SCHOOL MATHEMATICS NEWSLETTER

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APRIL 1978

Please ensure that every member of
your mathematics staff has an
opportunity to read this Newsletter.

MATHEMATICS SECTION

EDUCATION DEPARTMENT

HONG KONG

PREFACE

The principal objective of the School Mathematics Newsletter (S.M.N.) is to improve the teaching of school mathematics.

You will find a variety of articles in S.M.N. expounding views, theories, experiences and critiques together with an extensive assortment of information supplied by those directly involved. We hope to provide a vertiable pool of ideas for teachers to use, including recreational material. We also hope to create a challenge corner to include puzzles, problems and other investigations that may be of interest.

An important aspect of S.M.N. is the correspondence page. We wish to encourage people to express their views freely and hope to establish a forum in this respect. So if you have something to say or something to argue about, whatever your field in education, put your pen to paper and forward your correspondence to the Editor, School Mathematics Newsletter, Mathematics Section, Advisory Inspectorate, Education Department, Lee Gardens, Hong Kong.

We extend our thanks to all who have contributed to this month's issue.

F. Parkin

Conference on the Teaching of Mathematics in Secondary Schools

The Conference to be held on May 6, 1978 at Grantham College of Education, Gascoigne Road, is organised by the University Graduates Association of Hong Kong and the School of Education of the Chinese University of Hong Kong and sponsored by the Mathematics Section, Advisory Inspectorate, Education Department. The aim of the Conference is to provide a forum for teachers to exchange their ideas and experience on the teaching of Mathematics in secondary schools. The outline of the programme for the Conference is given below :-

9.00 - 9.30	Registration
9.30 - 9.45	Opening address
9.45 -10.45	Talk by Dr. S.C. Cheng
10.45 -11.00	Recess
11.00 -12.00	Workshops (1st round)
12.00 - 1.00	Lunch break
1.00 - 2.00	Film show
2.00 - 3.00	Workshops (2nd round)

Exhibition of books, teaching aids and calculators will be held from 11.00 a.m. to 5.00 p.m.

All teachers are welcome to attend.

For further details, please contact the Mathematics Section, Advisory Inspectorate, Education Department, Hong Kong. (Tel. 5-774001 Ext. 36).

Metrication Seminars for Teachers

A series of metrication seminars designed to promote the further use of metric units in the teaching of mathematics are being organized by the Mathematics Teaching Centre. They will include talks, discussions and exhibits.

At the time of writing, seven seminars, each attended by more than 120 primary school teachers, have already been held with the help of District Education Officers. Seminars of similar nature for secondary school teachers will be held in summer.

Metrication posters, metrication box and reference pamphlets will be distributed to the participants in the seminars.

W.C. WONG

A Programme for Primes

Mr. A.G. Brown
A.D. (C.I.S.)
Education Department

Editoral Note: Although $P(n) = [10^{2^n} A] - 10^{2^{n-1}} [10^{2^{n-1}} A]$ *
gives a 'formula' for the nth prime P(n), it is not a very useful one. To calculate P(n) from this formula, it is necessary to know the values of P(1), P(2), ..., P(n) and the value of A correct to 2^n decimal places. There are a number of similar formulae, but they all suffer from the same defect. There is as yet no formul which we can calculate the value of P(n) for any given n without previous knowledge of its value, and no rule for the prime which follows a given prime. By means of a programmable calculator (HP-25), Mr. Brown writes a programme with 49 programme steps which (i) generates all primes greater than or equal to 5, (ii) displays the next prime that follows a given number, and (iii) may be used to verify whether a given number is prime or not.

$$* \quad A = \sum_{m=1}^{\infty} P(m) 10^{-2^m} = .0203000500000000070 \ldots\ldots$$

$[x]$ represents the integral part of x.

J.S.

Calculator Model : HP-25

Instructions :

1. Store .2 in store 7.
2. Insert number to find next prime.
3. Press R/S to find next prime.
4. For completely new start go back to step 2.

Programme :

DISPLAY		KEY ENTRY	DISPLAY		KEY ENTRY
LINE	CODE		LINE	CODE	
00			25	02	2
01	14 71	f x = y	26	23 51 01	ST + 1
02	13 20	GTO 20	27	23 04	STO 4
03	01	1	28	05	5
04	51	+	29	23 03	STO 3
05	06	6	30	24 01	RCL
06	23 01	STO 1	31	24 03	RCL 3
07	71	÷	32	71	÷
08	14 01	f INT	33	14 73	f Last x
09	23 61 01	ST x 1	34	21	x ⇄ y
10	14 73	f Last x	35	14 41	f x < y
11	15 01	g Frac	36	13 48	GTO 48
12	24 07	RCL 7	37	15 01	g Frac
13	14 41	f x < y	38	15 71	g x=0
14	32	CHS	39	13 20	GTO 20
15	05	5	40	24 04	RCL 4
16	61	x	41	32	CHS
17	23 41 01	ST – 1	42	23 04	STO 4
18	23 02	STO 2	43	15 51	g x ≥ 0
19	23 51 02	ST + 2	44	23 51 03	ST + 3
20	24 02	RCL 2	45	02	2
21	32	CHS	46	23 51 03	ST + 3
22	23 02	STO 2	47	13 30	GTO 30
23	15 51	g x ≥ 0	48	24 01	RCL 1
24	23 51 01	ST + 1	49	31	ENTER

HOW GOOD ARE HONG KONG STUDENTS IN MATHEMATICS?

Dr. CHENG SHIU CHING

In Hong Kong we often heard that a mediocre student did well in Mathematics when studying abroad. This is not an isolated incident. If this is so, the phenomenon itself is worth investigation. To do this an international achievement measurement is needed.

One of the well known surveys of achievement in education across nations is the IEA—The International Study of Evaluation of Educational Achievement. This organisation, since its inception in 1962, has conducted several international surveys in educational achievements. In these surveys, it was found that achievements of students were related not only to a number of pedagogical factors such as course content, teacher qualifications and classroom schedules, but also to many socio-economic factors in the studied countries.

The IEA study on mathematics achievement was done in 1964 with twelve participating countries. They were Australia, Belgium, England, Federal Republic of Germany, Finland, France, Israel, Japan, The Netherlands, Scotland, Sweden and the United States of America.

In the survey, the target populations were the 13 year old students and the pre-university grade, since they represent the two major terminal points of education in these countries. The 13 year old population was further subdivided into 1a and 1b. The 1a population was for all students who were aged between 13:00-13:11 (year:month) at the date of testing, while the 1b population was for the grade level in which the majority of students were of age 13:00-13:11.

The test for 1b population consists of 70 questions. The following table shows its composition.

Topic	Number of questions
Basic arithmetic	13
Advanced arithmetic	18
Elementary algebra	12
Intermediate algebra	4
Euclidean geometry	13
Analytic geometry	1
Set	4
Affine geometry	3
Others	2
	Total 70

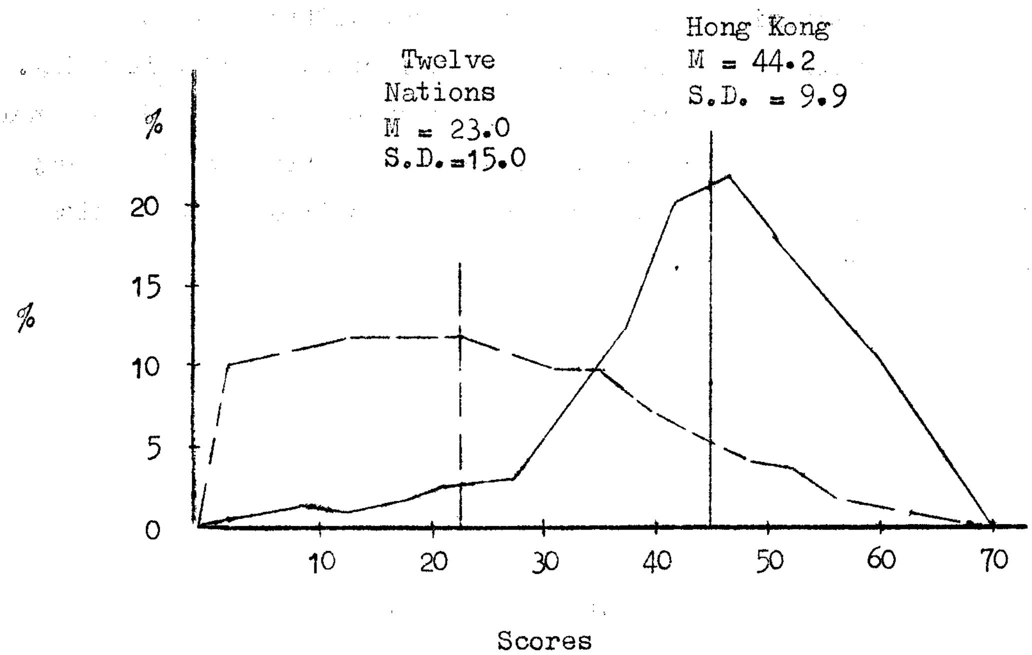
The test had been administered to 132,775 students from 5348 schools in the twelve nations. The result of the major findings were analysed and published in the two volume book edited by the director of this project Professor T. Husen. In 1970 the writer had the opportunity to invite a group of testers from the Third-year Mathematics Special Course in the Northcote College of Education to conduct this test. In the Hong Kong survey, 415 students from 14 schools were sampled. These included students from urban and rural schools, Chinese and Anglo-Chinese schools, grammar and technical schools.

As the test used was for the 1b population, our samples were students in classes in which the majority were of age 13:00-13:11. In the sample there were 215 male students and 200 female students. The result of the test is given in the distribution table below. For comparison, the statistics from the international inquiry are also listed.

Distribution of scores in percentage

Score	Hong Kong	Twelve nations total
0		1.0
1-5		10.0
6-10	0.5	11.0
11-15	0.2	12.0
16-20	0.7	12.0
21-25	2.5	12.0
26-30	3.4	10.0
31-35	7.7	10.0
36-40	12.0	7.0
41-45	18.3	6.0
46-50	20.7	4.0
51-55	16.9	3.0
56-60	10.4	1.0
61-65	5.1	0.5
66-70	1.4	0.1
Mean	44.2	23.0
S.D.	9.9	15.0

Distribution of Test Score in Hong Kong



The Hong Kong students made a mean score of 44.2 with a standard deviation of 9.9. This mean score is high in comparison with that obtained by the twelve nations. From the above table it can be seen that two thirds of the Hong Kong students scored between 40 and 60 out of a maximum score of 70. The overall mean is 21.2 marks higher than the average mark of the twelve nations. In terms of standard deviation this is 1.4 S.D.

At first, this figure seems unreasonably high and casts doubt on its validity. But as we look at it from the angle of selectivity of samples, this is not hard to explain. In 1970 there were only about 50% of the 13:00-13:11 age group children in schools in Hong Kong, while all the twelve nations nearly had 100% of the age group in schools.

Assuming the test population in Hong Kong represents the upper 50% of the age group, the comparison of the scores could be made more meaningfully with the upper 50% of the twelve nations achievements. The following table shows this comparison. In this table we could see that Hong Kong falls behind Japan and Israel and becomes the third highest nation in mathematics achievement.

Of course, achievement measurement as such should not be mis-interpreted as an international gallop. It is not difficult to see that score of achievement does not necessary reflect successful learning. In fact, any educational achievement is only an outcome of the educational complex formed by dozens of variables. As these variables are in no way equal in the investigated countries, we are in no position to say which are really the better achievers.

Comparison of Hong Kong Mathematics Achievement with the Upper 50% of Scores in 12 Nations

Population 1b.

Score	Aus	Bel	Eng	Fin	France	Ger	Isr	Jap	Net	Sco	Swe	USA	Total	Hong Kong
0														
1-5														
6-10														0.5
11-15	1.0										11.0			0.2
16-20	12.0		1.2		8.6				1.6	1.2	30.0	21.0		0.7
21-25	28.0		10.0		26.0				32.0	20.0	22.0	24.0	16.8	2.5
26-30	22.0		14.0	37.0	18.0	31.4			22.0	18.0	18.0	20.0	20.0	3.4
31-35	18.0	21.0	16.0	28.0	16.0	30.0	8.0	13.2	18.0	16.0	10.0	14.0	20.0	7.7
36-40	10.0	26.0	14.0	20.0	10.0	16.0	26.0	22.0	12.0	14.0	6.0	10.0	14.0	12.0
41-45	6.0	24.0	12.0	8.0	10.0	12.0	24.0	18.0	6.0	12.0	2.0	6.0	12.0	18.3
46-50	2.0	16.0	14.0	6.0	4.0	6.0	18.0	18.0	6.0	10.0	0.8	2.0	8.0	20.7
51-55	0.6	8.0	10.0	1.0	6.0	4.0	14.0	16.0	2.0	6.0	0.2	2.0	6.0	16.9
56-60	0.2	4.0	4.0	0.0	1.0	0.6	6.0	10.0	0.2	2.0		0.8	2.0	10.4
61-65	0.2	1.0	4.0		0.4		4.0	2.0	0.2	0.8		0.2	1.0	5.1
66-70	0.0	0.0	0.8		0.0			0.8					0.2	1.4
Mean	28.73	42.1	39.6	34.05	31.84	35.28	44.9	45.18	31.29	35.26	23.86	28.51	34.49	44.2

十進制的簡史

教育司署
數學科高級督學 黃煥章

十八世紀末，法國是西方文化的領導者。當時法國認爲度量衡的單位太混亂，又沒有標準，遂於1795年設一委員會以建立一個完全新的度量衡制度，這便是米突制（或後人稱爲十進制）的起源。

1799年，該委員會決定以地球的子午線爲標準，並以北極通過巴黎至赤道的距離的$\frac{1}{10,000,000}$爲一米。當時認爲該長度對日常生活的量度較爲適合。又以每邊$\frac{1}{10}$米的正立方體的容量（即1000立方厘米）稱爲一升；以$\frac{1}{1000}$升（即一立方厘米）的純水在4°C時的重量爲一克，作爲重量標準。並依此定義，用鉑銥合金鑄成標準米尺及千克法碼各一，藏於法國檔案局中。這便是十進制的開始。此制度有兩特點：一是體積與重量有密切的關係；二是化聚因數皆爲10的乘冪，在化聚各單位時，祇需移動小數點的位置。

法國實行米突制後，於1875年，經十七國同意，簽訂公約成立國際度量衡局(CGPM－General Conference on Weights and Measures)，並通過以米突制作爲國際度量衡的通用制度。當時的基本單位：長度爲米，重量爲千克，時間爲秒。

量度儀器及技術不斷進步，後來發現由北極通過巴黎至赤道的真實距離並非10,000,000米而應爲10,002,288.3米。所以一米實較地球子午線全長的$\frac{1}{40,000,000}$爲短，而1799年所訂定的定義便失去其意義。1964年國際度量衡代表決議承認一米的新定義爲氪86 (krypton 86) 在真空中發射的橙黃線波長的1 650 763.73倍。（此長度與法國在1799年用鉑銥合金鑄成而藏於法國檔案局的標準米尺長度相同）

在1927年，經過精確測量，發現標準千克法碼相當於1000.028立方厘米純水在4°C時的重量。所以千克的定義改爲相當於法國檔案局標準千克法碼的重量。但一千克與一升間的關係仍舊保持，因此一升亦改爲相當於1000.028立方厘米。但此項差異僅影響極精密的量度，在日常生活中，一升仍相等於1000立方厘米。

國際單位制 (SI Units－International system of metric weights and measures)

1960年因科技上的進步，量度時需要較大或較小的單位，於是各國科學家都希望有一種國際間公認的新量度單位以促進國際間的貿易及科技上的研究，於是以千進爲主的國際單位制便由此產生。茲列舉各單位如下：

類別	單位	符號
基本單位：		
長度	米	m
質量	千克	kg
時間	秒	s
電流	安培	A
熱力學的溫度	開氏度	K
發光强度	燭光	cd
輔助單位：		
平面角	弧度	rad
立體角	球面度（立體角單位）	sr

導出單位（祇列舉部份單位）：

面積	平方米	m^2
體積	立方米	m^3
速度	每秒＿米	m/s

該制度的特點：

（一）凡屬一般性的單位，英文名稱字首不需大寫；簡寫後不必加縮寫符號（即dot「·」）；衆數詞亦不用加「s」如6m，8kg等。

（二）凡用以紀念前人的單位，英文名稱字首不用大寫如ampere，watt等，然僅以簡寫代表該單位則需用大楷如A，W等。

（三）日常溫度計算時仍沿用攝氏。但水的冰點0°C相當於273.15K。

（四）在歐洲各國，逗點（，）的意義各有不同，故有三位加一逗點的記數法改用「離位法」，如2 314 297或0.196 25。

（五）此制度着重千進，其化聚因數祇須乘以或除以10^3。

如　長度　　1千米＝1×10^3米＝1000米

＝1000×10^3毫米

＝1 000 000毫米

重量　　1 000 000毫克

＝1 000 000÷10^3克

＝1 000克

＝1 000÷10^3千克

＝1千克

SETS OF NUMBERS WHOSE MEANS AND STANDARD DEVIATIONS ARE ROUND NUMBERS

by Greogry CHAN and Joseph SHIN

To calculate the standard deviation of a set of numbers, a student may

(a) use a calculator that has a built-in program for doing the job, or

(b) use the formula $s = \sqrt{\dfrac{\sum_{i=1}^{N}(x_i - \alpha)^2}{N} - (\bar{x} - \alpha)^2}$

where α is a suitable constant, or

(c) use the formula $s = \sqrt{\dfrac{\sum_{i=1}^{N} x_i^2}{N} - \bar{x}^2}$, or

(d) use the formula $s = \sqrt{\dfrac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N}}$

Method (a) gives us the answer in a short time without any labour in calculation. A student who uses this method to do his exercise only gains the skill of using the calculator. So, this method is only recommended when the student has well understood the meaning of standard deviation.

Method (b) is recommended in many books written before electronic calculators became cheap and popular. Examples of using method (b) are often shown in table forms with a methematical proof of the formula preceeding it. But when electronic calculators (without built-in program to calculate standard deviation) are available, the best constant for α

is zero and so the formula becomes the one in method (c). Hence, nowadays, method (c) is preferred to method (b). In fact, the formula in method (c) is much easier to apply than the formula in method (d).

Nevertheless, the formula in method (d) is the most instructive formula. It helps students to understand that the standard deviation of a set of numbers is the root mean square of the deviations of the numbers from the mean. Doubtlessly, it is the most crumsy formula to apply especially when the mean is not a round number. Because of the instructive value of the formula, it is recommended to encourage students to do a few examples using this formula. But, to releaf their difficulty in calculation, it is recommended to provide students with sets of numbers (not too many in number) whose means are round numbers. And students will appreciate if the standard deviations are also round numbers and they are told of this before they start calculating. So, when they find that their answers are not round numbers, they will look back and check their calculations, and if their answer is a round number, they are sure that the correct answer is obtained and would have some sense of satisfaction.

The following table lists several sets of numbers whose means are all zero and their standard deviations are round numbers.

	N	x_i	s_x
1	6	-4, -3, 0, 0, 2, 5	3
2	6	-4, -2, -2, 1, 2, 5	3
3	6	-3, -2, -2, 0, 1, 6	3
4	6	-3, -3, -2, 0, 4, 4	3
5	6	-4, -3, -3, -1, 5, 6	4
6	6	-5, -3, -3, 1, 4, 6	4
7	6	-5, -5, 0, 1, 3, 6	4
8	6	-7, -5, -1, 1, 5, 7	5
9	6	-6, -6, -2, 3, 4, 7	5
10	10	-5, -1, -1, 0, 0, 0, 1, 2, 2, 2	2
11	10	-3, -2, -2, -1, 0, 0, 1, 1, 2, 4	2
12	10	-6, -5, -4, -2, 0, 0, 3, 3, 5, 6	2

More examples could be constructed by transforming the above sets using the formula

$$y_i = ax_i + b.$$

Then, for the y's, the mean will be b and the standard deviation will be as_x . More examples could be constructed by pooling two or more sets whose means and standard deviations are the same.

On Standard Deviations

Ko Lo Suen
Mathematics Section, E. D.

Which of the following

$$\text{Standard Deviation} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\text{Standard Deviation} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

is correct ?

Normally in school, we are taught that the standard deviation of a set of numbers $x_1, x_2, \ldots\ldots, x_n$ are defined as

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

but if you happen to buy a calculator which has a key for stadard deviation then you may notice that the value of standard deviation is calculated from the formula

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

It is quite interesting to know why there is such a difference.

Let us now consider a population

$$(z_1, z_2, z_3, \ldots\ldots\ldots, z_m)$$

consisting of m numbers. The population mean μ , and population variance σ^2 are given as

$$\mu = \frac{1}{m}\sum_{i=1}^{m} z_i \quad , \quad \sigma^2 = \frac{1}{m}\sum_{i=1}^{m}(z_i - \mu)^2$$

A sample of n numbers $x_1, x_2, x_3, \ldots\ldots, x_n$ are selected from the population.

$(z_1, z_2, \ldots\ldots, z_m)$ population mean $= \mu$
population variance $= \sigma^2$

↓ a sample is selected

$(x_1, x_2, \ldots\ldots, x_n)$ sample mean $= \bar{x}$
sample variance $= s^2$

The sample mean $\bar{x}$ and the sample variance s^2 are calculated as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i ,$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

It is quite natural to ask whether or not $\bar{x}$, s^2 are reasonablc estimators of the population mean μ and the population variance σ^2 ? To define a resonable estimator, we imagine that for every sample of n numbers from the population, the mean and variance are calculated (actually the total no. of samples is $N = C_n^m$). Let those sample means and sample variances be

$$\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\ldots\ldots\ldots, \bar{x}_N$$

$$s_1^2, s_2^2, s_3^2, \ldots\ldots\ldots\ldots, s_N^2$$

It is natural to say that $\bar{x}$, s^2 are reasonable estimators of μ and σ^2 if

$$\text{Mean of}\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\ldots\ldots\ldots, \bar{x}_N\} = \mu ,$$

$$\text{Mean of}\{ s_1^2, {}_2^2, s_3^2, \ldots\ldots\ldots\ldots, s_N^2\} = \sigma^2$$

Unfortunately, only the first of the above holds. Thus s^2 is not a resonable estimator (in mathematical term, not an unbiased estimator) of the population variance σ^2. In any case, if we change the definition of s^2 as

$$\frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

then it becomes a reasonable estimator. Generally speaking, for any population, it is reasonable to use the sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

to estimate the population mean $\mu = \frac{1}{m}\sum_{i=1}^{m} z_i$ and to use the number

$$\frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

to estimate the population variance $\sigma^2 = \frac{1}{m}\sum_{i=1}^{m} (z_i - \mu)^2$. A mathematical proof is given below.

Suppose that $x : x_1, x_2, \ldots\ldots, x_n$ and $y : y_1, y_2, \ldots y_n$ are independent random samples from the population $z : z_1, z_2, \ldots\ldots, z_m$ with mean μ and variance σ^2. The following properties will be used and are listed for reference.

Definition: $E(x) = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$; $V(x) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Prob. 1 $E(ax) = a\,E(x)$
Prob. 2 $E(x+y) = E(x) + E(y)$
Prob. 3 $E(x^2) = V(x) + [E(x)]^2$
Prob. 4 $V(x+y) = V(x) + V(y)$
Prob. 5 $V(ax) = a^2 V(x)$

Firstly we are going to see that the mean of the sample means is equal to the population mean.

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \qquad E(\bar{x}) = E\left[\frac{1}{n} \sum_{i=1}^{n} x_i\right],$$

$$= \frac{1}{n} \sum_{i=1}^{n} E(x_i) = \frac{1}{n}(n\mu) = \mu \quad \text{by Prob. 2}$$

since each x_i is an independent random variable of the population.

We are now in a position to investigate the mean of the sample variances

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 ,$$

By Prob. 3, $$s^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2 ,$$

$$E(s^2) = \frac{1}{n} E\left[\sum_{i=1}^{n} x_i^2\right] - E(\bar{x}^2), \text{ by Prob.2,}$$

$$= \frac{1}{n} \sum_{i=1}^{n} E(x_i^2) - E(\bar{x}^2) \quad \ldots\ldots\ *$$

For each i, we have $$E(x_i^2) = V(x_i) + [E(x_i)]^2 \quad \text{by Prob. 3,}$$
$$= \sigma^2 + \mu^2$$

Also, $$E(\bar{x}^2) = V(\bar{x}) + [E(\bar{x})]^2, \text{ by Prob. 3,}$$
$$= V\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) + \mu^2,$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} V(x_i) + \mu^2 \qquad \text{by Prob. 4,5,}$$

$$= \frac{\sigma^2}{n} + \mu^2$$

On substituting into *, we have

$$E(s^2) = \frac{1}{n} \sum_{i=1}^{n} (\sigma^2 + \mu^2) - (\frac{\sigma^2}{n} + \mu^2)$$

$$= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2$$

$$= \frac{n-1}{n} \sigma^2$$

Equivalently, $E(\frac{n}{n-1} s^2) = (\frac{n}{n-1})(\frac{n-1}{n} \sigma^2) = \sigma^2$, by Prob. 1.

But $\frac{n}{n-1} s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$.

This means that $\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ is a better estimator of the population variance σ^2.

(Reproduced from "Clementi Middle School Golden Jubilee Issue")

數學教學上計算機應用舉例

一炎木一

隨着小型電子計算機的價錢日漸降低，愈來愈多學生擁有計算機，（筆者在月前曾見過一部有＋、－、×、÷、√‾和記憶系統的計算機標價只不過二十五元）；無可懷疑，計算機已代替了昔日的計算尺，並且在工商界和科學研究的各行各業中，受到廣泛的應用。從數學教育觀點看：

1. 計算機能快速運算多個位數字，方便而有效率，可以鼓勵學生學習數學快而準的精神。
2. 計算機能減少花費於繁雜計算工作上的時間，從而促使學生集中注意力去瞭解數學概念。
3. 計算機能鼓勵學生驗證的精神，探討和發現數學的規律，培養創造能力。
4. 計算機能處理真實的問題，鼓勵學生對量的變化提出估計和約值，擴大數學問題的內容和處理的技巧。

作爲數學教師，對些種價廉物美，日漸普遍的精巧設計，很值得考慮到課堂上如何利用它，發揮上述的優點，促進教學的效果。以下是筆者從各種書刊中看到的一些應用例子，提供給各位老師參考：

㈠　恒等式 $(a+b)(a-b)=a^2-b^2$ 的發現

甲　計算和觀察下列的乘積：

5 x 5 = ?	(25)	9 x 9 = ?	(81)
6 x 4 = ?	(24)	10 x 8 = ?	(80)
25 x 25 = ?	(625)	30 x 30 = ?	(900)
26 x 24 = ?	(624)	31 x 29 = ?	(899)

我們可以發現有如下的一個規格：

$$\square \times \square = \square^2$$

$$(\square + 1)(\square - 1) = \square^2 - 1$$

乙　再計算和觀察下列的乘積：

6 x 6 = ?	(36)	4 x 4 = ?	(16)
8 x 4 = ?	(32)	6 x 2 = ?	(12)
19 x 19 = ?	(361)	40 x 40 = ?	(1600)
21 x 17 = ?	(357)	48 x 38 = ?	(1596)

我們可以發現有如下的一個規格：

$$\square \times \square = \square^2$$

$$(\square + 2)(\square - 2) = \square^2 - 4 = \square^2 - 2^2$$

丙　再計算和觀察下列的乘積：

7 x 7 = ?	(49)	8 x 8 = ?	(64)
10 x 4 = ?	(40)	11 x 5 = ?	(55)
60 x 60 = ?	(3600)	18 x 18 = ?	(324)
63 x 57 = ?	(3591)	21 x 15 = ?	(315)

我們可以發現有如下的一個規格：

$$\square \times \square = \square^2$$

$$(\square + 3)(\square - 3) = \square - 9 = \square - 3^2$$

再進一進有：

若 $18 \times 18 = 324$，則 $22 \times 14 = 324 - 4^2 = 304$

若 $70 \times 70 = 4900$，則 $75 \times 65 = 4900 - 5^2 = 4875$

若 $40 \times 40 = 1600$，則 $46 \times 34 = 1600 - 6^2 = 1564$

我們可以歸納法爲以下一個規格：

若 $\square \times \square = \square^2$，則 $(\square + \triangle)(\square - \triangle) = \square^2 - \triangle^2$

採用代數符號來表示，使得平方差的恒等式：

$$(a + b)(a - b) = a^2 - b^2$$

(二) 平方根的概念認識：

設 a 爲一正實數，符號 $\sqrt{a}$ 是代表怎樣的一個數？根據定義，$\sqrt{a}$ 是方程 $x^2 = a$ 的一個正數解，即是 $(\sqrt{a})(\sqrt{a}) = a$ 。故此：

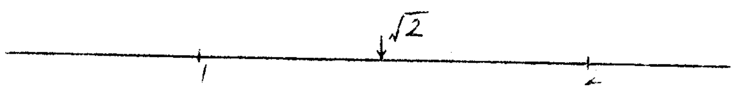
因爲 $4 \times 4 = 16$，所以 $\sqrt{16} = 4$；

因爲 $3 \times 3 = 9$，所以 $\sqrt{9} = 3$ 。

但如 $\sqrt{2}$，$\sqrt{3}$，$\sqrt{5}$，… 這些數又怎樣呢？現在以 $\sqrt{2}$ 爲例，我們對 $\sqrt{2}$ 作一些估值。首先小心觀察以下運算：

$1 \times 1 = 1$	太小
$2 \times 2 = 4$	太大
$3 \times 3 = 9$	太大

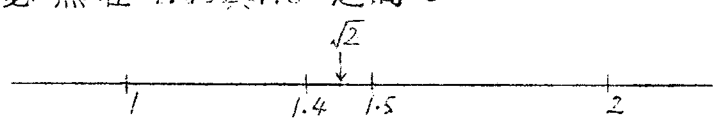
因爲 $\sqrt{2}$ 是具備有如此關係 $\sqrt{2}\,\sqrt{2} = 2$ 的一個正數，所以我們有充份理由假定 $\sqrt{2}$ 在 1 與 2 之間。



但 $\sqrt{2}$ 究竟有 1.1　1.2　　或是 1.9，我們再考慮以下乘積：

$1.1 \times 1.1 = 1.21$	太小
$1.2 \times 1.2 = 1.44$	太小
$1.3 \times 1.3 = 1.69$	太小
$1.4 \times 1.4 = 1.96$	太小
$1.5 \times 1.5 = 2.25$	太大

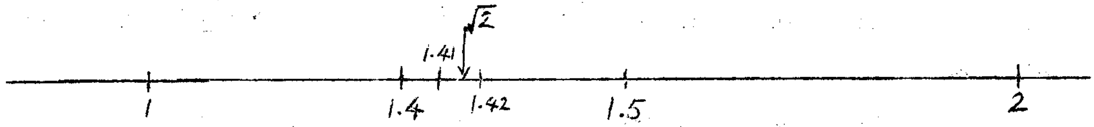
故此　必然在 1.45與1.5 之間。



再觀看以下乘積：

$1.41 \times 1.41 = 1.9881$	太小
$1.42 \times 1.42 = 2.0164$	太大

可以知道$\sqrt{2}$在 1.41 與 1.42 之間



依照以上方法繼續對$\sqrt{2}$作出估值，我們有

$\sqrt{2}$ 介於 1.414 與 1.415 間

$\sqrt{2}$ 介於 1.4142 與 1.4143 間

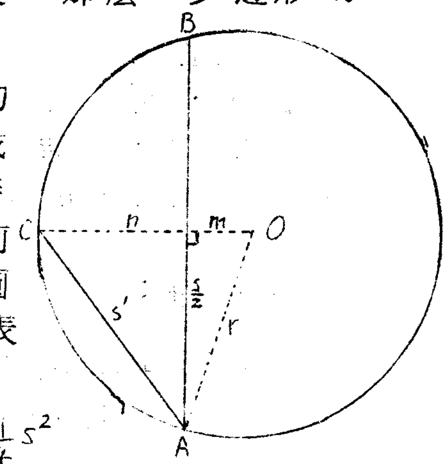
$\sqrt{2}$ 介於 1.41421 與 1.41422 間

$\sqrt{2}$ 介於 1.414213 與 1.414214 間

我們可以認識到對$\sqrt{2}$只能作一估值，而不能用一準確小數來表示。對$\sqrt{3}$和$\sqrt{5}$亦可用相同的方法得到一些估值。總的來說，$\sqrt{a}$未必能用一個小數來表示，它只具備一個根本的關，就是$\sqrt{a}\times\sqrt{a}=a$。

(三) π 的約值計算

我們知道π是圓周的周圍長與直徑的比率，在這裏，我們準備用一內接正多邊形的周界代替圓周長，當內接正多邊形的邊數增加時，多邊形的周界便逼近圓周長，那麼，多邊形的周界與直徑比率便是π的一個約值。



見右圖，設 AB 為一內接正多邊形的一邊，其長度 s 為已知，現在弧 $\widehat{AB}$ 上找一中點 C，聯結 AC，AC 便是另一內接正多邊形的一邊，此多邊形的邊數較先前一個增加了一倍。設 AC 的長度為 s'，圓的半徑為 r，我們尋求一個利用 s 和 r 表示的 s' 的長度公式：

首先由畢氏定理，在 $\triangle ACD$ 內，$m^2=r^2-\frac{1}{4}s^2$

所以 $m=\sqrt{r^2-\frac{1}{4}s^2}$

另外，因為 $OC=r$，所以 $n=r-m=r-\sqrt{r^2-\frac{1}{4}s^2}$

因而有 $n^2=r^2-2r\sqrt{r^2-\frac{1}{4}s^2}+r^2-\frac{1}{4}s^2$

再從 $\triangle ADC$ 內，由畢氏定理 $s'^2=n^2+\frac{1}{4}s^2$

代入 n^2 之後，我們有 $s'^2=2r^2-2r\sqrt{r^2-\frac{1}{4}s^2}$

$$=2r^2-r\sqrt{4r^2-s^2}$$

所以

$$s'=\sqrt{2r^2-r\sqrt{4r^2-s^2}} \qquad (*)$$

如果假定 $r=1$，內接的正六邊形的邊長便為 1，周界長便為 6，周界與直徑的比例是 $6/2=3$，這便是π的第一個約值。如果此內接正六邊形依上述方法平分，便得一內接正十二邊形，長度便是：

$s'=\sqrt{2-\sqrt{4-1}}$ （因為內接六邊形的 $s=1$ 和 $r=1$ ）

$\doteq 0.517638$

周界與直徑的比率是 $0.517638 \times 12 \div 2 = 3.10583$，這便是 π 的第二個約值。

下表是應用公式(*)九次後的數值紀錄，我們可以看到周界與直徑的比率愈來愈接近 π 的真值。在表中，符號 S_n 是代表內接單位圓內的正 n 多邊形的一邊長。

多邊形	一邊長度	周界	周界／直徑
S_6	1.0000000	6.0000000	3.0000000
S_{12}	0.5176381	6.2116572	3.1058286
S_{24}	0.2610524	6.2652576	3.1326288
S_{48}	0.1308063	6.2787024	3.1393512
S_{96}	0.0654382	6.2820672	3.1410336
S_{192}	0.0327235	6.2829120	3.1414560
S_{384}	0.0163623	6.2831232	3.1415616
S_{768}	0.0081812	6.2831616	3.1415808
S_{1536}	0.0040906	6.2831616	3.1415808

以上三個例子所涉及的數學內容均包括在目前中學數學課程範圍內，對這些繁複數目作規則性的運算，使用計算機是很自然的；其實，現有的課程有很多好的教材是值得利用計算機以增進教學的效果和解決學生在學習上的困難。

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A Problem on Locus

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Given: (i) A fixed line L,
(ii) A fixed point A outside L,
(iii) If B is any point on L, construct the equilateral triangle ABP.

Find the locus of P.

[Solution] Without loss of generality, let L be the x-axis, and A be the point (0,1). Let the co-ordinates of B be (a,0), say, where a is a real number. Then we form the equilateral triangle ABP, and let the co-ordinates of P be (x,y).
Thus

$$\sqrt{a^2+1} = \sqrt{x^2+(y-1)^2} = \sqrt{(x-a)^2+y^2}.$$

By eliminating a, we get

$$2x\sqrt{x^2+y^2-2y} = x^2+y^2-1$$

so that

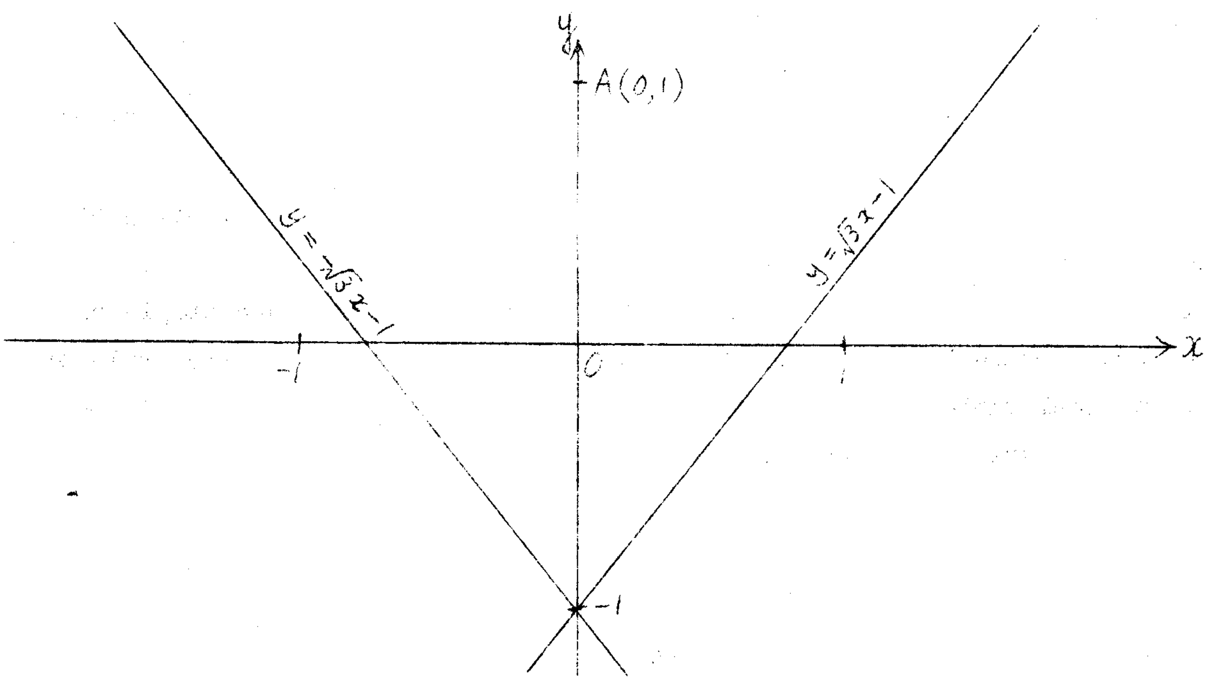
$$4x^2(x^2+y^2-2y) = (x^2+y^2-1)^2$$

i.e. $3x^4-y^4+2x^2y^2-8x^2y+2x^2+2y^2-1 = 0$,
which is a quartic equation.

However, this equation can be rewritten into

$$(\sqrt{3}x+y+1)(\sqrt{3}x-y-1)(x^2+(y-1)^2) = 0$$

which represents a pair of straight lines passing through (0,1) with slope $-\sqrt{3}$ and $\sqrt{3}$ respectively, and a point of sigularity, viz. (0,1).
Hence, the locus of P is a pair of straight lines.



A SURVEY ON THE ATTITUDE OF SCHOOL CHILDREN TOWARDS THE USE OF CALCULATORS IN SCHOOLS

JOSEPH SHIN

The survey, in the form of a questionaire, was made last year on four classes of fourth formers (N = 152) in Clementi Middle School. A copy of the questionaire (translated into English) is given in the appendix.

I. Analysis of replies

The sample consists of four Form 4 classes, two Science (74 boys and girls) and two Arts (78), giving a sample size of 152. The replies show no significant difference between the Science and Arts classes, though one would think the former had more occasions to do calculations than the latter. However, all children have to take mathematics irrespective of whether they take other science subjects or not, and it seems that the possible use of calculators in mathematics alone is enough to make the Arts classes think favourably of it.

More than 77% of children in the sample have some experience using calculators and 60% of them or their families own a calculator, while less than1% of them have a slide rule and almost all of them prefer a calculator to a slide rule.

It is interesting that the more advanced calculators have a greater appeal than those that can only handle simple arithmetic, though the latter are more widely advertised in newspapers. The preference for advanced types springs partly out of a wish to have the best; however, there is a genuine desire to get rid of involved computations with a calculator and to do away with mathematical tables (Questions 6 and 11).

It also appears (Questions 9 and 10) that children do not find the idea of allowing the use of calculators in public examinations objectionable; indeed, over 76% think candidates may bring their own calculators.

To summarize, the calculator has become a common calculating aid among school children. Because of its cost and limited speed, the slide rule has never been popular. An electronic calcualtor, even a simple one, can work very much faster and children like it better than a slide rule or mathematical tables. They see it as a useful aid and think it should be allowed in public examinations.

II. Mathematical Tables vs calculators - a reflection

Some people are justifiably suspicious of the role of calculators in mathematics education - will they not impair children's grasp of fundamental operations like addition and multiplication ? To many people the multiplication table is something to be memorized at all cost and to be known backwards. To them calculators are not without objection because they enable children to do multiplications accurately and quickly without requiring them to memorize their tables.

I think the multiplication table is useful only because we cannot always carry a calculator around. If one day every watch is also a calculator (which is very likely to happen, given the present trend of miniaturization of calculators and quartz watches), much of the usefulness of the multiplication table (which is its portability) will have been lost. The multiplication table illustrates no mathematical principle (except the commutative law of multiplication, perhaps) and a man who knows $3 \times 6 = 18$ by heart is not necessarily a better mathematician than one who works it out the long way, as $(6 + 6) + 6 = 12 + 6 = 18$.

Besides, one can learn the multiplication table with the help of a calculator by checking one's memory against the LED display. (A type of calculator advertised in Dollar Saver displays YES and NO instead of numbers. A child enters a calculation on the machine and also its answer, which he got by his own calculation. If the answer is right the machine displays YES; if NO he has to do it again. Of course this kind of toy is not a 'calculator' in the strict sense).

With Logarithms, mathematical tables may be better than calculators in eliciting the mathematics behind their use. As for compound interest, it is more than likely that involved computation will distract children from the mathematics, which is just that the accumulated sum (interest + original deposit) follows a geometrical progression. (Because they are living in an age of 'buy now pay later', calculations with hire-purchase repayments is of some practical importance. However, such work is mathematically barren and can easily become boring if too much time is spent on it. A calculator enables one to get the answer without the obsession of the computation.)

The strength of logarithm tables as a learning aid is its limited range: To find log 314.2, one has to write it as $\log (3.142 \times 10^2) = \log 3.142 + \log 10^2 = 2 + \log 3.142$, thus reinforcing two theorems of log calculation. (This necessity to transform a number becomes a disadvantage when it is only the numerical answer that matters.) This limitation also gives trigonometric tables some merit. Thus

$$\sin 120^\circ = \sin (180^\circ - 60^\circ) = \sin 60^\circ = 0.8660$$

is more than a mere calculation.

Apart from this merit (which arise out of the weakness of tables as a computational tool), tables cannot stand up favourable against electronic calculators educationally, not to say as computational tools. The idea of a limit and convergence, for example, is better brought out with a calculator than laboriously with tables.

There are other surprises. If we press 1 (radian) and then cos, and then the cosine of the display, then cos that, etc., the sequence of displays oscillates slightly and eventually stabilizes at .7390851332. This is interesting, for not only have we solved the equation $\cos x = x$, we are also inspired to investigate the sequence

$$\cos x, \cos \cos x, \cos \cos \cos x, \ldots$$

Given x, must this sequence converge ? If yes, to what limit ? Etc.

With a programmable calculator the mathematics master has an even more powerful teaching aid. Programming is now something within reach, and children can learn the technique of writing programs and through them learn many other things, as, approximating to the area of a circule by a sequence of regular polygons and hence calculating π to any desired degree of accuracy, finding the 100th prime, doing a numerical integration, comparing an approximation $dy = f'(x)\,dx$ with $\Delta y = f(x + \Delta x) - f(x)$, generating a set of random numbers (possible with some models) to simulate a random experiment and even nuclear disintegration, etc.

In short, the calculator is not just a computational tool, good as it is at that. It is a useful teaching aid for the exposition of certain topics, and its use makes it possible to teach others that are otherwise difficult or impossible to teach. If one remembers that the one single merit of mathematical tables is also a drawback, and that calculators can do no harm to the learning of 'pure' algebra and geometry, calculators will have an appeal over and above their ability to do calculations quickly.

Reference

1. A.G. Brown, Electronic calculators and their use in education.

APPENDIX

THE QUESTIONAIRE

Replies are shown in the form 'm + n', where m is the number of replies from the Science classes and n the number from the Arts classes.

1. Have you ever used an electronic calculator?

☐ Yes	☐ No
56 + 62 = 118	18 + 16 = 34

2. Do you or any member of your family own a calculator?

☐ Yes	☐ No
45 + 46 = 91	29 + 32 = 61

3. Do you want to have a calculator to yourself?

☐ Yes	☐ No
72 + 76 = 148	2 + 2 = 4

If yes, how much are you willing to pay for it?

☐ Under \$50	☐ \$50 – \$100	☐ Over \$100
10 + 11 = 21	48 + 59 = 107	12 + 8 = 20

Which of the following types will you buy, if cost is no problem ?

2 + 1 = 3	☐ Simple calculators (+ − × ÷ and perhaps one key for a constant)
32 + 52 = 84	☐ Scientific calculators (+ − × ÷ $\sqrt{x}$ lnx sin cos etc.)
37 + 25 = 62	☐ Programmable scientific calculators

4. Do you think schoolchildren should be allowed to use calculators in School?

☐ Yes	☐ No
61 + 68 = 129	11 + 10 = 21

If yes, when should they be allowed?

☐ F.1	☐ F.2	☐ F.3	☐ F.4 and above
5 + 2 = 7	0 + 1 = 1	15 + 25 = 40	54 + 50 = 104

5. Do You think children must first be thoroughly at home with the four operations of arithmetic before they can be allowed to use calculators?

☐ Yes ☐ No

72 + 75 = 147 2 + 3 = 5

6. In your opinion, working on complicated arithmetical operations

9 + 12 = 21 ☐ will strengthen your understanding of mathematics

65 + 66 = 131 ☐ tends to distract you from the essence of an argument

7. Do you think all schoolchildren should should learn to use calculators?

☐ Yes ☐ No

70 + 72 = 142 4 + 6 = 10

8. Should they all have a calculator that they can bring to school for daily use?

☐ Yes ☐ No

61 + 67 = 128 13 + 11 = 24

9. Should the use of calculators be allowed in the Certificate Examination?

☐ Yes ☐ No

59 + 69 = 128 15 + 9 = 24

10. If suitable restrictions are made on the types allowed, will it be fair to candidates to allow them to bring their own calculators to public exams. ?

☐ Yes ☐ No

56 + 60 = 116 18 + 18 = 36

11. Do you think calculators are going to replace mathematical tables?

☐ Yes ☐ No

58 + 61 = 119 16 + 17 = 33

12. Do you or your family own a slide rule?

☐ Yes ☐ No

7 + 6 = 13 67 + 72 = 139

13. Have you ever used a slide rule?

☐ Yes ☐ No

11 + 10 = 21 63 + 68= 131

14. If you are given a choice between a slide rule and an electronic calculator, which will you prefer to use?

☐ Slide rule ☐ Calculator

2 + 0 = 2 72 + 78 = 150

Short Notes on Two Experiments with Prime Numbers

L.L. Li

1. By way of introduction : a prime number is a natural number not divisible by any other natural number with the trivial exception of 1 and itself; 1 is not taken as a prime.
   Of classical interest in the Theory of Prime Numbers is the Goldbach Conjecture which claims that every even number is decomposable into the sum of two prime numbers. For example:
   1000 = 479+521, 1002 = 499+503,
   2000 = 769+1231 and 2002 = 971+1031

2. There are other decomposition "theorems", some proved while others are verified up to some large numbers. Proofs, in general, are extremely difficult.
   The author would like to add two more decomposition rules (believed to be original) just to further complicate the situation.

   Rule 1 Every prime ≥ 7 is decomposable into the sum of double a prime and another prime, i.e. prime = $2P_1 + P_2$

   Rule 2 Every prime ≥ 19 is decomposable into the sum of double a prime and triple another prime, i.e. prime = $2P_1 + 3P_2$

3. Rule 1 is numerically verified up to 937 and Rule 2 verified up to 307 by the author using a slow computer.
   The decompositions are not unique, e.g.,
   101 = 2x2+97 = 2x11+79 = 2x17+67
   = 2x29+43 = 2x41+19 = 2x47+7

   according to Rule 1, and, e.g.,
   101 = 2x7+3X29 = 2x31+3x13
   = 2x43+3x5

   according to Rule 2; the number of possible representations in both decompositions has the tendency to increase as the original prime gets larger.

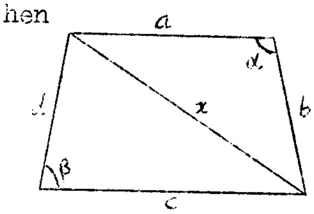
4. The interested reader may like to examine whether the above decomposition rules for prime numbers can also be applied to sufficiently large odd numbers.

Are Circumscribable Quadrilaterals Always Inscribable?

Joseph Shin,

A quadrilateral which can be circumscribed to a circle is said to be circumscribable. A quadrilateral which can be inscribed in a circle is said to be inscribable. How would you answer this question: Are circumscribable quadrilaterals always inscribable? You might find it interesting to pause in your reading at this point. Certainly squares are circumscribable as well as inscribable. Isosceles trapeziums also favour the affirmative. However, what happens if the figure is not so "regular"? After a moment's thought, you will find out that the answer is negative.

The problem then remains; what conditions must be placed additionally on the circumscribable quadrilateral in order that it is inscribable? One way of attacking a problem such as this is to use a method suggested by George Polya : take the problem as solved. Suppose the circumscribable quadrilateral as shown in the figure is also inscribable, then



$$a + c = b + d,$$

so that $(a - b)^2 = (c - d)^2,$

giving $a^2 + b^2 - 2ab = c^2 + d^2 - 2cd$ .... (1)

and $\alpha + \beta = 180^\circ$ .... (2)

Now $a^2 + b^2 - 2ab\cos\alpha = x^2 = c^2 + d^2 - 2cd\cos\beta$

so that $a^2 + b^2 - 2ab\cos\alpha - (a^2 + b^2 - 2ab) = c^2+d^2 - 2cd\cos\beta - (c^2+d^2-2cd)$

giving $2ab(1 - \cos\alpha) = 2cd(1 - \cos\beta)$

i.e. $ab(1 - \cos\alpha) = cd(1 + \cos\alpha).$

This result is not nice since it contains an angle α. We prefer a neat and compact expression. By considering the area of the quadrilateral, we have :

$$4A^2 = (2A)^2$$

$$= (ab \sin\alpha + cd \sin\beta)^2$$

$$= (ab + cd)^2 \sin^2\alpha \qquad \text{since } \sin\alpha = \sin\beta$$

$$= a^2b^2\sin^2\alpha + c^2d^2\sin^2\alpha + 2abcd \sin^2\alpha$$

$$= ab\ (1-\cos\alpha)\ ab\ (1+\cos\alpha) + cd\ (1+\cos\alpha)\ cd\ (1-\cos\alpha) + 2abcd \sin^2\alpha$$

$$= cd\ (1+\cos\alpha)\ ab\ (1+\cos\alpha) + ab\ (1-\cos\alpha)\ cd\ (1-\cos\alpha) + 2abcd \sin^2\alpha$$

$$= abcd \left[(1+\cos\alpha)^2 + (1-\cos\alpha)^2 + 2 \sin^2\alpha \right]$$

$$= 4abcd$$

$$\therefore A^2 = abcd.$$

This result is beautiful! Thus we may want to predict the following propositions :

Proposition 1 : A circumscribable quadrilateral with its area equals the square root of the product of its four sides is inscribable.

Proof : Since the quadrilateral is circumscribable, $a + c = b + d$, so that

$$a^2 + b^2 - 2ab = c^2 + d^2 - 2cd.$$

Now, $a^2 + b^2 - 2ab \cos\alpha = x^2 = c^2 + d^2 - 2cd \cos\beta$, so that

$$a^2 + b^2 - 2ab \cos\alpha - (a^2 + b^2 - 2ab) = c^2 + d^2 - 2cd \cos\beta - (c^2 + d^2 - 2cd)$$

$$\therefore ab\ (1 - \cos\alpha) = cd\ (1 - \cos\beta)$$

Since $A = \sqrt{abcd}$,

$$4\ abcd = (2A)^2$$

$$= (ab \sin\alpha + cd \sin\beta)^2$$

$$= a^2b^2 \sin^2\alpha + c^2d^2 \sin^2\beta + 2abcd \cdot \sin\alpha \sin\beta$$

$$= ab\ (1 + \cos\alpha)\ ab\ (1 - \cos\alpha) + cd\ (1 - \cos\beta)\ cd\ (1 + \cos\beta) + 2abcd \sin\alpha \sin\beta$$

$$= ab\ (1 + \cos\alpha)\ cd\ (1 - \cos\beta) + ab\ (1 - \cos\alpha)\ cd\ (1 + \cos\beta) + 2abcd \sin\alpha \sin\beta$$

$$= abcd\left[(1+\cos\alpha)(1-\cos\beta)+(1-\cos\alpha)\cdot(1+\cos\beta)+2\sin\alpha\sin\beta\right]$$

$$= abcd\left[2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta\right]$$

$$= abcd\left[2-2\cos(\alpha+\beta)\right]$$

This implies that $4 = 2 - 2\cos(\alpha+\beta)$

Hence, $\alpha + \beta = 180^{\circ}$

Proposition 2 : A circumscribable quadrilateral is inscribable if and only if its area equals the square root of the product of its four sides.

Proof : Combine the above discussion and proposition 1.

A lesson based on the material outlined above will be of value to students in several ways. Not only it gives a necessary and sufficient condition for circumscribable quadrilaterals to be inscribable, but also it demonstates that a proposition that is not true may sometimes be salvaged by imposing the correct additional conditions to make it true. Polya's method serves as an introduction to mathematical discovery. It is hoped that this note may suggest paths to discovery to be traveled by students.

How Many Primitive Pythagorean Triples Are Themselves in Arithmetic Progression

For positive integers a, b, c we shall call the ordered triple (a,b,c) a Pythagorean triple provided $a^2 + b^2 = c^2$. A Pythagorean triple (a,b,c) is called primitive if a and b are relatively prime. The purpose of this article is to prove

Theoram (3,4,5) is the only primitive Pythagorean triple which is itself in arithmetic progression.

Suppose (a,b,c) is a Pythagorean triple which is itself in aritmetic progression with common difference x, then

$$c - b = x, \quad c - a = 2x$$

Let $y = a - x$, then a,b,c may be represented by

$$a = x + y, \quad b = 2x + y, \quad c = 3x + y \qquad (G)$$

Since (a,b,c) is Pythagorean,

$$(x + y)^2 + (2x + y)^2 = (3x + y)^2$$

so that

$$y = 2x$$

Interested readers may verify that by choosing a positive integer x and letting $y = 2x$, (G) generates exactly once all Pythagorean triples which are themselves in arithmetic progression. It is interesting to note that a, b are relatively prime if and only if $x = 1$ (and hence $y = 2$). Now for $x = 1$, $y = 2$, $a = 3$, $b = 4$ and $c = 5$

Thus we come to the conclusion that (3,4,5) is the only primitive Pythagorean triple which is itself in arithmetic progression. All Pythagorean triples which are themselves in arithmetic progression are of the form (3k, 4k, 5k) where k may be any positive integer.

Comparison between C.D.C. and Amalgamated Syllabus

Algebra	C.D.C.	Amalgamated
Numbers : Conversion of denary numbers into binary numbers and vice-versa	V	
Positive and negative	V	V
Formulae : their manipulation and numerical applications	V	V
Percentage : converting fractions to percentage and vice-versa	V	V
percentage in everyday problems : interest rate, growth and depreciation, profit and loss, discount, etc.	V	V
Approximation and measurement	V	V
Significant figures	V	V
Polynomials in one variable not higher than the third degree : Simple operation with polynomials	V	V
Fractorization of polynomials, e.g. $px^2 + qx + r = (bx + k)(mx + n)$	V	V
Fractorization of $ac + bc$, $a^2 - b^2$, $a^2 \pm 2ab + b^2$	V	V
$a^3 \pm b^3$		V
Notation of function	V	V
Remainder Theorem	V	V
L.C.M., H.C.F. of polynomials	L.C.M. only	V
Algebraic expressions	V	V
Binomial expansions with integral indices	V	
Simple algebraic fractions	V	V
Ratio, Proportion	V	V
Variation		V
Solution of Linear equations in one unknown	V	V
simultaneous linear equations in two unknowns	V	V
quadratic equations in one unknown	real roots only	V
Relations between roots and coefficients of quadratic equations in one unknown	as exercises	V
Solution of simultaneous equations (one linear and one quadratic in two unknowns)	V	V
Simple problems leading to quadratic and simultaneous equations	V	V
Distinctions between equations and identities	V	V

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	C.D.C.	Amalgamated
Elementary measuration of and formulae for rectangle, triangle, parallelogram, trapezium polygon, circle, rectangular block, prism, cylinder, pyramid, right circular cone and sphere	V	V
Similar plane figures and solids	V	V
Relation of area and volume to their corresponding dimensions	implied	V
Linear inequalities in one or two variables, their graphical representation and application to simple practical problems such as Linear Programming	V	V
Quadratic inequalities in one variable	V	V
Laws of rational indices	V	V
Calculation using common logarithms.	V	V
Equations with unknown indices	as exercises	V
Quadratic surds, rationalisation, easy equations with the unknown under a radical sign	V	V
A.P. and G.P.	V	V
Insertion of arithmetic and geometric means	V	V
Sum of A.P. and G.P. to n terms	V	V
Summation of G.P. to infinity		V
Graphical representation of polynomials	V	
Location of roots by graphical methods	V	
Iterative methods	V	
Graphs of linear and quadratic functions, travel graphs	V	V
Geometry		
Angles at a point	V	V
Sum of angles of a triangle and of other convex polygons.	V	V
Parallel lines	V	V
Proportional division of transversals by parallel lines	V	V

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	C.D.C.	Amalgamated
Similar triangles	V	V
Congruent triangles	V	V
Isosceles triangles	implied	V
The distance of a point from a straight line	V	V
Parallelograms	implied	V
Mid-point and intercept Theorem		V
Pythagoras' Theorem and its converse	V	V
Perpendicular bisector and angle bisector	V	V
Circles, arcs, chords	V	V
Angles in a segment	V	V
Cyclic quadrilateral	V	V
Tangents	V	V
Angles in the alternate segment		

Trigonometry

Measure of angles in degrees and in radians	V	V
Length of arc and area of sector of a circle	V	V
The function of sine, cosine, tangent and their graphs in the interval 0 to 2π	V	V
The relation $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\sin^2 \theta + \cos^2 \theta = 1$	V	V
Solution of right-angled triangles, with simple applications	V	V
Easy problems in three dimensions soluble by analysis into right-angled triangles	V	V
Sine and Cosine formulae and their applications	V	V
Area of a triangle as $\frac{1}{2}$ ab Sin C.	V	V
Easy trigonometric equations (solutions in the interval 0 to 2π).		V

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	C.D.C.	Amalgamated
True bearings.	V	
Co-ordinate Geometry		
Rectangular co-ordinates in 2-dimensional space	V	V
Ordered pairs.	V	V
Distance between two points	V	V
Points dividing line segments in a given ratio	implied	V
Equations of a straight line	V	V
Slope (gradient) of a straight line	V	V
Perpendicularity	V	V
Intersection	implied	V
Equations of circles; co-ordinates of centre; length of radius	V	V
Intersection of straight lines and circles	as exercises	V
Equation of parabola $y^2 = 4ax$	V	
Statistics		
Collection and organisation of numerical data, and their graphical representation by bar charts	V	V
Pie charts, histogram	V	V
Frequency polygons and curves	polygons only	V
Cumulative polygons and curves	polygons only	V
Calculation of the mean	V	V
Determination of the median	V	V

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	C.D.C.	Amalgamated
Calculation and use of standard deviation as a measure of dispersion	V	V
Simple ideas of probability with application of the addition law and multiplication	V	V
law to easy problems	V	V
3 - σ test	V	?
Recreational Mathematics	V	

1. Pretty series

a) $\sum_{i=1}^{\infty} (3i+1)\, x^{i} = \frac{1}{2}$, when $x = 2/27$
(D. F. Ferguson, Mathematical Gazette)

b) $\sum_{i=1}^{\infty} \binom{4i}{2i} x^{2i} = 13/7$, when $x = 6/25$

(D. G. Tahta, Mathematical Gazette)

2. Schur's inequality

a) If $\mu \geq 0$ and x, y, z are all positive, then
$x^{\mu}(x - y)(x - z) + y^{\mu}(y - z)(y - x) + z^{\mu}(z - x)(z - y) \geq 0$
(G. N. Watson, Mathematical Gazette)

b) Let $f(t)$ be a positive function of t, monotone or convex, in some interval and let x, y, z belong to this interval. Then, unless $x = y = z$,
$f(x)(x - y)(x - z) + f(y)(y - z)(y - x) + f(z)(z - x)(z - y) > 0$
(E. M. Wright, Mathematical Gazette)

3. A test for divisibility by 19 Cross off the last digit and add to the number remaining twice the digit crossed off. If the result is divisible by 19, so is the original number.
(J. Kashangaki, Mathematical Gazette)
e.g. To test whether 1032099 is divisible by 19, we proceed as follows :

$$
\begin{aligned}
1032099 &\longrightarrow 103209 + 2\times 9 = 103227 \\
&\longrightarrow 10322 + 2\times 7 = 10336 \\
&\longrightarrow 1033 + 2\times 6 = 1045 \\
&\longrightarrow 104 + 2\times 5 = 114 \\
&\longrightarrow 11 + 2\times 4 = 19
\end{aligned}
$$

This shows that 1032099 is divisible by 19.

4. $a_n \longrightarrow 0$ is not sufficient to ensure the convergence of $\sum a_n$

Let $a_n = \mathrm{Log}\left(1 + \frac{1}{n}\right)$, then $a_n \longrightarrow 0$

However,
$$
\begin{aligned}
\sum_{1}^{N} a_n &= \sum_{1}^{N} \log\left(1 + \frac{1}{n}\right) \\
&= \sum_{1}^{N} \log\left(\frac{n+1}{n}\right) \\
&= \sum_{1}^{N} (\log(n+1) - \log n) \\
&= \log(N+1) \\
&\longrightarrow \infty.
\end{aligned}
$$

(P. H. Cody, Mathematical Gazette)

5. <u>Quadratic polynomials and prime numbers</u>

$x^2 + x + 41$, prime for $-40 \leq x \leq 39$

$x^2 - 79x + 1601$, prime for $0 \leq x \leq 79$

$x^2 - x + 41$, prime for $-39 \leq x \leq 40$

$x^2 + 79x + 1601$, prime for $-79 \leq x \leq 0$

$x \; - 2999x + 2248541$, prime for $1460 \leq x \leq 1539$

6. <u>Trigonometrical factors and identities</u>

(i) $1 + \sin 2x = (\cos x + \sin x)^2$

(ii) $1 + \sin 3x = (1 - \sin x)(1 + 2\sin x)^2$

(iii) $1 - \sin 3x = (1 + \sin x)(1 - 2\sin x)^2$

(iv) $1 + \cos 3x = (1 + \cos x)(1 - 2\cos x)^2$

(v) $1 - \cos 3x = (1 - \cos x)(1 + 2\cos x)^2$

(vi) $\sin 3x + \cos x = (\sin x + \cos x)(\sin 2x + \cos 2x)$

(vii) $\sin 3x - \cos x = (\cos x - \sin x)(\sin 2x - \cos 2x)$

(viii) $\cos 3x + \sin x = (\cos x - \sin x)(\cos 2x + \sin 2x)$

(ix) $\cos 3x - \sin x = (\cos x + \sin x)(\cos 2x - \sin 2x)$

(x) $\sin 3x + \cos 3x = (\cos x - \sin x)(1 + 2\sin 2x)$

(xi) $\cos 3x - \sin 3x = (\cos x + \sin x)(1 - 2\sin 2x)$

(xii) For positive integers n and p,

$$\frac{\sin (n+p)x + \cos nx}{\cos (n+p)x + \sin nx} = \tan px + \sec px.$$

(Walter F. Grieve, Mathematical Gazette)

7. <u>$\log(-1) = 0$?</u>

$\log(-1) = \log\left(\frac{1}{-1}\right) = \log 1 - \log(-1) = 0 - \log(-1) = -\log(-1)$

$\therefore 2\log(-1) = 0$,

i.e. $\log(-1) = 0$. (A. J. Howie, Mathematical Gazette)

8. <u>Power - sums of a sequence</u>

a) If $C \sum_{i=1}^{n} i^{p} = \left(\sum_{i=1}^{n} i^{q} \right)^2$

where p, q are positive integers and C is a constant,
then $p=3$, $q=1$ and $C=1$. (S. M. Edmonds, Mathematical Gazette)

b) If for all integers $n \geq 1$ we have $x_n > 0$ and

$$\sum_{i=1}^{n} x_i^3 = \left(\sum_{i=1}^{n} x_i \right)^2$$

then $x_n \equiv n$. (F. Gerrish, Mathematical Gazette)

9. Magic squares All magic squares of the third order possess the property that the sums of the squares of the numbers in the "outside" columns or rows are equal. (D. B. Eperson, Mathematical Gazette)

10. Sums of two squares An integer is the sum of two squares if and only if it may be expressed in the form $2^n (4N+1)$, where n is any integer and N is the sum of two triangular numbers.
(A. Sutcliffe, Mathematical Gazette)

11. Powers of $\sum_{i=1}^{n} i$

Denote $\sum_{i=1}^{n} i^k$ by S_k. Then

a) $4 S_1^3 = 3 S_5 + S_3$

b) $2 S_1^4 = S_7 + S_5$ (Edmonds, Mathematical Gazette)

12. $\pi = 0$?

$$\frac{\pi}{2} = \int_{-1}^{1} \frac{dx}{1+x^2}$$

$$= -\int_{-1}^{1} \frac{dy}{1+y^2} \qquad \left(\text{put } x = \frac{1}{y} \right)$$

$$= -\frac{\pi}{2}$$

$\therefore \pi = 0.$ (Richard Beetham, Mathematical Gazette)

13. $$2\int \sec^3 x\,dx = \int \left[\sec x \,(1 + \tan^2 x) + \sec^3 x \right] dx$$

$$= \int \sec x\,dx + \int (\sec x \tan^2 x + \sec^3 x)\,dx$$

$$= \log (\sec x + \tan x) + \int d(\sec x \tan x)$$

$$= \log (\sec x + \tan x) + \sec x \tan x + C$$

$$\therefore \int \sec^3 x\,dx = \tfrac{1}{2}\left[\log (\sec x + \tan x) + \sec x \tan x + C \right]$$

(M. A. Jerome, Mathematical Gazette)

14. Number curiosity

$1^5 + 9^5 + 4^5 + 9^5 + 7^5 + 9^5 = 194979$

$2^8 + 4^8 + 6^8 + 7^8 + 8^8 + 0^8 + 5^8 + 1^8 = 24678051$

15. <u>Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ...</u>

Observe that $2 \times 3 = 1 + 1^2 + 2^2$,

$3 \times 5 = 1 + 1^2 + 2^2 + 3^2$,

$5 \times 8 = 1 + 1^2 + 2^2 + 3^2 + 5^2$,

Can this pattern be generalized ?

16. <u>Circle passing through three points</u>

Find the equation of the circle which passes through P (3,3), Q (6,4) and R (7,1)

[Solution] The circle on QR as diameter has the equation

$$(x - 6)(x - 7) + (y - 4)(y - 1) = 0,$$

i.e. $$x^2 + y^2 - 13x - 5y + 46 = 0.$$

The straight line QR has the equation

$$y - 4 = \frac{4 - 1}{6 - 7}(x - 6)$$

i.e. $3x + y - 22 = 0$

Now $x^2 + y^2 - 13x - 5y + 46 + k(3x + y - 22) = 0$.....(*) represents circles through Q and R for values of the parameter k. Choose k to make (*) pass through P; that is

$$3^2 + 3^2 - 13(3) - 5(3) + 46 + k(3(3) + 3 - 22) = 0$$

$$\therefore k = 1.$$

Thus the required equation is

$$x^2 + y^2 - 13x - 5y + 46 + (3x + y - 22) = 0$$

i.e. $$x^2 + y^2 - 10x - 4y + 24 = 0$$

(Clifford Bond, Mathematical Gazette)

PROBLEM CORNER

1. If a number N, having nr digits, is divisible by P, where P is any factor of $(10^r - 1)$, then any number with the same digits cyclically permuted will also be divisible by P.
(S. Parameswaran, Mathematical Gazette)

2. If $a + b + c = 0$ and $x + y + z = 0$, then
$$4(ax + by + cz)^3 - 3(ax + by + cz)(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - 2(b - c)(c - a)(a - b)(y - z)(z - x)(x - y) = 54\, abcxyz$$

3. Let a_n denote the n-th Fibonacci number, so that
$$a_1 = 1,$$
$$a_2 = 1,$$
$$\vdots$$
$$a_{n+1} = a_n + a_{n-1}$$
Then (i) $a_{2n} = a_{n+1}^2 - a_{n-1}^2$
(ii) $a_{2n+1} = a_{n+1}^2 + a_n^2$ (W. A. Capstick, Mathematical Gazette)

4. In a Pythagorean triangle, the length of one of its sides is divisible by 5. (Y. U. Bashir, Mathematical Gazette)

5. I am at a corner where I have the choice of two independent bus systems, buses on each running at ten-minute intervals. How long may I expect to wait for a bus ?
(ANS : 3 1/3 minutes)

6. If f(x) is differentiable and f'(x) is continuous in $a \leq x \leq b$, and $f'(a) = f'(b) = 0$, then there is at least one θ in $a < \theta < b$ such that
$$\frac{f(\theta) - f(a)}{\theta - a} = f'(\theta)$$
(T. M. Flett, Mathematical Gazette)

7. If a triangle be similar to the triangle formed by its medians, then the sum of the squares on two sides of the triangle is equal to twice the square on the third side.
(A. A. K. Ayyangar, Mathematical Gazette)

8. To solve $x^2 - px + q = 0$ graphically, plot the points (0,1) and (p,q) on squared paper. Draw the circle with these two points as diameter. Then this circle meets the x-axis in the points whose x-coordinates are the roots of the equation.
(J. W. Hesselgreaves, Mathematical Gazette)

9. Find a quadratic equation $x^2 + ax + b = 0$ whose two roots are the coefficients a, b. (Teiji Nakazawa, Mathematical Gazette)
(ANS : $x^2 + x - 2 = 0$)

10. Given an equi-arm balance and weights of 1, 3, 9 and 27 units, how to weigh 38 units if you are allowed to put the weights in either scale-pan.

11. Suppose that you have three boxes, one containing two black marbles, one containing two white marbles, and the third, one black marble and one white marble. The boxes are labelled according to their contents —— BB, WW, and BW. Someone has switched the labels so that every box is now incorrectly labelled. By drawing a marble from an appropriate box, you can determine the contents of all three boxes. Which box must be chosen ? Why ?

12. Find out the unknowns :

```
    L A B E L
        A L L
+     S E A L
-------------
    B A S E S
```

(ANS :

```
   37413
     733
+   9173   )
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   47319
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