2016 年國際數學奧林匹克通訊 IMO 2016 NEWSLETTER

第五期
Nov 15
5th ISSUE

數聞 IMOment

中國剩餘定理
CHINESE REMAINDER THEOREM

挑戰園地
CHALLENGE CORNER

笑一笑
LAUGH OUT LOUD

領隊於 IMO 2015 之見聞 (上)
IMO 2015 REPORT – LEADER'S PESPECTIVE (I)

納什之後
NASH AND BEYOND

2016 年第五十七屆國際數學奧林匹克籌備委員會
Organising Committee of the 57th International Mathematical Olympiad 2016

2016 年第五十七屆
國際數學奧林匹克籌備委員會

香港將於 2016 年 7 月主辦第五十七屆國際數學奧林匹克（IMO），迎接來自超過 100 個國家和地區的中學生數學精英。希望《數聞》可在我們邁向 2016 年 IMO 期間帶動同學和公眾對數學的興趣，更希望這種氣氛歷久不衰。

歡迎讀者向《數聞》投稿。文章須為原著，以中文或英文寫成（或兩種文本兼備），長度為一至四頁（就一種語言而言），並以電郵附件方式傳送至 info@imohkc.org.hk，或郵寄至 九龍油麻地 405 號九龍政府合署 4 樓 403 室 教育局數學教育組《數聞》編輯，標題為「Submission to IMOment」。

Hong Kong is proud to be hosting the brightest secondary school mathematics talents from over 100 countries and regions at the 57th International Mathematical Olympiad (IMO) in July 2016. We hope that IMOment will promote interest in mathematics among students and the public in this period leading up to IMO 2016, and beyond.

Readers are welcome to submit articles on mathematics and/or mathematical Olympiad to IMOment. Submissions should be original, one to four pages in length in either Chinese or English (or both), and should be sent by attachment to an email to info@imohkc.org.hk, or be mailed to Rm . 403, 4/F, Kowloon Government Offices, 405 Nathan Road, Kowloon, titled "Submission to IMOment."

Organising Committee of the 57th International Mathematical Olympiad 2016

第五期內容
Contents of 5th Issue

WWW.IMO2016.ORG

領隊於 IMO 2015 之見聞 (上)

IMO 2015 Report Leader's Perspective (I)

/羅家豪 LAW KA-HO

2015 年國際數學奧林匹克（IMO 2015）於 7 月 4 日至 16 日在泰國清邁舉行。以下為香港代表隊隊員名單：

- 張偉霖（伊利沙伯中學，中五）
- 郭敏怡（浸信會呂明才中學，中四）
- 李信明（中華傳道會安柱中學，中四）
- 董鑫泉（喇沙書院，中六）
- 吳詩通（香港國際學校，中四）
- 于鎧瑋（喇沙書院，中四）

當中，張偉霖及于鎧瑋是去年港隊隊員，其他四位則屬首次參賽。

由於香港將主辦 IMO 2016，故本港派出十四位觀察員陪同代表隊、領隊及副領隊前赴泰國。

IMO 2015 was held in Chiang Mai, Thailand from Jul 4 to 16. The Hong Kong team consists of the following students:

- CHEUNG Wai Lam (Queen Elizabeth School, Form 5)
- KWOK Man Yi (Baptist Lui Ming Choi College, Form 4)
- LEE Shun Ming Samuel (CNEC Christian College, Form 4)
- TUNG Kam Chuen (La Salle College, Form 6)
- WU John Michael (Hong Kong International School, Form 4)
- YU Hoi Wai (La Salle College, Form 4)

Cheung and Yu were in the IMO team last year, while the rest are first-timers.

Since Hong Kong will host IMO 2016, we sent a total of 14 observers in addition to the contestants, the leader and the deputy leader.

時隔超過二十年後，美國隊在本屆 IMO 終於重奪桂冠（美國隊對上一次在 IMO 掄元，正是 1994 年香港主辦的那一屆）。中國隊以 4 分之差屈居次席，但仍以超過 20 分距離拋離其後的南韓、北韓和越南隊 —— 不約而同都是亞洲隊伍！

For the first time in over two decades, the United States went to the top of the scoreboard. (The last time this happened was in Hong Kong during IMO 1994!) China was behind by a margin of 4 points. The next few teams were more than 20 points behind, and they were all Asian teams --- South Korea, North Korea and Vietnam.

此外，澳洲、伊朗及新加坡隊均打入前十名，主辦國泰國則排名 12。俄羅斯隊排名第8，歷來首次跌出六強及未能摘金。

Australia, Iran and Singapore all went into top 10. The host Thailand was 12th. Russia, ranked 8th, went out of top 6 and went home without any gold medal for the first time in history.

香港代表隊的六名學生分別得到 20、20、18、16、14 和 13 分，共獲得兩面銀牌及三面銅牌，另一名學生則取得優異獎。其中兩名學生以一分之差未能取得高一級的獎牌，實屬可惜。港隊以總成績 101 分排第 28 位，屬香港近年來較低的排名。

The six students from Hong Kong scored 20, 20, 18, 16, 14 and 13 points respectively, thus brought home 2 silver and 3 bronze medals. The other student obtained an Honourable Mention. It was a bit unfortunate that two students were just one point away from upgrading their medals. The total score of 101 puts the Hong Kong team in the 28th place, which is a relatively low position compared with recent years.

由於今年的隊伍資歷較淺（五位獎牌得主明年仍有資格參賽，而其中四位亦有資格參加 IMO 2017），所以賽果尚算合理。但從結果中可見，港隊在幾何題的表現實在有很大的進步空間。今年 IMO 的第四題是第二天比賽中一道較為淺易的幾何題，可是港隊中只有一半隊員答對。相反，不少整體排名比香港低的隊伍在此題均有不錯的發揮，其中五隊更全隊取得滿分（事實上，在整體排名前五十名的隊伍中，香港隊在這題的總分排名為倒數第二）。似乎港隊在幾何方面的表現相對略為遜色（在 IMO 的四大範疇中，我本人最弱的一環也是幾何），因此我很自然想到一個問題：各隊伍在不同範疇的相對強弱，跟各國學校課程的差異是否有關？

Considering that this is a very young team (all five medallists are still eligible next year and four of them will still be eligible for IMO 2017), the result is reasonably acceptable. However, it is quite clear that there is much room for improvements in their performance in geometry, as reflected in Problem 4, which is an easy geometry problem in Day 2. In this question a lot of lower-ranking teams did better than Hong Kong, in which only half of our team members solved it. Five of these teams even scored full marks. (In fact, if only the top 50 teams are considered, the total score of the Hong Kong team in this question would be second bottom.) Apparently the Hong Kong team is relatively weak in geometry. (I myself am weakest in this among the four areas of IMO problems as well.) So a question naturally came up to me --- is there something to do with the school curriculum which makes some countries relatively stronger in certain areas?

在閉幕典禮上，唯一取得滿分的參賽者獲頒特別獎。這位貌似華人的加拿大隊選手今年已是第六次參加 IMO，並取得個人第五面金牌，刷新了 IMO 史上的個人奪金紀錄。

In the Closing Ceremony, the one contestant who got a perfect score was invited onto the stage for a special award. He was from the Canadian team, apparently of Chinese origin. This was his sixth IMO and his fifth gold medal, which updated IMO history.

在惜別晚宴上，以色列的領隊獲頒一個看來是「金咪獎」的獎項。據我理解，此獎項乃頒發給在領隊大會上發言最多的領隊，但到底賽會是如何統計各人發言次數的呢？

I seemed to see that a golden microphone was presented to the leader of Israel at the Farewell Party. Apparently this is for the leader who speaks the most during the Jury Meetings. How did the organiser do the counting?

最後，我還想提一提有關 IMO 的各項軼事。

Finally I include some miscellaneous snapshots from various parts of the IMO.

領隊需把試題從英文翻譯成自己學生所選的語言，然後翻譯版本會供所有領隊參閱並投票核准。這是一個學習不同語言的好機會。原來在各國的語言中，表示「角」的符號（在 第 3題中出現）均有所不同。有些語言的「角」符號在字母（HQA）的左方，有些在右方，有些甚至在上方！而「角」符號本身亦有幾個不同版本。有興趣的讀者可到 IMO 官方網站（www.imo-official.org）進一步探索。

After leaders translated the papers into their own languages, the translated papers were posted for everyone to review and approve. This provides a good chance to learn the different languages. It turned out that there were several different ways to represent the angle sign (which appeared in Problem 3) in different languages. Some had a symbol on the left of the angle （HQA）, some on the right, and some above! There were also different symbols for `angle'. The interested reader may visit the IMO official website (www.imo-official.org) to explore further.

在官方的英文版本試題中，採用的不等式符號是 ⩽，而非 ≤（反方向的符號亦然）。有趣的是少數的翻譯版本用了 ≤，這意味著那些領隊在翻譯時自己把方程重新輸入一次，而非直接編輯英文版的原稿！

In the official English paper, the symbol ⩽ was used instead of ≤ (same for the symbol in the opposite direction). Interestingly, in a small number of translated papers, the symbol became ≤. This means those leaders actually retyped the whole equation instead of just copied it from the source file of the English version!

跟大部分其他數學比賽一樣，IMO 不容許參賽者使用計算機。原來現在連量角器也不能使用，但我記得多年前（當我還是參賽者時）大會並沒有此規定，那麼到底量角器是從甚麼時候開始被禁的呢？

As in most other mathematical competitions, calculators are not allowed for the IMO. I noticed that protractors have also been disallowed nowadays. As such rule did not exist many years ago (when I was a contestant), I wonder when this rule was first enforced.

量角器之所以被禁是因為它屬於「量度工具」，不過參賽者可以使用「繪圖工具」，例如圓規。奇怪的一點是間尺（不是沒有刻度的直尺！）也可以使用 —— 但間尺不也是「量度工具」嗎？

While protractor is disallowed as it is a `measuring instrument', `drawing instruments' are allowed. In particular, contestants could bring compasses. However, rulers (not straight edges!) are also explicitly specified as allowed. Aren't rulers also measuring instruments?

大會有一個沿用多年的電腦系統，負責處理IMO的不同範疇，例如編製協調分數的時間表、計算賽果及製作名牌等。若我們明年能使用這套系統，應可避免一些麻煩，當然這也會帶來一定的花費。

There is a computer system which had been in use for many years. The system would take care of different aspects of the IMO, such as working out coordination schedules, generating results, facilitating the printing of name tags, etc. We may be able to save a lot of potential troubles if we use the same system next year, but of course at a cost.

我無法在此將所有想法一一說明。我相信 IMO 2015 帶給我們的特別體驗將有助完善 IMO 2016的準備工作，並熱切期待香港在來年成功舉辦這項盛事。

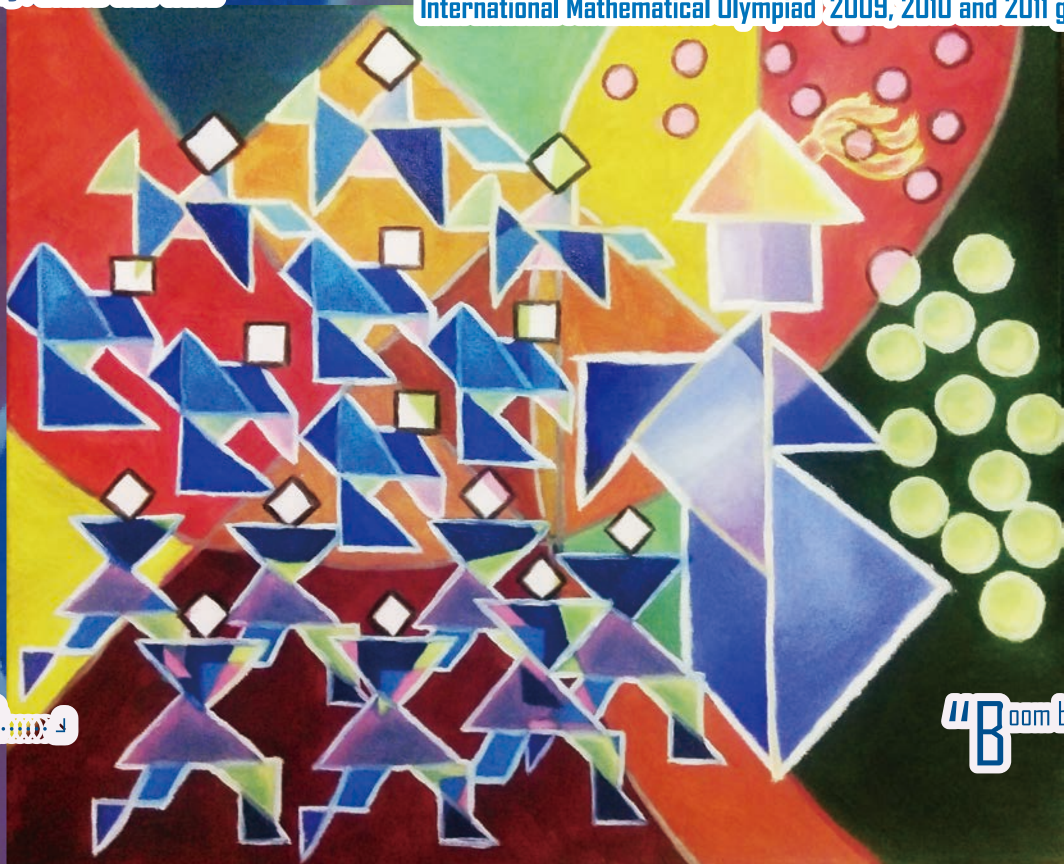
There were certainly a lot of other things which I could not list one by one. I believe that our different experience in IMO 2015 would help a lot towards better preparing for IMO 2016, and I look forward to a successful event next year.

(待續...)

(To be continued...)

中國剩餘定理
CHINESE REMAINDER THEOREM

/程德永 CHING TAK-WING
2009、2010 及 2011 年國際數學奧林匹克金牌得主
International Mathematical Olympiad 2009, 2010 and 2011 gold medalist



「隆 隆隆......」

"Boom boom boom..."

戰鼓聲響起，敵人轉眼將至，到底是奮勇抗敵，出門迎戰，還是避其鋒銳，暫且閉門不出？大將軍韓信必須在瞬間作出抉擇，而關鍵在於是本隊士兵是否足夠。因此韓信立即召集所有士兵，清點數目。可是士兵有數千之眾，如何能夠迅速點數清人數？

韓信先命令士兵每 3 個站成一排，記下餘下的人數，然後依次令他們每 5 個、每 7 個、每 11 個和每 13 個站成一排，同樣記下每次剩餘的人數，最後韓信利用這些數據，推算出士兵的總人數。由於士兵訓練有素，整個程序很快就完成了，大將軍也能在短時間內作出相應策略。

這其實是一個數學問題，簡而言之，若我們知道一個正整數除以某些數時（例如上述的 3、5、7、11 和 13 所得的餘數，我們能否得知這個正整數的值？相傳西漢開國名將韓信經常以此方法點兵，其真實性自然不可考究，

Amid the war drum's beating, the enemies are drawing near. To fight, or not to fight? General Han Xin must count his soldiers and make a decision. But how can he count the thousands of soldiers in a split second?

Han orders the soldiers to stand in rows of 3, and notes the remainder. Next, he in turn asks them to stand in rows of 5, of 7, of 11 and of 13, also noting the remainder in each case. Then using these figures, Han deduces the total number of soldiers. Thanks to the soldiers' discipline, the whole process is completed quickly, and the General is able to make a decision in time.

Actually, this is a mathematical problem. Basically, if we know the remainders when a positive integer is divided by some numbers (such as 3, 5, 7, 11 and 13 in the above example), can we find out the value of the positive integer? Legend has it that the famous general Han Xin of the Western Han dynasty often used this method to count his soldiers, though the authenticity of this story is uncertain.

不過在成書距今至少一千五百年的中國算術著作《孫子算經》中，已確切記載了類似的數學問題和答案。鑑於這個歷史原因，現在解決這類問題的方法被統稱為「中國剩餘定理」。

現在我們正式介紹中國剩餘定理。為了方便，我們以同餘式 $n \equiv a \pmod m$ 表示整數 n 除以 m 時的餘數是 a（更廣義地，當 n 和 a 除以 m 時的餘數相同，我們也可以用這個同餘式表達）。定理指出：對於任意整數 $a_1, a_2, \cdots, a_k$ 及兩兩互質的正整數 $m_1, m_2, \cdots, m_k$（即當中任意兩者均沒有大於 1 的公因數），必存在無窮多個正整數 n 滿足所有同餘式

$$n \equiv a_1 \pmod{m_1}$$
$$n \equiv a_2 \pmod{m_2}$$
$$\cdots$$
$$n \equiv a_k \pmod{m_k}$$

此外，我們亦可證明 n 的所有可能值，除以乘積 $m_1 m_2 \cdots m_k$ 時餘數均相等，而所有具備此性質的數均為 n 的可能值 。

利用同餘式的運算，雖然確實有一些正式做法求出 n 的可能值，但通常比試驗需要花更多的時間。例如對於同餘式

$$n \equiv 1 \pmod 3$$
$$n \equiv 3 \pmod 5$$

我們由第一式知 $n = 1, 4, 7, 10, 13, \cdots$，當中 13 同時滿足第二式，故由中國剩餘定理，我們立刻知道 n 可以是任意除以 $3 \times 5 = 15$ 時餘 13 的整數。當然，以這種方式計算絕對是對解題者計算能力和數字觸覺的考驗，要像韓信一樣在短時間內正確地解題，可不是人人都能做得到的！順帶一提，雖然 n 有無窮多個可能值，但若我們知道 n 值大概的範圍，依然可以由此推出其唯一數值，例如韓信對其軍隊士兵數目當然是心中有數，而利用此方法可得出一個更確切的數目。

很多時候我們在應用中國剩餘定理時，需要的只是同餘方程組的解 n 的存在性，往往並不用計算 n 的值。例如：是否存在無窮多組 10 個連續正整數都是合成數呢？答案是肯定的，我們來看看利用中國剩餘定理的證明。

What we do know is that a selfsame problem and its answer was written at least 1500 years ago in the ancient mathematical text The Mathematical Classic of Sunzi. For this historical reason, this problem solving method is now called the Chinese remainder theorem.

Now let us state the Chinese remainder theorem formally. For convenience, we use the congruence relation $n \equiv a \pmod m$ to mean that the integer n leaves a reamainder of a when divided by m. (More generally, the congruence relation holds whenever n and a leave the same remainder when divided by m.) For any integers $a_1, a_2, \cdots, a_k$ and pairwise relatively prime positive integers $m_1, m_2, \cdots, m_k$ (no two of which have a common factor greater than 1), there exists infinitely many positive integers n satisfying the set of congruence relations

$$n \equiv a_1 \pmod{m_1}$$
$$n \equiv a_2 \pmod{m_2}$$
$$\cdots$$
$$n \equiv a_k \pmod{m_k}$$

Further, we can prove that when divided by the product $m_1 m_2 \cdots m_k$, all the feasible values of n have the same remainder, and all numbers having this property are feasible values of n.

To find the possible values of n, although there are some proper methods in modular arithmetic, they are usually more tedious than a direct search. For example, for the set of congruence relations

$$n \equiv 1 \pmod 3$$
$$n \equiv 3 \pmod 5$$

the first congruence relation restricts us to $n = 1, 4, 7, 10, 13, \cdots$. Among these values, 13 satisfies the second congruence relation too. So by the Chinese remainder theorem, the feasible values of n are exactly the integers that leave a remainder of 13 when divided by $3 \times 5 = 15$. Of course, this kind of calculation demands computational prowess and a sense of numbers. Not everybody can solve the problem as quickly as Han Xin did! In passing, note that although n has infinitely many feasible values, we may pin down a unique value provided that we know an approximate range of n. Han Xin must have had a rough idea of his number of soldiers, and used the above method to get a more accurate figure.

Oftentimes when we apply the Chinese remainder theorem, we are only concerned with the existence rather than the value of n. For example, are there infinitely many sets of 10 consecutive positive integers that are all composite? The answer is yes. Let us look at a proof via the Chinese remainder theorem.

假設該 10 個連續數為 $n+1, n+2, \cdots, n+10$ ，若要求 $n+1$ 是合成數，最簡單方法是令 $n+1$ 是某個合成數 m_1 的倍數，換言之，n 除以 m_1 的餘數是 -1（或 m_1-1）。同樣道理，對每個 $n+k$ ，為使它是某個合成數 m_k 的倍數，我們需要 n 除以 m_k 的餘數是 $-k$ ，利用同餘式表達，即

$$n \equiv -1 \pmod{m_1}$$
$$n \equiv -2 \pmod{m_2}$$
$$\cdots$$
$$n \equiv -10 \pmod{m_{10}}$$

由中國剩餘定理，當 $m_1, m_2, \cdots, m_{10}$ 兩兩互質時（注意 $m_1, m_2, \cdots, m_{10}$ 是由我們選擇的），相應的正整數 n 必定存在，且有無窮多個。對於每一個這樣的 n ，連續數 $n+1, n+2, \cdots, n+10$ 就分別是 $m_1, m_2, \cdots, m_{10}$ 的倍數。這個方法雖然不能簡單地給出 n 的可能值，但容許我們較有自由地選擇 $m_1, m_2, \cdots, m_{10}$ 的部分因數，靈活程度相對較大。

再來看看另一道差別不大的問題：是否存在無窮多組 10 個連續數，當中恰好有一個是質數？例如我們要求 $n+10$ 是質數， $n+1, n+2, \cdots, n+9$ 是合成數。利用同樣方法，我們得知存在整數 a ，使得 $n = m_1m_2\cdots m_9 + a,\ 2m_1m_2\cdots m_9 + a, \cdots$ 皆可令 $n+1, n+2, \cdots, n+9$ 為合成數，但我們能否在當中選出一些值，令 $n+10$ 是質數？這裏就必須用到另一個高等數論定理：對所有等差數列 $d+h, 2d+h, 3d+h, \cdots$ ， 只要 d 和 h 為互質的正整數，該數列中必存在無窮多個質數。這是著名的狄利克雷定理 (Dirichlet' s theorem)，利用它就可輕易解決上述問題（取 $d = m_1m_2\cdots m_9$ 及 $h = a+10$ ）。

現在問題來了，我們能否找到無窮多組 10 個連續數，使得當中恰好有兩個是質數？答案就不得而知了，唯一確定的是，假如答案是肯定的，那麼我們就知道存在無窮多對質數，其差不超過 10 ，而這個看似簡單的結果，卻是迄今為止卻還未能證明的！目前已知結果是存在無窮多對質數，其差不超過 246 ，而這類結果也是近年才被證明的。

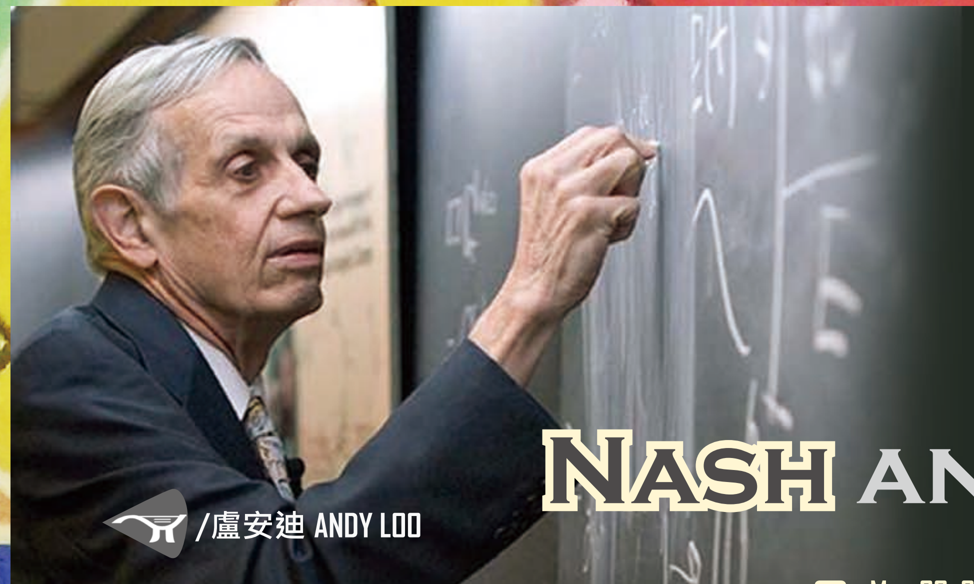
Let the 10 consecutive numbers be $n+1, n+2, \cdots, n+10$. To make $n+1$ composite, the easiest way is to make it a multiple of some composite number m_1. In other words, the remainder when n is divided by m_1 is -1 (or, equivalently, m_1-1). Likewise, for every $n+k$, to make it a multiple of some composite number m_k, we need n to leave a remainder of $-k$ when divided by m_k. Written in congruence relations:

$$n \equiv -1 \pmod{m_1}$$
$$n \equiv -2 \pmod{m_2}$$
$$\cdots$$
$$n \equiv -10 \pmod{m_{10}}$$

By the Chinese remainder theorem, when $m_1, m_2, \cdots, m_{10}$ are pairwise relatively prime (note that we can pick $m_1, m_2, \cdots, m_{10}$ at will), a corresponding positive integer n must exist, and there are infinitely many of them. For every such n, the consecutive numbers $n+1, n+2, \cdots, n+10$ are multiples of $m_1, m_2, \cdots, m_{10}$ respectively. Although this method cannot directly give feasible values of n, it gives us more flexibility in choosing factors of $m_1, m_2, \cdots, m_{10}$ and hence in choosing n.

Next, let us look at a rather similar problem: Do there exist infinitely many sets of 10 consecutive positive integers where exactly one is a prime? For example, we may require $n+10$ to be a prime and $n+1, n+2, \cdots, n+9$ to be composite. By the same reasoning as before, we know that there exists an integer a such that $n = m_1m_2\cdots m_9 + a,\ 2m_1m_2\cdots m_9 + a, \cdots$ all make $n+1, n+2, \cdots, n+9$ composite. But can we select some of them that make $n+10$ prime? Here we invoke another theorem in advanced number theory: For any relatively prime positive integers d and h, the arithmetic sequence of integers $d+h, 2d+h, 3d+h, \cdots$ contains infinitely many primes. This is the famous Dirichlet's theorem, which enables us to solve the above problem easily (by taking $d = m_1m_2\cdots m_9$ and $h = a+10$).

Here comes the real difficult problem: Can we find infinitely many groups of 10 consecutive positive integers such that in each group, exactly two numbers are primes? The answer is unknown. The only thing we know is that if the answer is yes, then we know there exist infinitely many pairs of primes that differ by at most 10. This result may sound simple, but it remains an open problem to prove or disprove it! The best result known so far is that there are infinitely many pairs of primes whose difference is at most 246, and even this result was only proved in recent years.



納什之後
NASH AND BEYOND

/盧安迪 ANDY LOO

2015年5月23日，世界失去了一顆美麗心靈。我自高中時主持他在香港的公開對話而與他相識，後來有幸在普林斯頓大學的課堂內外向他學習。約翰．納什教授的離世，令我尤感難過。

納什在博弈論——研究決策的學問——作出了開創性的貢獻。在一場博弈中，每人選擇一個策略。各人所選策略的組合決定博弈的結果。每人從結果得到一定的「功用」，即他對結果滿意、享受的程度。

1950年，尚為博士生的納什建立了後來被稱為「納什均衡」的概念：給定一場博弈，如果在各人的某個策略組合中，沒有人可以在其他人所選的策略保持不變下，通過改變自己的策略而增加自己的功用的期望值，那麼這個策略組合就是一個納什均衡。換言之，假如再進行一次同樣的博弈，如果每人都相信其他人不會改變其策略，那麼他自己也沒有動機去改變自己的策略。

為闡明這個概念，讓我們看看其中一個最著名、最基礎的博弈：男女大戰。假設一對情侶——約翰和愛麗——打算約會。約會場合有兩個選擇：一場數學講座或一場電影。因為某些原因，他們還未說好去哪裏約會，但現在已不能再溝通了。於是，他們須各自決定前往何處。他們在不同結果下得到的功用如下表：

		愛麗	
		講座	電影
約翰	講座	愛麗：1 約翰：2	愛麗：0 約翰：0
	電影	愛麗：0 約翰：0	愛麗：2 約翰：1

On May 23, 2015, the world lost a beautiful mind. Having known him since high school when I hosted an open dialogue with him in Hong Kong, and having learned from him both in and out of class at Princeton University, I was particularly saddened by the passing of Professor John Nash.

Nash is best known for his pioneering work in game theory, the study of strategic decision making. In a game, each player chooses a strategy. The combination of chosen strategies determines the outcome of the game. We often consider the utility that each player gets from the outcome, which measures how much he enjoys the outcome.

In 1950, Nash (then a PhD student) established a concept that would later be called Nash equilibrium: Given a game, a combination of strategies chosen by the players is a Nash equilibrium if, supposing the game is played again, no single player can increase his expected utility by deviating from his current strategy, assuming that all other players' strategies remain unchanged. In other words, if the same game is played once more, and if every player believes that other players will not change their strategies, then he himself also does not have an incentive to change his own strategy.

To illustrate this idea, let us look at one of the most famous and fundamental games – battle of the sexes. Suppose a couple, John and Alicia, are going on a date. There are two options: a mathematics lecture and a movie. For some reason, they haven't agreed on where to go and they cannot communicate now. So, each of them must individually decide where to go. Their utility in different outcomes is represented in the following table:

		Alicia	
		Lecture	Movie
John	Lecture	Alicia: 1 John: 2	Alicia: 0 John: 0
	Movie	Alicia: 0 John: 0	Alicia: 2 John: 1

相片來源 /Photo Source: ICTP Photos Archieves


相片來源 /Photo Source: Princeton University, Office of Communications, Denise Applewhite

如果兩人都去聽數學講座（因而在那裏一起約會），約翰會較享受這場約會。如果兩人都去看電影，愛麗會較享受。如果兩人去了不同地方，約會便會告吹，兩人都毫不享受。

If they both go to the mathematics lecture (and hence meet and date there), John will enjoy the date more than Alicia will. If they both go to the movie, Alicia will enjoy it more. If they go to different places, their date will be spoiled and they won't enjoy it at all.

顯然，「兩人都去聽講座」是一個納什均衡，因為假若愛麗去聽講座，如果約翰不去聽講座，他的功用會從一個正數下降至 0，而非增加。同理，愛麗亦不能通過改變策略而增加自己的功用。同理，「兩人都去看電影」亦是一個納什均衡。

Obviously, both going to the lecture is a Nash equilibrium of this game, because assuming that Alicia goes to the lecture, if John did not go to the lecture, he would decrease his utility from a positive number to 0, instead of increasing it. Similarly Alicia also cannot increase her utility by unilaterally deviating. By the same token, both going to the movie is a Nash equilibrium too.

然而，這場博弈也有混合策略的納什均衡。每人可以有一定的概率去聽講座，否則便去看電影。假設約翰有 2/3 機會去聽講座，1/3 機會去看電影（可把約翰的策略想成：擲一顆骰子，如果擲到 1、2、3 或 4，便去聽講座，如果擲到 5 或 6，便去看電影）；愛麗有 1/3 機會去聽講座，2/3 機會去看電影。這是否一個納什均衡？

Yet, this game may also have mixed strategy Nash equilibria. That is, a person's strategy may be to go to the lecture with a certain probability and go to the movie otherwise. Suppose John goes to the lecture with probability 2/3 and the movie with probability 1/3 (we may think of John's strategy as: roll a dice, and go to the lecture if the dice roll gives a 1, 2, 3 or 4, and go to the movie if the dice roll gives a 5 or 6), and Alicia goes to the lecture with probability 1/3 and the movie with probability 2/3. Is this a Nash equilibrium?

假若愛麗採用上述策略，約翰可否通過改變策略而增加自己的功用的期望值？設想約翰有 p 機會去聽講座， 機會去看電影。考慮四個結果的概率，可知約翰的功用的期望值必為

Assuming Alicia is using the above-mentioned strategy, can John increase his expected utility by deviating? Suppose John went to the lecture with probability p and the movie with probability . Considering the probability of the four outcomes, John's expected utility would always be

$$p\cdot\frac{1}{3}\cdot 2+p\cdot\frac{2}{3}\cdot 0+(1-p)\cdot\frac{1}{3}\cdot 0+(1-p)\cdot\frac{2}{3}\cdot 1=\frac{2}{3}$$

因此，約翰不能通過改變策略而增加自己的功用的期望值。愛麗也同樣不能──讀者可驗證，無論她如何改變策略，她的功用的期望值總是 2/3。因此，我們找到了一個混合策略的納什均衡，其中約翰和愛麗的功用的期望值都是 2/3。

So John cannot increase his expected utility by deviating from his present strategy. Neither can Alicia, as the reader may similarly verify that her expected utility would always be 2/3 no matter how she deviated. Thus, we have indeed found a mixed strategy Nash equilibrium, where both John and Alicia have an expected utility of 2/3.


相片來源 /Photo Source: Princeton University, Office of Communications, Denise Applewhite

事實上，不僅這場男女大戰有納什均衡，納什在 1951 年證明了任何有限個參與者、有限個純策略（但參與者可採用混合策略）的博弈，都有至少一個納什均衡──這個成果在 40 多年後為他贏得了諾貝爾經濟學獎。

In fact, not only does battle of the sexes has Nash equilibria, but Nash proved in 1951 that any game with finitely many players and finitely many pure strategies (where players may use mixed strategies) has at least one Nash equilibrium – a result that would earn him an Economics Nobel Prize more than 40 years later.

接著，我們考慮一個稍為不同的劇本：假設約翰和愛麗的朋友──哈諾──挺身相助這對情侶。哈諾分別向兩人建議一個地點。有 1/2 機會，他會向兩人都建議去聽講座。有 1/2 機會，他會向兩人都建議去看電影。約翰和愛麗都知道哈諾的上述習慣，並可各自選擇聽從哈諾的建議，或採用任何其他策略。

Next, we consider a slightly different scenario: Suppose John's and Alicia's friend, Harold, comes to the couple's assistance. Harold makes recommendations to the couple (separately) on where to go. With probability 1/2, he recommends both to go to the lecture. With probability 1/2, he recommends both to go to the movie. Both John and Alicia know this habit of Harold. John and Alicia may choose to follow Harold's recommendation or to use any other strategies.

其實，「兩人都永遠聽從哈諾的建議」是一個均衡：假設哈諾建議約翰去聽講座。由於約翰知道哈諾作出建議的習慣，所以約翰確定哈諾一定也建議愛麗去聽講座。如果愛麗聽從哈諾的建議，約翰確實應該去聽講座。同理可知，約翰和愛麗在任何情況下都不應單方面偏離哈諾的建議。

Actually, it is an equilibrium for both of them to always follow Harold's recommendation: Suppose Harold recommends John to go to the lecture. Knowing Harold's recommendation habit, John is certain that Harold must also be recommending Alicia to go to the lecture. So, assuming Alicia follows Harold's recommendation, John should indeed go to the lecture. By similar analyses, we see that neither John nor Alicia should unilaterally deviate from Harold's recommendation in any case.

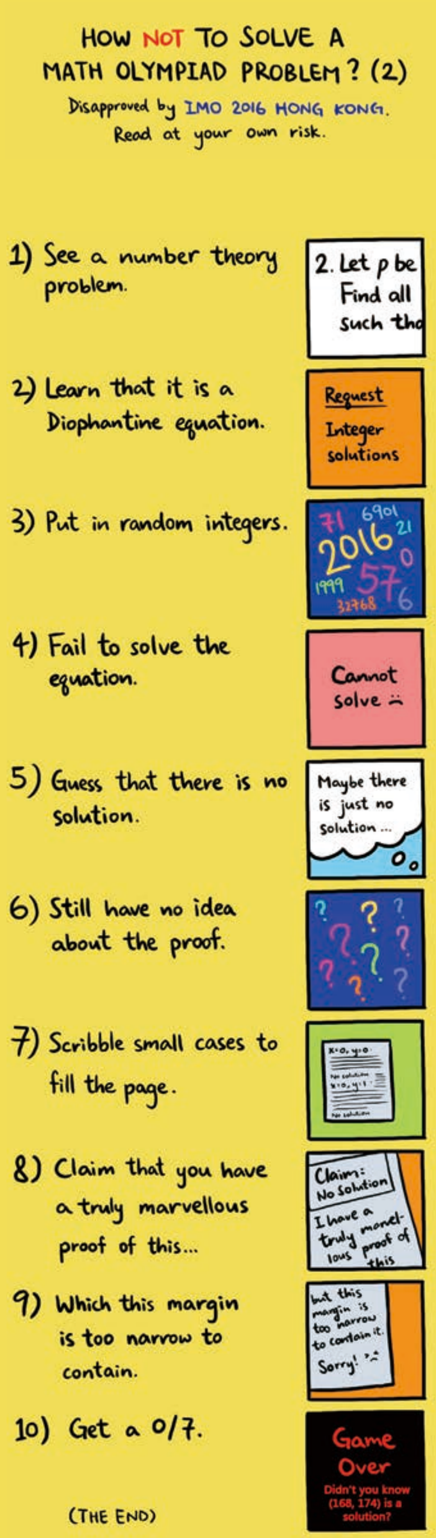
在這個均衡中，約翰的功用的期望值是多少？由於約翰有 1/2 機會跟愛麗一起聽講座，有 1/2 機會跟愛麗一起看電影，故他的功用的期望值為 1.5。同理，愛麗的功用的期望值也是 1.5，這比先前沒有哈諾的協助時的混合策略納什均衡中更高！

What is John's expected utility in this equilibrium? As John will be at the lecture with Alicia with probability 1/2 and at the movie with Alicia with probability 1/2, it follows that his expected utility is 1.5. Similarly Alicia's expected utility is also 1.5, which is higher than they could have gotten in the mixed strategy Nash equilibrium without Harold's help!

這種每人可根據一些信號而選擇策略的均衡，稱為「協調均衡」，並且是納什均衡的眾多推廣和變化之一。此外，人們還通過考慮多階段的博弈、容許對博弈有不確定性，以及放寬參與者的理性條件等等，把納什均衡發揚光大，開拓新天。約翰．納什教授的真知灼見，在他生前身後，都會永遠照亮人間。

This kind of equilibrium, where each player can choose his strategy based on some signal, is called correlated equilibrium, which is one of the many generalizations and variations of Nash equilibrium. Other efforts to expand on the concept of Nash equilibrium include considering games with multiple stages, allowing for uncertainties about the game, and relaxing rationality conditions on players. All these attest to the wondrous world of possibilities inspired by Professor John Nash in his lifetime and beyond.

笑一笑 Laugh Out Loud



挑戰園地 Challenge Corner

過往挑戰園地的解答及得獎名單，可見:

For the solutions and list of awardees of the Challenge Corner of the past issues, please see:

http://www.edb.gov.hk/tc/curriculum-development/kla/ma/IMO/IMOment.html

1. 若 x、y 和 z 為實數，使得 $\frac{x-y}{z}+\frac{y-z}{x}+\frac{z-x}{y}=2015$，求 $\frac{x-y}{z}\cdot\frac{y-z}{x}\cdot\frac{z-x}{y}$ 的所有可能值。

If x, y and z are real numbers such that $\frac{x-y}{z}+\frac{y-z}{x}+\frac{z-x}{y}=2015$, find all possible values of $\frac{x-y}{z}\cdot\frac{y-z}{x}\cdot\frac{z-x}{y}$.

2. 一個三角形的邊長分別為 $\sqrt{5}$、$\sqrt{10}$ 和 $\sqrt{13}$，試求它的面積。（提示：構作一個正方形。）

Find the area of a triangle whose side lengths are $\sqrt{5}$, $\sqrt{10}$ and $\sqrt{13}$ respectively. (Hint: Construct a square.)

3. 求以下方程的所有整數解：

Find all integer solutions to the following equation:

$$\frac{1}{x}-\frac{1}{y}=\frac{1}{2015}$$

4. 參見本期《納什之後》一文中關於協調均衡的討論。假設文中描述的哈諾作出建議的習慣不變，試找出除了文中指出的那個協調均衡之外，該博弈的另一個協調均衡。

Refer to the discussion of correlated equilibria in the article Nash and Beyond in this issue. Given Harold's recommendation habit described in the article, find a correlated equilibrium of the game other than the correlated equilibrium stated in the article.

歡迎香港中學生讀者電郵至 info@imohkc.org.hk 提交解答（包括證明），並於電郵中列明學生中英文姓名、學校中英文名稱及學生班級。每一名學生只可發送一份電郵。首 20名 答對最多題目的同學將獲贈紀念品，但每間學校最多有 3 名同學得獎。解答可以中文或英文提交。打字及掃描文件皆可接受。得獎者將於下一期公布。2016 年第五十七屆國際數學奧林匹克籌備委員會對本活動安排有最終決定權。如有疑問，可電郵至 info@imohkc.org.hk 查詢。

Hong Kong secondary school student readers are welcome to submit solutions (with proofs) via email to info@imohkc.org.hk, specifying the student's name in Chinese and in English, the school's name in Chinese and in English, and the student's class in the email. Each student may send at most one email. Souvenirs will be awarded to the first 20 students solving the most questions on the condition that each school can have at most 3 awardees. Solutions can be submitted in Chinese or English. Both typed and scanned files are acceptable. The awardees will be announced in the next issue. The decision of the Organising Committee of the 57th International Mathematical Olympiad on any matter of this activity is final. Enquiries may be emailed to info@imohkc.org.hk.