Solutions for 2nd issue of IMOment

挑戰園地
Challenge Corner

1. 希淳和彥琪打開了 n 包糖果，每包有 n 顆糖果。希淳先拿 10 顆，然後彥琪拿 10 顆，然後希淳拿 10 顆，然後彥琪拿 10 顆，如此類推。最後，輪到彥琪拿糖果時，只剩下少於 10 顆，於是彥琪便拿了所有剩下的糖果。希淳總共比彥琪多拿多少顆糖果？

   Helsa and Kiki open n bags of candies. Each bag has n candies. Helsa takes 10 candies, then Kiki takes 10 candies, then Helsa takes 10 candies, then Kiki takes 10 candies, and so on. Finally, when it is Kiki's turn to take candies, there are fewer than 10 candies left, so Kiki just takes all the candies left. In total, how many more candies does Helsa take than Kiki?

   解答： 由於$(n+10)^2 = n^2+20n+100$，我們只需檢驗1^2、2^2、……、10^2，便可發現 n^2被 20 除的餘數只能是 0、1、4、5、9 或 16。換言之，若 n^2的十位數為奇數，其個位數必為 6。故此，答案為$10-6=4$。

   Solution: Since $(n+10)^2 = n^2+20n+100$, we need only examine 1^2, 2^2, … 10^2 to find that the remainder when n^2 is divided by 20 can only be 0, 1, 4, 5, 9 or 16. In other words, if the tens digit of n^2 is odd, its units digit must be 6. So the answer is $10-6=4$.

2. 已知 n 為大於 1 的正整數，而邊長為 n、n+1、n+2 的三角形的面積為整數。求 n 的兩個可能值（包括證明）。

   For a positive integer n greater than 1, the area of a triangle with side lengths n, n+1 and n+2 is an integer. Find, with proof, two possible values of n.

   解答：注意到 3 是 n 的一個可能值。此外，若把一個邊長為 5、12、13 的直角三角形和一個邊長為 9、12、15 的直角三角形結合（以長為 12 的邊為公共邊），可得一個邊長為 13、14、15，且面積為整數的三角形。所以 13 是 n 的另一可能值。（如有其他可能值，亦可接受。）

   Solution: Note that $n=3$ works. Also, by combining a right triangle with side lengths 5, 12, 13 and a right triangle with side lengths 9, 12, 15 (with the edge of length 12 as the common edge), we get a triangle with side lengths 13, 14, 15 and an integral area. So $n = 13$ is another possibility. (Other possibilities, if any, are acceptable too.)

3. 在社交網站 Mathbook，每名用戶都有一張個人資料圖片和一張封面圖片。有些用戶的個人資料圖片相同，但不是所有用戶的個人資料圖片都相同。有趣的是，如果兩名用戶的個人資料圖片不同，他們的的封面圖片便相同。求證：所有用戶的封面圖片都相同。

   On the social networking website Mathbook, each user has a profile picture and a cover picture. Some users have identical profile pictures, but not all users have identical profile pictures. Interestingly, any two users who don't have identical profile pictures have identical cover pictures. Prove that all users have identical cover pictures.

   解答：我們只需考慮任何兩名用戶 A、B 的封面圖片都相同。我們只需考慮 A、B 的個人資料圖片相同的情況。由於不是所有用戶的個人資料圖片都相同，故存在用戶 C，其個人資料圖片與 A、B 不同。於是 A、C 的封面圖片相同，B、C 的封面圖片相同。所以 A、B 的封面圖片相同。

   Solution: We need only prove that any two users A and B have identical cover pictures. We need only consider the case that A and B have identical profile pictures. Since not all users have identical profile pictures, there exists a user C who does not have identical profile pictures with A and B. Then A and C have identical cover pictures, and B and C have identical cover pictures.

4. 設 x、y、z、w 為正實數。求證以下不等式。（提示：試用本期有關 Muirhead 不等式的文章中介紹的方法！）

   If x, y, z and w are positive real numbers, prove the following inequality. (Hint: Try the method introduced in the article on Muirhead's inequality in this issue!)

$$(x+y+z+w)^4 \le 64\left(x^4+y^4+z^4+w^4\right)$$

   解答：注意到兩邊的展開式共有相同數目的對稱總和（這可由代入 $(x,y,z,w)=(1,1,1,1)$ 得知）。由於 $(4,0,0,0)$ 比所有總和為 4 的單調遞減數列優化，故由 Muirhead 不等式可知右邊必大於或等於左邊。

   Solution: Note that the two sides, upon expansion, consist of the same number of symmetric sums. (This can be seen by plugging in $(x,y,z,w)=(1,1,1,1)$.) Since $(4,0,0,0)$ majorizes all monotonic decreasing sequences with sum 4, we know by Muirhead's inequality that the right-hand side must be greater than or equal to the left-hand side.