_0\sin\omega t$$

$$I=\frac{dQ}{dt}=\frac{d(CV)}{dt}=C\frac{d}{dt}(V_0\sin\omega t)=CV_0\omega\cos\omega t\text{......(1)}$$

Let $I_0=CV_0\omega$, $\therefore V_0=I_0\left(\frac{1}{\omega C}\right)$

Define capacitive reactance $X_C=\frac{1}{\omega C}$

Moreover, $V=V_0\sin\omega t=V_0\cos\left(\omega t-\frac{\pi}{2}\right)$......(2)

Comparing (1) & (2), V lags I by $\frac{\pi}{2}$.

Similarly, for an a.c. circuit with an inductor,

$I = I_0 \sin \omega t$......(3)

$$\varepsilon = -L\frac{dI}{dt} = -L\frac{d}{dt}(I_0 \sin \omega t) = -LI_0\omega \cos \omega t$$

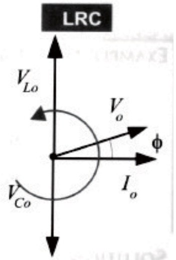
Voltage across the inductor to oppose the induced (or back) e.m.f. $V = -\varepsilon = LI_0\omega \cos \omega t$

Let $V_0 = I_0(\omega L)$

Define inductive reactance $X_L = \omega L$

Moreover. $V = V_0 \cos \omega t = V_0 \sin\left(\omega t + \frac{\pi}{2}\right)$.....(4)

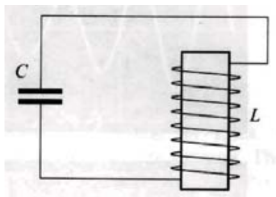
Comparing (3) & (4), V leads I by $\frac{\pi}{2}$.



So we have the phasor model for LCR circuit as shown in the figure.

(c) Parallel LC circuit

Consider an inductor and a charged capacitor with charge Q_0. Assume that there is no external voltage and no resistance.



Potential difference in the circuit:

$V_L + V_C = 0$

$$L\frac{dI}{dt} + \frac{Q}{C} = 0$$

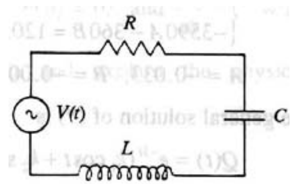
$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

Define $\omega^2 = \frac{1}{LC}$. The solution of the 2nd order differential equation $\frac{d^2Q}{dt^2} = -\omega^2 Q$ is $Q = Q_0 \cos \omega t$ (initial condition $Q = Q_0$ is applied).

Moreover, compare $\frac{d^2Q}{dt^2} = -\omega^2 Q$ and $\frac{d^2x}{dt^2} = -\omega^2 x$, this is a simple harmonic motion. The "oscillating" component is the charge. The phenomenon is also called electrical oscillation.

(d) LCR circuit

Consider an a.c. circuit with an inductor, a resistor and a capacitor



Potential difference in the circuit:

$V_L + V_R + V_C = V$

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = V_0 \sin \omega t$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{V_0}{L}\sin \omega t$$

This is a non–homogeneous 2nd order differential equation. The complementary solution Q_c depends on the roots of the characteristic equation $\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$......(**) and the particular solution Q_p takes the form $Q(c_1 \cos kQ + c_2 \sin kQ)$ (for complex roots in (**)) or the form $(c_1 \cos kQ + c_2 \sin kQ)$ (for real roots in (**)).

III. Waves

1. Beats

Consider 2 waves with same amplitude A but slightly different frequencies f_1, f_2

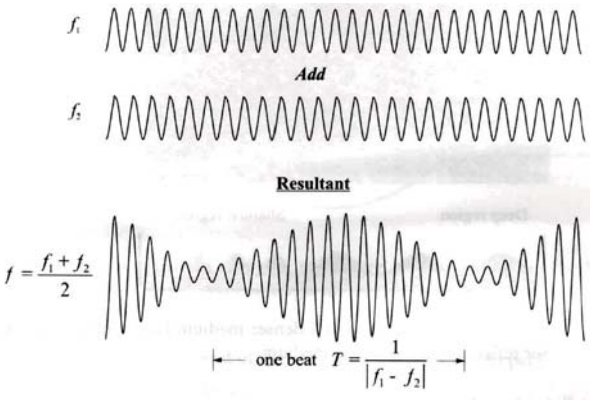
$s_1 = A\sin\omega_1 t \qquad s_2 = A\sin\omega_2 t \qquad$ where $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$

Superposition of the 2 waves:

$$s = s_1 + s_2 = A(\sin\omega_1 t + \sin\omega_2 t) = 2A\sin\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_1-\omega_2)t}{2}$$

Where $|\omega_1 - \omega_2|$ has a smaller frequency, hence larger period.

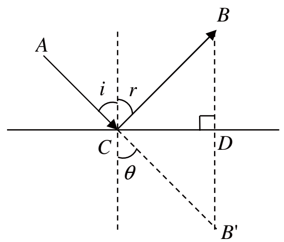
Define $|f_1 - f_2|$ as the beat frequency.



2. Law of Reflection

When light rays travel from A to B through a reflective surface at point C, it follows from Fermat's Principle of Least Time that total distance travelled should be minimum. First of all, reflect point B about the surface to B'. The required distance is the same as $AC + CB'$. By Triangular Inequality, the total length is shortest when A, C and B' are collinear.

$\therefore i = \theta$ (vert. opp. $\angle$ s)



$$\because \tan\angle BCD = \frac{BD}{CD} = \frac{B'D}{CD} = \tan\angle B'CD$$

$\therefore \angle BCD = \angle B'CD$

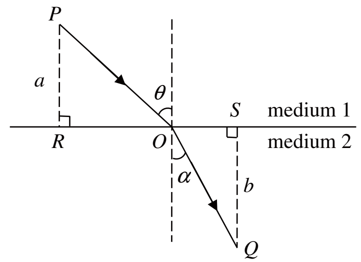
$r = 90° - \angle BCD = 90° - \angle B'CD = \theta$

$\therefore i = r$

3. Law of Refraction (Snell's Law)

It follows again from Fermat's Principle of Least Action.

In the figure, a light ray passes through medium 1 with velocity v_1 m/s from a fixed point P, which is a m above the water surface. The light ray reaches a point O on the boundary. Then it passes through the medium 2 with velocity v_2 m/s to a fixed point Q, which is b m below the water surface. R and S are 2 points on the boundary which are closest to points P and Q respectively. Let the angle of incidence and angle of refraction be θ and α respectively.



$$RS = OR + OS = a\tan\theta + b\tan\alpha$$

$$\frac{d}{d\theta}(RS) = \frac{d}{d\theta}(a\tan\theta + b\tan\alpha)$$

$\because P$ & Q are fixed points, $\therefore RS$ is constant.

$$\therefore \frac{d}{d\theta}(a\tan\theta + b\tan\alpha) = 0$$

$$a\sec^2\theta + b\sec^2\alpha\frac{d\alpha}{d\theta} = 0$$

$$\frac{d\alpha}{d\theta} = -\frac{a\sec^2\theta}{b\sec^2\alpha} \text{......(1)}$$

Let T seconds be the time to travel from P to Q via O.

$$T = \left(\frac{a}{\cos\theta}\right)\div v_1 + \left(\frac{b}{\cos\alpha}\right)\div v_2 = \frac{a}{v_1}\sec\theta + \frac{b}{v_2}\sec\alpha$$

$$\frac{dT}{d\theta}=\frac{a}{v_1}\sec\theta\tan\theta+\frac{b}{v_2}\sec\alpha\tan\alpha\frac{d\alpha}{d\theta}......(2)$$

Put (1) in (2) $$\frac{dT}{d\theta}=\frac{a}{v_1}\sec\theta\tan\theta+\frac{b}{v_2}\sec\alpha\tan\alpha\left(-\frac{a\sec^2\theta}{b\sec^2\alpha}\right)$$

$$=\frac{a}{v_1}\sec\theta\tan\theta-\frac{a\sec^2\theta\tan\alpha}{v_2\sec\alpha}$$

$$=\frac{a\sin\theta}{v_1\cos^2\theta}-\frac{a\sin\alpha}{v_2\cos^2\theta}$$

$$=\frac{a}{\cos^2\theta}\left(\frac{\sin\theta}{v_1}-\frac{\sin\alpha}{v_2}\right)$$

When T is minimum, $\frac{dT}{d\theta}=0$, $\therefore\frac{\sin\theta}{v_1}=\frac{\sin\alpha}{v_2}$

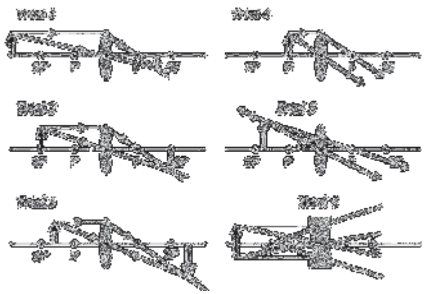
$$\frac{c}{v_1}\sin\theta=\frac{c}{v_2}\sin\alpha$$

$$n_1\sin\theta=n_2\sin\alpha$$

where $n=\frac{c}{v}$ is the refractive index of a medium.

4. Convex Lens

An object is placed at a distance of u cm in front of a convex lens with constant focal length f cm. A screen is placed v cm at another side of the lens so that a real image is formed. Find the minimum distance between the object and the screen in terms of f.



Method 1

In order to obtain real image with finite image distance, $u>f$.

By lens formula, $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$.

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$

$$\frac{1}{v}=\frac{u-f}{uf}$$

$$v=\frac{uf}{u-f}$$

Let y cm be the distance between the object and the screen.

$$y=u+v$$

$$y=u+\frac{uf}{u-f}$$

$$y=\frac{(u^2-uf)+uf}{u-f}$$

$$y=\frac{u^2}{u-f}$$

$$\frac{dy}{du}=\frac{(u-f)(2u)-u^2(1)}{(u-f)^2}$$

$$\frac{dy}{du}=\frac{2u^2-2uf-u^2}{(u-f)^2}$$

$$\frac{dy}{du}=\frac{u^2-2uf}{(u-f)^2}$$

$$\frac{dy}{du}=\frac{u(u-2f)}{(u-f)^2}$$

When $\frac{dy}{du}=0$, $u=2f$ or 0 (rej.)

By 1st derivative test, y is a minimum when $u=2f$.

$\therefore v = \dfrac{(2f)(f)}{2f - f} = 2f$.

Minimum distance $= 2f + 2f = 4f$.

It is the situation when the object and the image are of the same size.

<u>Method 2</u>

Let $u = xf$, where x is a real number.

As the image is real, $x > 1$.

By lens formula, $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{f}$.

$$\frac{1}{xf} + \frac{1}{v} = \frac{1}{f}$$

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{xf}$$

$$\frac{1}{v} = \frac{x-1}{xf}$$

$$v = \frac{x}{x-1} f$$

The distance between the object and the screen is $u + v$

$$= xf + \frac{x}{x-1} f$$

$$= \frac{x(x-1) + x}{x-1} f$$

$$= \frac{x^2}{x-1} f$$

Let $y = \dfrac{x^2}{x-1}$, where $x > 1$.

$$\frac{dy}{dx} = \frac{2x(x-1) - x^2(1)}{(x-1)^2}$$

$$= \frac{x^2 - 2x}{(x-1)^2}$$

$$= \frac{x(x-2)}{(x-1)^2}$$

When $\dfrac{dy}{dx} = 0$, $x = 2$ or 0 (rej.)

By 1st derivative test, y is a minimum when $x = 2$.

$\therefore$ the distance is minimum when $u = 2f$, i.e. when $v = 2f$.

min. distance $= 2f + 2f = 4f$.

It is the situation when the object and the image are of the same size.

IV. Matter

<u>1. Gas</u>

Ideal gas law: $PV = nRT$

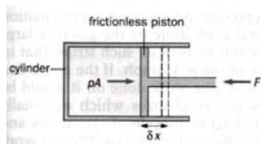
Work done by the gas in an isothermal process, i.e. constant temperature:

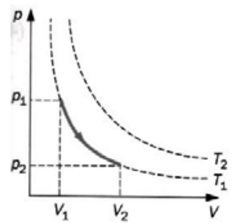
W.D. $= \int F dx = \int \dfrac{F}{A} \cdot A dx$

$= \int P dV = \displaystyle\int_{V_1}^{V_2} \frac{nRT}{V} dV$

$= nRT[\ln V]_{V_1}^{V_2} = nRT(\ln V_2 - \ln V_1)$

$= nRT \ln\left(\dfrac{V_2}{V_2}\right)$





2. Radioactive Decay

Let N be the number of nuclei. The rate of decay of nuclei $\frac{dN}{dt} \propto N$.

Moreover, N should be decreasing.

$\therefore \frac{dN}{dt} = -kN$, where k is called the decay constant.

$$\int_{N_0}^{N} \frac{dN}{N} = -k\int_{0}^{t} dt$$

$$[\ln N]_{N_0}^{N} = -kt$$

$$\ln N - \ln N_0 = -kt$$

$$\ln \frac{N}{N_0} = -kt$$

$$\frac{N}{N_0} = e^{-kt}$$

$$N = N_0 e^{-kt}$$

Let $t_{\frac{1}{2}}$ be the half life.

$$\therefore \frac{N_0}{2} = N_0 e^{-kt_{\frac{1}{2}}}$$

$$2 = e^{kt_{\frac{1}{2}}}$$

$$kt_{\frac{1}{2}} = \ln 2$$

$$t_{\frac{1}{2}} = \frac{\ln 2}{k}$$

Discussion

1. Mathematics Background in Secondary School Physics curriculum

(a) Trigonometry

Radian Measure, Arc Length & Area of Sectors,

Compound Angle Formula, Double Angle Formula, Sum–and–Product Formula

(b) Vectors

Operations of Vectors, Scalar and Vector Products

(c) Limits

Intuitive concepts of Limits, basic calculations of limits (up to HKCEE A. Math or HKDSE Math Module)

(d) Differentiation

Concepts of Derivatives from Limits, calculations of derivatives (up to HKDSE Math Module 2 level)

(e) Integration

Indefinite and definite integration (concepts and calculations up to HKDSE Math Module 2 level)

(f) Complex Numbers

Standard Form & Polar Form, Operations of Complex Numbers

(g) Power Series

Taylor's Series of Elementary Functions

(h) Ordinary Differential Equations

1st order: Separable Variables

2nd order: Homogeneous and Non–Homogeneous Equations, Concept of Complimentary and Particular Solutions, Linear with Constant Coefficients

2. Advantages of Incorporating Mathematics in Physics Curriculum

(a) Some quantitative results which can't purely determined by Physical Laws (or by merely Physics Curriculum) can be explained with the help of Mathematics.

e.g. half life of a radioactive nuclei, the time variations of voltage, current and charge in RC, LR and LRC circuit.

(b) Physical Laws can be written in a more succinct way.

e.g. In electromagnetic induction, the direction of magnetic force/induced current is determined by Fleming's Left/Right Hand Rule, and magnitude by formula $F = BIl\sin\theta$. However, the direction and magnitude in both phenomena can be determined by $\vec{F} = \ell\bar{I}\times\vec{B}$.

(c) Mathematical knowledge comes in a more natural way.

e.g. The definitions of scalar and vector products, without the direct application in mechanics and electromagnetism in Physics, are rather strange.

(d) Compare and contrast Mathematics and Physics

Currently a lot of students regard Mathematics and Physics as nearly the same subject, mainly due to lots of computations in Physics problems. Indeed students should be able to realize that 2 subjects, though inter–related in many aspects, still have their own characteristics. Mathematics starts from several definitions, axioms and then the whole structure with Theorems are constructed. Physics, however, starts from natural phenomenon and several simple laws are concluded to explain them. In some cases, quantitative laws are written, which is mathematical, yet they are for modeling the real world, themselves can't be regarded as definitions nor axioms.

3. Difficulties in "Mathematical Physics"

(a) Inevitably students are required to have a good grasp of the knowledge in Mathematics in order to appreciate the application in Physics. Yet under current situation, it is not compulsory for students taking Physics as an elective subject in HKDSE to also take either Module 1 or 2 in Mathematics.

(b) The sequence in teaching content should be revised so as to make the curriculum flow more natural. For example, in past HKCEE, students taking A. Math learnt Calculus during F.4 second term. Relevant topics in Physics should be put after that.

(c) The HKDSE syllabus has been under revised and some parts in the curriculum in Modules are removed starting from 2016 HKDSE. If further changes are made, the relevant part in Physics curriculum relying on the removed portion in Mathematics curriculum will encounter problems.