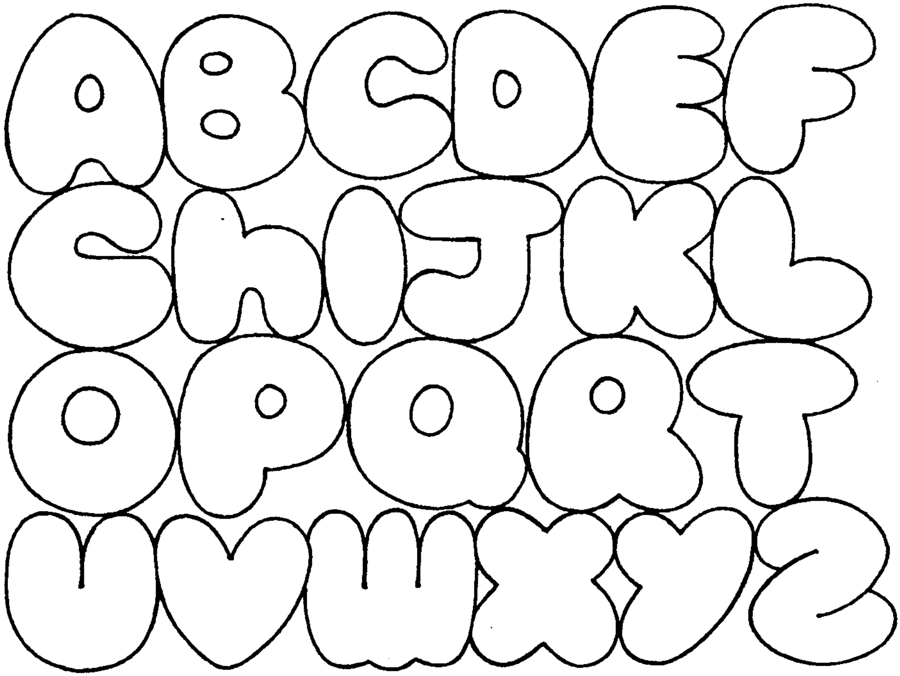

SCHOOL MATHEMATICS NEWSLETTER

February 1987

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Mathematics Section, Advisory Inspectorate
Education Department, Hong Kong

Please ensure that every member of your mathematics staff has an opportunity to read this Newsletter.

The views expressed in the articles in this Newsletter are not necessarily those of the Education Department, Hong Kong.

Theme of cover design : Missing SMN ? It has been two years since the issue of the last SMN. If you want SMN published more frequently, send in contributions.

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FOREWORD

In this issue of the School Mathematics Newsletter (SMN), we have brought to you articles on various ideas & experiences on mathematics. We have also included an introduction on the International Mathematical Olympiad and a report on a survey on basic numeracy, together with some mathematical puzzles. We hope you will find the contents of this issue both informative and useful.

Some of the articles in this issue are contributions from classroom teachers of primary and secondary schools, who would like to share their experience in mathematics teaching with readers of the SMN. We do hope that teachers will continue to support the SMN by sending in more contributions for inclusion in the future issues of the Newsletter, thus helping to make the SMN an even more useful venue for the exchange of ideas & experiences.

We are thankful to all those who have contributed to this issue of the SMN. I would also like to thank my colleagues who have helped to produce this Newsletter. We need your support to make possible the future issues of the SMN. Please send your contributions directly to the Mathematics Section of the Advisory Inspectorate. Suggestions and comments on both the layout and contents of the Newsletter are also welcome. The success of the SMN depends on your valuable support.

C. P. Poon

Principal Inspector (Mathematics)

國際奧林匹克數學競賽簡介

香港數學學會

一、前　　言

數學是學習科學知識的重要工具，也是人類心智與創造力的表現。在今天的社會中，數學的方法和術語，已滲透到日常生活的各方面。數學除了可以幫助解決物理、電子、工程及太空科學的問題之外，在工商管理、經濟學、生化學、心理學、社會科學甚至醫學上也有極廣泛的應用。隨着電腦科學的發展，數學的應用在近年來越來越普遍，所有基本科學，都已利用數學的術語和規律闡釋。可以說，不學習好數學，根本無法掌握先進的科技知識。在這現代化的社會中，數學已變成一種國際通用語言。一切科技訊息，都必須通過數學語言來傳遞。數學上的成就，是將人類社會推向現代化的原動力。因此世界上許多國家，除了大力推行數學的普及教育之外，更不遺餘力的去發掘和培養數學人才。

二、數學競賽起源

發掘數學人才的方法，最直接有效的方法之一就是激勵青年學生對數學的興趣，在競賽中選拔人才。最早期的數學競賽，溯於一八九四年。在匈牙利物理數學學會主辦。它是爲祝賀著名物理學家依埃特沃司（ Baron Lorand Evatös ）榮任該會會長及匈牙利教育部長而設的，競賽者都是投考大學一年級的考生。這個競賽舉辦得非常成功，歷年以來，競賽的優勝者，

先後在數學上大放光芒，例如菲查（Fejér）、馮卡文（Von Kármán）、哈爾（Haar）、尼茲（Riesz）、雷鐸（Radó）、斯茲歌（G. Szegö）、泰拉（Teller）、厄杜斯（Paul Erdös）等及其他人仕，均成爲國際上負有盛名的數學家，尤其是厄杜斯，更被喻爲當今首屈一指的數學大師。由於他們的成就，令匈牙利這個國家，在近代數學發展和應用方面，擔任一個重要角式。

三 國際奥林匹克競賽成立經過

在匈牙利的影響下，東歐國家亦紛紛舉辦數學競賽。一九五九年，羅馬尼亞正式邀請了匈牙利、蘇聯、東德、保加利亞、波蘭及捷克等舉行七國數學競賽。這次比賽，羅馬尼亞勇挫匈牙利，獲得團體冠軍。因此在一九六〇年羅馬尼亞再度舉辦此項競賽。這次比賽被形容爲一次富有刺激性的比賽，由捷克險勝，匈牙利與羅馬尼亞則並列亞軍。比賽之後，東歐國家則議決每年輪流舉辦一次競賽，藉以提高及刺激中學生對數學的興趣。自此以後，蘇聯在一九六三年至一九六六年，連續奪得冠軍，此項成就，引起世界各國的注視。南斯拉夫於一九六七年主辦此項競賽時，邀請了西歐英國、法國、意大利及瑞典四國參加，結果仍由東歐國家囊括三甲，蘇聯奪得冠軍，東德、匈牙利分佔二、三位。之後數年，蒙古、比利時、荷蘭、奥大利、古巴及芬蘭先後加入角逐。冠軍則由匈牙利與蘇聯包辦。

四 美國的參加及影響

美國於一九七四年應邀派隊前往東德接受挑戰，仍受挫於蘇聯。但美國隊經過組織與嚴格訓練之後，在一九七七年，終於在東歐的對手中脫穎而出，擊敗蘇聯，取得冠軍，而英國亦擊敗匈牙利取得季軍。接着，美國取得一九八一年的主辦權，有

更多國家參加競賽。在競賽的同時，並召開數學教育研討會，討論數學普及教育及培養數學人才的方法。到一九八五年，參加國際奧林匹克數學競賽的國家已達三十八個，成爲國際文化交流的一項重要事項。目前強隊是蘇聯、匈牙利、美國、羅馬尼亞及西德等國。但個別優秀人才、在其他國家的選手中不斷湧現。

五 中國的參加及越南的表現

一九八五年的競賽，於七月二十一、二十二日在芬蘭舉行，中國首次派隊列席，取得一項個人二等獎，一項個人三等獎。本屆前十名的名次分別是羅馬尼亞、美國、匈牙利、保加利亞、越南、蘇聯、西德、東德及英國。其中匈牙利的Geza Kos 及羅馬尼亞的Daniel Tatura 二人表現優異，同獲滿分。而美國年僅十六歲的學生 Jeremy Kahn ，則第三次代表美國出賽，他前兩屆獲個人二等獎，本屆則獲一等獎，名列第十四名。值得注意的是，過去五年來，越南隊均有非常強勁的表現，亦曾獲得個人滿分獎。

六 一九八八年澳洲的競賽

一九八六年及一九八七年的競賽，已分別定於在波蘭及古巴舉行。澳洲則申請爲一九八八年的主辦國。由於一九八八年是澳洲建國二百週年紀念，澳洲政府對此項比賽非常重視，列爲重要慶典活動之一。相信屆時將有更多國家參加此項富有意義的國際競賽。

七 參賽方法

參加比賽的國家，將被邀請選擬三至五題考題，送交主辦

國的籌備委員會（主辦國無權出題）。收集試題後，委員會將從中挑選約二十題考題，並加以修訂；然後再由各國領隊組成的評議會閉門商討及修改，最後訂出六題考題，分爲兩卷，每卷考試時間爲 4—4½ 小時，分兩天早上舉行。每題滿分是七分。。每隊將派六人出賽；試題的範圍環繞中學數學主要內容，包括數論、代數、幾何及組合數學。此項競賽，要求競賽選手對數學方面有卓越才能和廣博知識，因此考題均比較艱深，着重考驗選手的抽象思維與演繹方法。歷屆試題的答案，美國數學會、蘇聯及匈牙利等國均有彙編出版。目前加拿大數學會出版的 Crux Mathematicrum 數學期刊，提供了不少世界各國的數學競賽資料，它是國際奧林匹克數學競賽的重要刊物。而美國的陸軍研究中心、海軍研究中心、國家科學基金及許多商業機構，則紛紛撥款資助美國隊的培訓計劃。加拿大、澳洲、法國、西德、東歐國家與中國等，對有天才的年青數學家作出多方面的鼓勵並撥出款項將數學教育推廣到各個層面。適合中學生與大學生閱讀的數學雜誌與刊物，近年來亦出版了不少。

八 競賽評閱標準

奧林匹克數學競賽大會的官方語言現在是德、法、英、俄四國文字，但各國選手可用自己本國語言作答，如有需要，該國領隊必需翻譯試題爲該國語言，並呈交大會存案及審核，試卷的評審，先由該國之領隊與副領隊先行批閱其本國選手之試卷，然後由主辦國之代表團逐題覆核及聆聽。審核團分爲六組，每組由數名數學教授組成，專職負責一題題目。評分之標準與名次，則由各國領隊組成之大仲裁會裁定。

九 領隊與副領隊職責

另外，大會規定，參加國家，最少委派出領隊與副領隊各一人。領隊應於賽前數天到達主辦國某處秘密地點，閉門參加

評審會議並作試題練習。因此領隊多爲大學數學教師，副領隊多由教育部人員或中學數學教師擔任；副領隊的職責主要是帶領選手，照顧飲食起居及提供訓練，心理輔導等。一般來說，副領隊與選手亦於賽前約三天到達，居住於酒店，以適應時差、飲食及環境。比賽開始後，則由主辦國招待食宿（約四至五天）。至於來回機票，則由參賽國家自行負責，參賽國亦可多派一、二名觀察員隨行。一切規則，與奧林匹克運動會大同小異，惟領隊則需於競賽開始後方可與其本身隊伍會合，以防洩漏試題內容。此外，主辦國一般尙舉辦數學研討會及展覽會，討論提高數學教育的方法及培養數學天才的指導。由於每年在不同的國家競賽，大會提供了年青數學家建立國際友誼與溝通的機會。因此，國際奧林匹克數學競賽，實在是一項具有深遠意義的文化活動。

十、各國數學競賽活動及刋物

歷年來，英、美、法、蘇、德等國，均有舉辦數學競賽；中國近年來，亦在各省市積極推行此種競賽。發掘出的優勝者，加以培訓後多成爲傑出的科學家、數學家及主管人才。美國最近出版了" Who's who of U.S.A. Mathematical Olympiad Participants "一書詳列了這些選手的出路，說明了這批選手，都是當今美國科學界及工程界的精英。過去二十年來，在國際奧林匹克數學競賽中有優異表現的選手，許多都成爲國際有名的數學家。

十一、香港籌備參與經過

香港現在已成爲一個中西文化匯點的國際大都會，在科技日新月異的今天，沒有足夠的數學人材，就談不上甚麼管理現代化，我們只能盲目地接受別人的經驗與知識而不能創新與發揮。因此發掘數學人才，提高本港市民對數學的認識與興趣，實在非常重要。香港數學學會於一九八五年，得到中學數學老師

熱烈支持與響應，和香港教育界合作，正式成立了香港國際奧林匹克數學競賽委員會，準備在一九八七年在本港舉行數學競賽。如果舉辦成功，希一九八八年派隊參加在澳洲舉行的國際奧林匹克數學競賽。美國國家隊的總教練格林聖教授（M.S. Klamkin）及他的助手，已很熱心地答應協助香港擬定首次試題及提供訓練經驗，希望我們有一個良好的開始。

爲着順利執行籌備工作，在一九八五年七月八日，由香港國際奧林匹克數學競賽委員會的成員，另組成一執行委員會。其中包括有香港中文大學、香港大學、香港理工學院、羅富國教育學院等講師、教育署輔導視學處數學組督學、香港考試局及中學數學主任、香港數理教育學會成員等共十六人，專責統籌一切有關事宜，並議定工作進度如下：

1985－86年　　籌備工作

1987年

87年 1月	邀請各中學提名學生參加香港選拔賽
87年 7月	舉行香港選拔賽並選出約三十名候選隊員參加集訓
87年 8月	第一次集訓
87年 12月	第二次集訓

1988年

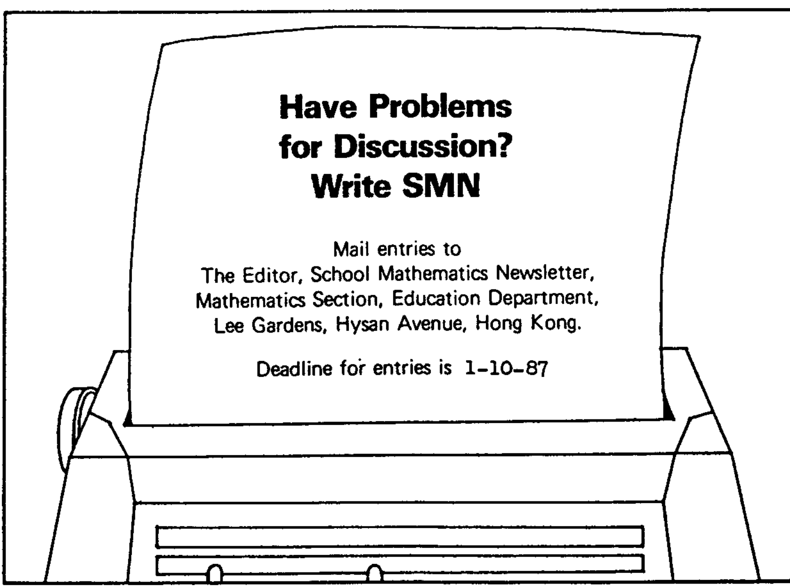
88年 7月	由候選隊員中選出約6名隊員參加在澳洲舉行之第廿九屆國際奧林匹克數學競賽

三　希望各界人仕資助

我們希望熱心人仕及社會賢達，慷慨解囊，集腋成裘，全力資助此項有益有意義的長遠文化活動，激發香港學生對數學的興趣，建立數學風氣，促進數學研究。事實上，發掘與培養人才，是香港人對下一代的責任與工作，也是促進香港穩定與繁榮的基礎。希望大家有錢出錢，有力出力，將此項與培養香港

人才有關的活動成功地辦好。讓本港青年，與外國青年有同等的機會，享有一個富有挑戰與創造性的豐盛人生，將來爲香港貢獻所長。

一切建議及通信，請寄香港大學數學系轉交香港國際奧林匹克數學競賽執行委員會主席收。



數學—科學的語言

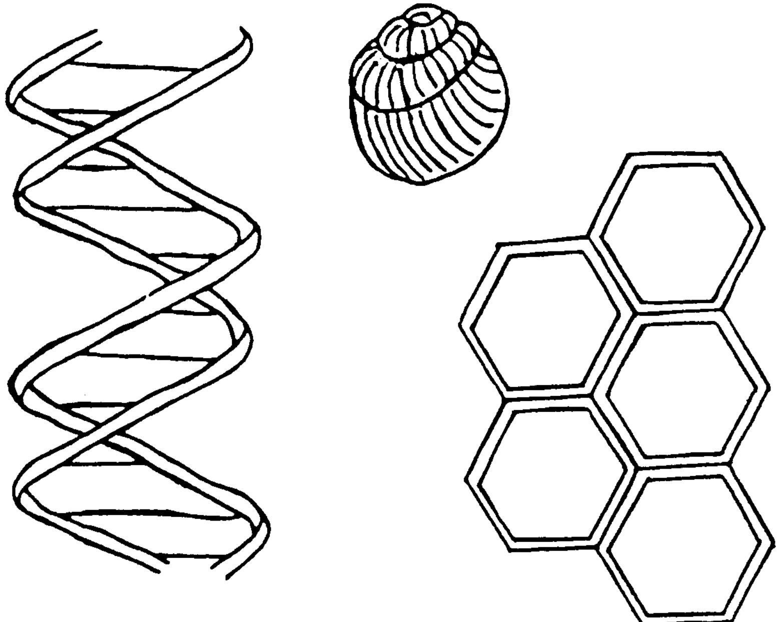
蕭文强博士

兩年前我在理學會主辦的「理學節」中給了一個講座，題目是「數學一科學的語言」。如今適值理學會編輯邀我給年刊撰稿，我便把當時的講稿拿來整理一番，希望仍能引起大家的興趣。這篇文章提的疑問比答案還多，可以說只是個人在思想學習的摸索歷程上的札記而已。

※※※※※※※※※※

你以為數學弄什麼的？「不要騷擾我，數學與我何干？」「聽說數學教人怎樣逢賭必贏，你不如教教我。」「懂得數學便懂得解答很多絞腦汁的遊戲。」這類答覆是出於對數學的誤解。「在日常生活中常常用數學，例如買菜用多少錢？每月的收入減去支出還剩下多少？」「科技發展倚靠數學，科學研究使用數學，即使企業管理也借助數學。」「數學就是解方程、計算積分、作圓作三角……。」這類答覆依然是片面而已。其實數學有它更廣闊的天地，根本上你是生活在數學世界之中。不信嗎？你看：太陽、月亮、地球基本上都是個球體；氣球、肥皂泡、露珠也是個球體；化學晶體的結構是有幾何對稱的多面體；鸚鵡螺的剖面呈現一條對數螺綫；海星的形狀是個五角星；牽牛花纏繞籬笆生長是沿着一條螺旋綫；螺絲帽是個正六角形；雷達天綫是個拋物面；鐵閘由很多菱形組成；橋樑的支架是個三角形；很多裝飾圖案基本上是互相扣合的正三角形、正四方形和正六角形；光綫反射時入射角等於反射角；行星繞日運行的軌道是個橢圓；支配遺傳的 DNA 分子結構是條雙螺旋綫；蜂巢由很多正六角形組成，而且每個蜂房的底部是由三個全等的菱形組成，每個菱形的角都是 109°28′ 和 70°32′。這樣看來，數學不單是工具，它簡直就是大自然的語言！當你意識到這一點的時候，你也許會產生以下的疑問：為什麼這個世界充

滿數學？爲什麼數學對於解釋這個世界的事物是這樣有效？數學是根本存在於這個世界之中，抑或只是人類自由思維的產物？若是前者，如何解釋近代數學的衆多人爲的公理化系統？若是後者，以我們在宇宙中只不過有如滄海一粟之渺小，又怎能以憑空想像出來的東西描述實際的世界？



數學的世界

這些疑問，一直困擾着無數的哲學家、科學家和數學家。愛因斯坦（EINSTEIN）曾經說過：「宇宙間使人最不理解的事情就是它是可以理解的。」在一個題爲「幾何與經驗」的講座上，他提出上面的問題，而他的答覆是一句著名（但不易明白）的話：「就數學定理之涉及實在來說，它們並非是可靠和必然的；就其可靠性和必然性來說，它們並不涉及實在。」另一位物理學家維格納（WIGNER）也曾給了一個講座，題目是「數學在自然科學上令人難以理解的有效性」，數學家電腦學家漢明（HAMMING）也曾給了一個題目很相似的講座，叫做「數學的令人難以理解的有效性」，他們都針對上面的問題作討論。我的學識、才能、見地、修養顯然遠不及以上幾位著名的學者，我對上面的問題也沒有什麼創見，在這裡我只打算就着一些科學史上的例子跟

大家談談這些問題。限於學識背景，我提的例子都是數學或物理學方面的，但那當然不等於說數學只對物理學有用，也不等於說數學只對自然科學有用。

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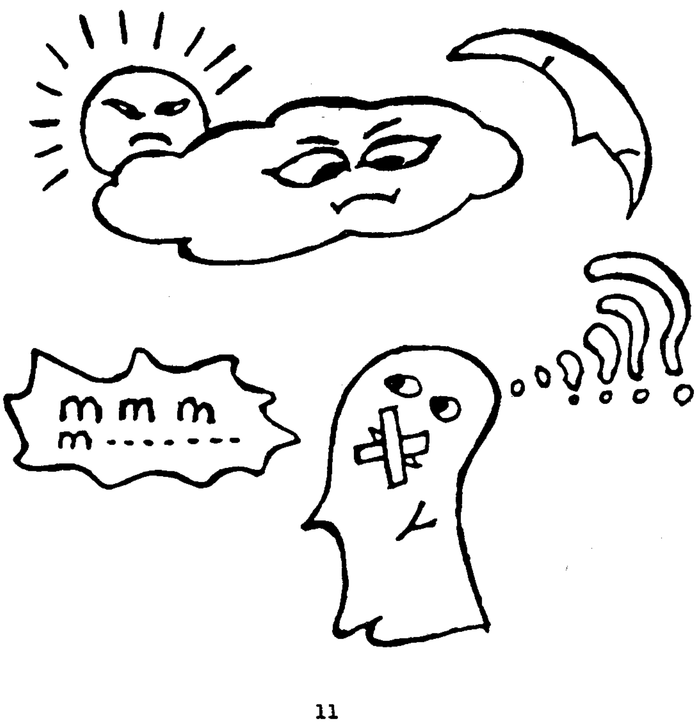
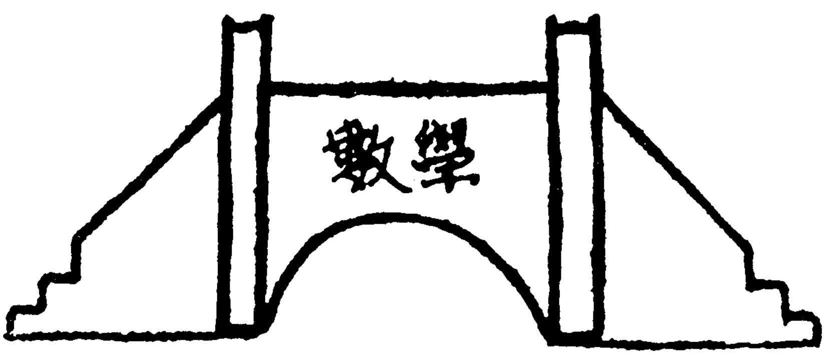
上古時代的人對大自然各種現象既驚訝也敬畏，認爲那是天神的旨意，所謂「天尊不可問」。然而人類畢竟是有思維能力的動物，漸漸天也可問了。公元前四世紀我國戰國時代的偉大詩人屈原便寫了一首『天問』，開首便說：「曰：遂古之初，誰傳道之？上下未形，何由考之？冥昭瞢闇，誰能極之？馮翼惟像，何以識之？」跟着他提出了 172 個問題，都是關於自然現象、社會現象、古代歷史、神話傳說，大膽地懷疑傳統的觀念。很可惜，這種熱切求眞知的慾望在古代中國可沒有引起廣泛的影響。雖然古代中國在科學和技術上作出了不少貢獻。但要追溯現代科學思想的精神源頭，我們需要把故事背景移到西方的希臘。



在公元前六世紀，有些希臘哲學家首先提出以理性認識世界的主張。例如泰勒斯（THALES）認爲世上萬物是由物質構成，而水乃萬物本原。畢達哥拉斯（PYTHAGORAS）在意大利南部開壇講學，吸引了一批信徒，尋且成爲一幫又是學術性、又是政治性、又是宗教性的神秘組織，被後世稱爲畢氏學派。他們認爲萬物本原不是物質而是抽象的數（指正整數），瞭解數即能瞭解天下萬物。他們正確地看到很多事物都具有數的關係，可惜却把數片面地誇大了，提出「數即一切」的信念。於是一方面他們有不少荒誕的說法，例如1代表理性、2代表女、3代表男、4代表公正、5（＝2＋3）代表婚姻等等。但另一方面他們也做了很多重要的數學工作，並且試圖用數學來描述自然現象。譬如他們發現如果撥動一根弦綫，它發出的音調隨弦綫的長度變更，又如果撥動幾根長度互相成整數比率的弦綫，它們會發出和諧悅耳的音調。（這個弦綫振動問題，過了二千多年後再度成爲熱門話題，吸引了不少十八和十九世紀著名數學家的注意。到了那個時候，因爲數學有了更深刻的發展，人們才對這回事明白得更加透徹。）



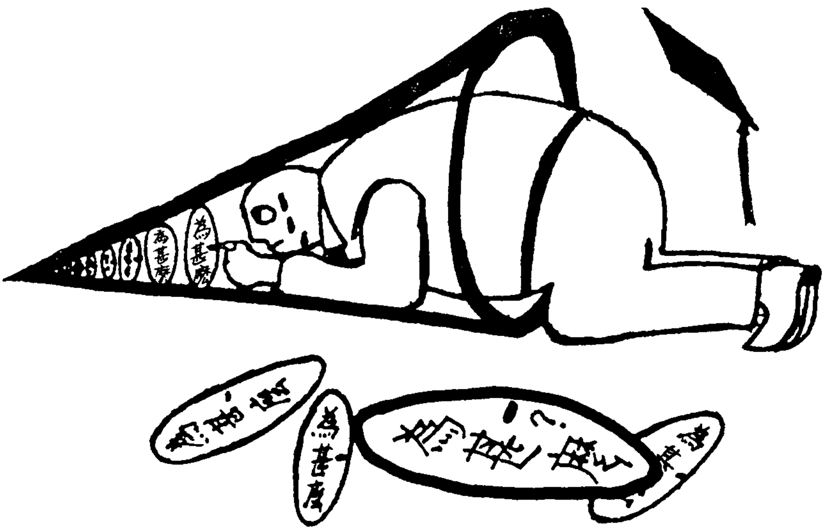
到了公元前四世紀，希臘哲學家柏拉圖（PLATO）更進一步認爲大自然就是數學。他認爲客觀世界是變化無常而不實在的，只有理念世界才是永恒不變而實在的。眞理是人對理念的認識，這種認識不能倚靠知覺，只能倚靠思維，而數學就是橋樑。柏拉圖有句名言：「神永遠按幾何（指數學）規律辦事。」意思就是說只有靠數學才能認識理念世界。他的學生亞里士多德（ARISTOTLE）後來另立門戶，提出不同的見解。他認爲客觀世界由物質構成，我們必須憑知覺去認識客觀

世界，數學可以幫忙我們瞭解客觀世界，但我們不能藉它去尋求最終的原因。

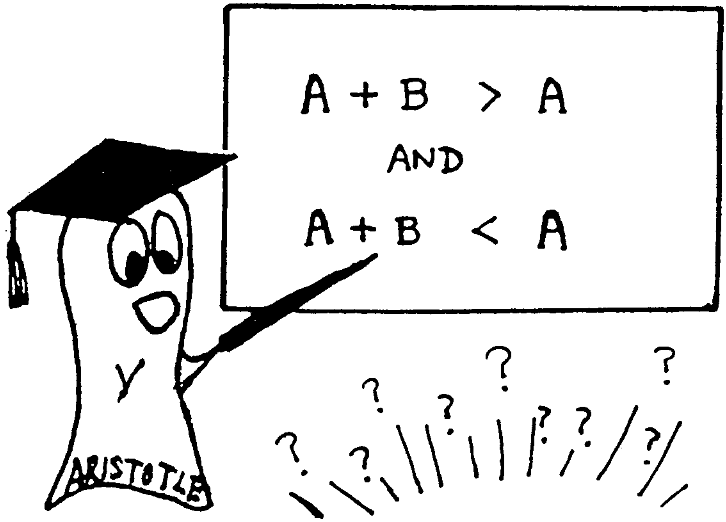
雖然亞里士多德的想法是比較接近現代科學思想精神，但現代科學思想的眞正開始，還得再等二千多年。在十七世紀初意大利科學家伽里略 (GALILEO) 提出他的主張，認爲必須從大自然錯綜複雜的現象中抓緊兩個基本概念，就是物質和運動；科學研究必須按照數學研究的方法，從一些基本原理出發，用推理演繹尋求新的結果。當然，如果單單如此，他的主張便和亞里士多德的主張沒有分別了。伽里略超越前人之處，在於他尋求基本原理的辦法。亞里士多德脫離不了柏拉圖的影響，認爲眞理獨立於客觀世界而存在，所以尋求眞理不靠知覺只靠思維。例如他認爲有兩種運動，就是自然運動和人爲運動。任何運動都受到動力和阻力的作用，沒有動力物體便不動，沒有阻力物體瞬刻便走完路程。

在自然運動中，動力來自物體的重量，阻力來自它通過的介質，因此他得出結論：物體自由下落時，重的比輕的下落快些。伽里略却認爲基本原理只能由觀察客觀世界而得，即是從實驗而來，而且從基本原理推論得來的結果，也只能從實驗中去驗證它是否正確。同時，因爲伽里略看重數學的力量，他提出一個跟前人極端不同的見解。這個見解可以說是現代科學最重大的發現：自然科學不是着眼於「爲什麼？」而是着眼於「怎麼樣？」，換句話說，應該尋求定量的描述而不是定性的解釋。不過，當我們累積足夠多「怎麼樣？」的解答時，

我們便有種感覺我們明白了「爲什麼？」吧。舉一個例子，爲什麼自由物體下落？因爲地球有吸力；爲什麼地球有吸力？因爲天下萬物都有引力；爲什麼有萬有引力？至少在目前階段沒有人能答覆這個問題。即使將來當引力子和引力波的研究進展到某個階段，使我們能答覆這個問題的時候，又將會有一層更深刻的問題被揭露出來的。但是，我們知道萬有引力服從平方反比定律，由此我們能夠推論天體如何運行，甚至預測天象。而且這些預測是可觀察和可量度的。因爲它們是這樣準確，我們便有種感覺我們明白了天體運行這個自然現象了。



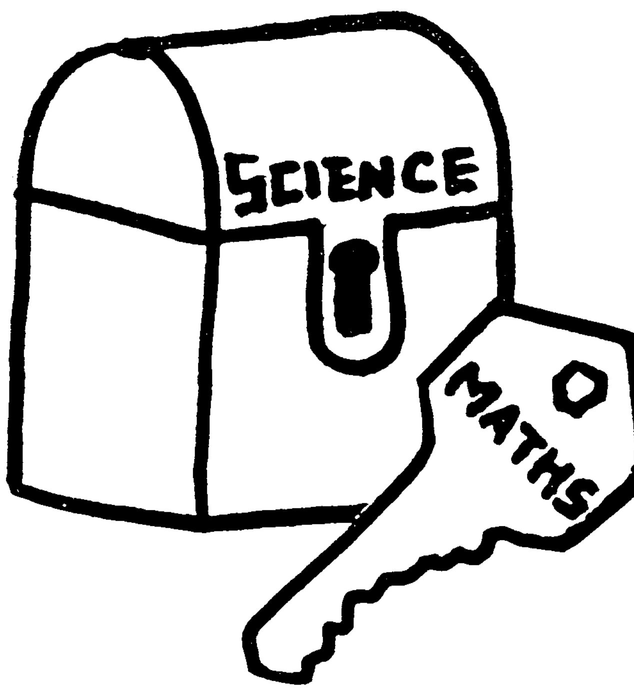
伽里略非常重視數學上的推論，最著名的一個例子便是推翻亞里士多德關於物體下落的論斷。通常聽到的故事，說伽里略跑上比薩斜塔放下兩枚重量不等的鐵球，它們同時墮地。沒有史實證據確定他曾做過這實驗，而有或沒有做過這實驗倒不是最重要的，重要的是他從理論上已經得到物體自由下落的速度與重量無關這個結論。他的證明簡煉精悍，是一個漂亮的「反證法」示範：如果A比B重，按照亞里士多德的說法，B比A下落慢些。於是B把A＋B（A和B縛在一起）的下落速度拖慢，所以A＋B比A下落慢些。但另一方面A＋B比A重，按照亞里士多德的說法，A＋B却比A下落快些。這是互相矛盾的，所以亞里士多德的說法不成立。



自從伽里略提出他的見解後，數學成爲瞭解自然最有力的幫手。他在1610 年說了一句有名的話：「大自然的奧秘都寫在這本永遠展開在我們面前的偉大書本裡，如果我們不先學曉它所用的語言就不能理解它……。這本書是用數學的語言寫成的。」十七、十八和十九世紀初期的數學家差不多都同時是科學家、天文學家，因此那個時代的數學發展跟自然科學發展是分不開的。牛頓 (NEWTON) 創立了微積分，但他也以天體力學、光學著稱後世；達朗貝爾 (D 'ALEMBERT)、歐拉 (EULER)、伯努利 (BERNOULLI)、傅里葉 (FOURIER) 對數學分析有重大貢獻，但他們也解決了弦綫振動、熱傳導的問題；拉格朗日 (LAGRANGE)、拉普拉斯 (LAPLACE) 是十八世紀後期的數學大師，但前者也以解析力學聞名於後世，後者也是當時最偉大的天文學家；高斯 (GAUSS)被公認爲有史以來最偉大的數學家之一，差不多在每個重要的數學領域上都留下功績，但他也對電磁學作出貢獻，而且還是哥廷根天文台主任。

到了十九世紀中葉，數學與科學還是攜手並進，相輔相成。一個最好的例子便是麥克斯韋 (MAXWELL) 在 1865 年前後的工作，把當時所知的電磁現象通過數學處理總結爲著名的麥克斯韋微分方程組。其中最有趣的一條方程是關於磁場的旋量，原本那是基於安培(AMPERE)

由實驗得來的定律，但按照該定律寫下來的方程却導致數學推論上的矛盾！爲了補救這個漏洞，麥克斯韋替方程加了一項，稱它作「位移電流」。其實早在 1820 年丹麥科學家奧斯忒已經發現傳導電流可產生磁場，但麥克斯韋加上的一項，却說明了除此以外，變化的電場也可產生磁場。不單止此，他還從方程組推論變化的電場在其附近產生變化的磁場，這變化的磁場又在其附近產生變化的電場，而這變化的電場又在其附近產生變化的磁場，……。這樣繼續下去，變化一路往外擴散，有如水波，就叫做電磁波。麥克斯韋還計算了電磁波的傳播速度，發現它竟然等於光速，於是他聲稱光是一種電磁波。單就理論方面而言，麥克斯韋的工作把電、磁、光和波運動有機地連繫起來，已經相當了不起；何況它還促使科學家在實驗室裡找尋電磁波，過了二十四年後赫兹 (HERTZ) 找着了它，再過了八年後馬可尼 (MARCONI) 利用它實現了無綫電通訊的夢想。麥克斯韋



在物理學史上更扮演了一位承前啓後的角色，因爲正是他的電磁學說帶來的新疑難引致愛因斯坦創立他的相對論。難怪赫兹讚嘆地說:「我們有個感覺，這些數學方程有它們獨立的存在和自身的理解力。它們比我們聰明，甚至比發現它們的人還要聰明，因爲我們從它們所取的比放進去的還要多。」

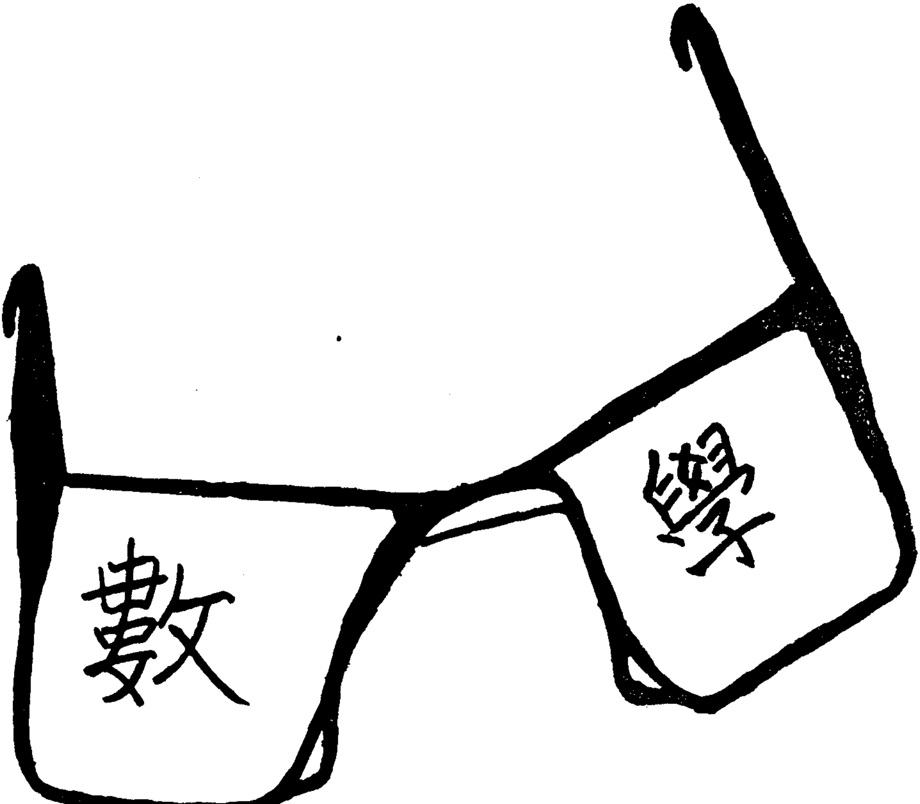
當然，還有很多別的例子。譬如群論由解代數方程而來，但近代基本粒子的研究少不了它；綫性代數由解綫性方程組和綫性微分方程組而來，但近代量子力學的研究少不了它；非歐幾何由考慮歐氏幾何的公理而來，但相對論的研究少不了它；微分幾何和微分拓朴由研究曲綫曲面而來，但近年來宇宙論和規範場的研究少不了它。而且，數學與自然科學的關係，不是單方面而已，數學也欠了自然科學的人情。從古至今，很大部份的數學都爲了認識自然而發展起來，自然科學也繼續刺激着數學深入的探討，例如最近規範場的研究便帶來低維流形方面的新發現了。

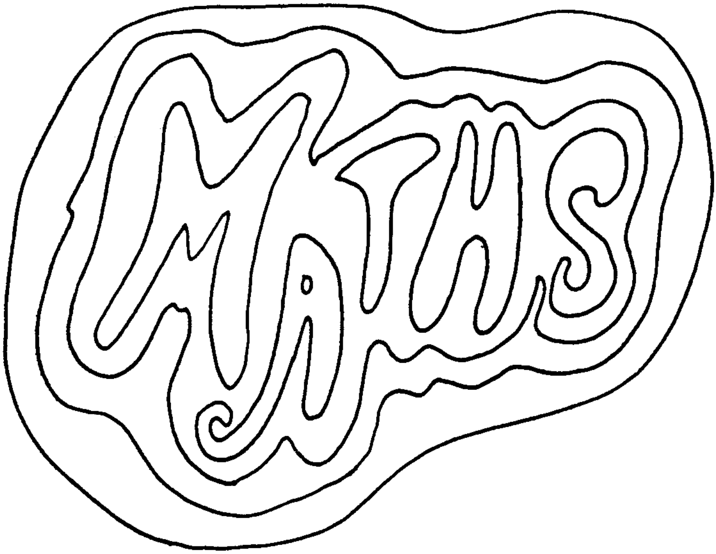
在十九世紀中葉以後，數學史上接二連三地發生了幾樁具革命意義的大事，改變了數學家對數學的傳統看法，也間接帶來數學與自然科學之間某種程度的疏遠。讓我們只看其中一樁，就是非歐幾何的誕生。從古希臘至十九世紀初，人們一直把歐氏幾何看作理所當然，認爲它不只描述了我們所處的空間，甚至只有它才是眞正的幾何。十八世紀後期的德國哲學家康德 (KANT) 便認爲歐氏幾何是獨立於我們經驗之外而存在的眞理。高斯是第一個從數學意義上理解非歐幾何的人，但他因爲害怕傳統壓力一直不敢發表他的學說，直到 1830 年左右當鮑耶 (BOLYAI) 和羅巴契夫斯基 (LOBACHEVSKY) 分別發表了他們的非歐幾何學說後，他才在寫給鮑耶父親的信上透露自己早在三十年前已經做了同樣的工作！非常簡略地說，他們的幾何與歐氏幾何最不同的地方，在於通過一條直綫以外的一點有多於一條與該直綫平行的直綫。因爲這緣故，服從這種幾何的空間便有很多稀奇古怪的性質，與我們的直觀毫不相符。起初連數學家也懷疑這樣的幾何會否自相矛盾，到了十九世紀後期，好幾位數學家利用歐氏幾何建立非歐幾何的模型，意思是說要是你接受歐氏幾何，你也得接受非歐幾何。

這些匪夷所思的幾何，使數學家意識到數學不是尋求「眞理」，數學只是從某些假設出發邏輯地推論某些結果。如果假設成立，則推論的結果也成立，如此而已；至於假設是否成立，那可不干數學的事。數學家索性把假設叫做公理，公理無所謂眞或假，我們只問公理是否協調，即是它們之間有沒有自相矛盾。爲了解決這個數學協調性的問題，德國數學大師希耳伯特 (HILBERT) 在本世紀初提出一個著名（但後來被證明是行不通）的方案，後來被稱爲「形式主義」。很可惜，每種主張最易爲一般人所接納的說法，往往是所謂「庸俗化」了的說法，數學上「形式主義」的「庸俗化」說法，就是「數學是建立空中樓閣」。一般人以爲只要提出幾條邏輯上沒有自相矛盾的公理，便是建立了一個數學系統；從這幾條公理推論定理，便叫做數學！推至極端的說法，數學是個符號遊戲！如果眞的是這樣，便的確令人難以理解爲何數學是這樣有效了。但其實數學創造必須求助於知覺和直覺想像，公理也是這樣醞釀而產生的。

不過，數學也的確有它「自由」的一面，正如數學家康托爾（CANTOR)的名言說：「數學的本質就是它的自由。」雖然我們很多時候倚靠客觀世界作嚮導，但不容否認有不少數學概念跟實際事物的關係，至少不是直接的。例如你在街上走不會碰見多項式，更加不會碰見虛數！這便引起一個疑問：會不會是我們戴上有色的「數學眼鏡」來看這個世界？我們利用自由思維創造數學來解釋事物的話，會不會我們看見的只是我們想像中的呢？讓我舉一個例子：打開一本化學手册或者物理手册，數數有多少常數的頭一位有效數字是 1、2、……、8、9。你以爲它們平均地出現嗎？錯了，差不多百分七十的常數的頭一位有效數字是 1、2、3 或 4，只有百分三十左右的常數的頭一位有效數字是 5、6、7、8 或 9 吧。是否大自然對 1、2、3、4 有偏愛呢？我們仔細地分析這個問題，便知道並非是大自然有所偏愛，而是由於我們定義的十進制地位記數法，使大自然看似有所偏愛吧！（頭一位有效數字不大於 N 的機會並不是 N／9，而是 $\log_{10}(N+1)$。當 N 是 4 時，$\log_{10}(4+1)$ 約爲 0.7。這個有趣的事實是美國天文學家紐康 (NEWCOMB) 在 1881 年發現的，後來在 1938 年被美國一位工程師本福 (BENFORD) 重新再發現。）

這個「數學眼鏡」的說法聽起來是否有點可怕呢？然而想深一層，科學何嘗不是這樣子。我們從衆多的表面現象中企圖抽取一些簡單的原理，這就是統一的過程。再從這些原理出發推論更多的現象，這就是預測的過程。統一加上預測，便編織成一套理論。愛因斯坦在一篇題爲「物理與實在」的文章結尾時說：「物理是一套有邏輯的思想體系……，它的基礎彷彿不能從歸納經驗而得，而只能從自由創作而來。」不過他也接着說：「對這個體系的判斷，是基於從知覺去驗證那些推論出來的結果，……」所以沒有一套理論是不可推動的權威，正如伽里略說：「在大自然的法令面前，人間的權威算得什麼？」正是這樣，人類對自然的認識逐步深化，永無盡頭，而在這探索過程中，數學一直起着重大的作用。





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以上簡略地敘述了歷史上一些數學家和科學家對數學和科學的看法，但對開首提到的問題好像仍沒有答覆。或者你會問：「那麼你個人的看法是什麼？」不如先講一個故事。托爾斯泰 (TOLSTOY) 小說『安娜・卡列尼娜』的主人公臥軌自殺了，托爾斯泰的朋友埋怨作者對她太殘忍，讓她得到這樣的結局。你猜托爾斯泰怎樣回答呢？他說他筆下的人物往往做出違反他本意的事情，連他自己也不喜歡，但他們做的却是實際生活中常發生的事，是他們應該做的事。雖然人物是由作者塑造出來，但連作者對此也感無可奈何！我覺得數學創造與此有異曲同工之妙。有人曾問哲學家懷德海 (WHITEHEAD)：「你以爲那樣較重要，事物呢？抑或理念？」他不假思索便答道：「當然是關於事物的理念囉。」我不妨「狗尾續貂」，提出還有關於理念的事物這一回事！舉一個例子：變量 X 是一個概念，變量 X 的變化關係 $ax^2 + bx + c$ 是一個概念，方程 $ax^2 + bx + c = 0$ 是一個概念，滿足方程的根是一個概念，但好比方寫下一條方程 $10x^2 - 6x + 2 = 0$ ，它有或沒有（實）根却不再由你決定了。又或者二次方程 $ax^2 + bx + c = 0$ 有多少個根，也不再由你決定。從你定義什麼叫做二次方程開始，這回事已

經有了客觀存在的答案。我打算說的，就是數學雖然是人類自由思維的產物，但它也有實在的意思，而且在一定程度上這種思維反映了客觀世界的素材，只不過它們是曾經加工抽象的。但概念一經形成，它却彷彿有了自己的生命，向着它應該生長的方向生長。

要是這樣想，數學對自然科學的有效性，也就不是那麼令人難以理解。固然，數學史上也有過很多例子，數學上的學說比它在實際世界的應用早了百多年甚至幾千年，前者如群論，後者如圓錐曲綫。然而，如果我們知道數學是研究量與形的性質、變化、關係、圖案、規律，而如果我們也相信宇宙是和諧、簡美的話，這一點也就變得不是完全（或者仍有一點點！）那麼神秘玄妙了。所以數學發展的過程，既有其他學科和社會活動對它的影響，也有自身內部問題給它的刺激。片面追求「立竿見影」的應用或者沉醉於「唯理論高」的說法，都是不健康的態度。

至於數學能帶引我們至那一個地步呢？我喜歡十九世紀法國數學家伽羅瓦（GALOIS）的話：「這門科學（指數學）是人類思維的結晶，命定是用以探討眞理而不是用以知悉它，是用以尋求眞理而不是得到它。」

算式中的名數

楊延壎

首先正名。

很多老師把例如「32 厘米」裡的「厘米」稱爲名數，這是值得商榷的。

名數有別於不名數：

不名數用來表示數，例如 32、½、0.75、…

名數用來表示量，例如 32 厘米、½ 小時、0.75 升、…

名數由「名」和「數」兩部分組成：

因此，假如學生在書寫上例中漏去「厘米」而只寫下「32」，我們不應把這說是「漏掉名數」而應是「漏掉單位」。

「32 厘米」是單名數，「4 米 32 厘米」是複名數。(在使用十進制單位時，一般主張用單名數而不用複名數記數，因此，4 米 32 厘米不如寫成 4.32 米或 432 厘米。)

一向以來，香港教師對學生在書寫名數上要求過於嚴格，而且往往訂下一些諸如「名數除名數得不名數（或異名數）」的清規戒律；學生犯了些微錯誤，往往遭受過份的懲罰（改正、扣分、……）。此外，學生的作業與教師的批改都在這方面花費很多精力與時間，這往往是得不償失的。

近年來，世界很多地區對名數的書寫採取比較開放的態度，香港有一部份課本與學校也開始把書寫格式簡化。以下舉一些例題作比較：

沿用寫法	簡化寫法
例(1) 同學43人，老師2人一同去旅行，參加旅行的共有多少人？	
共有　43人＋2人＝45人	共有　43＋2＝45（人）
例(2) 每株花有5朵，8株共有花多少朵？	
共有花　5朵×8＝40朵	共有花　5×8＝40（朵）
例(3) 故事書96頁，4天讀完，平均每天讀多少頁？	
平均每天讀　96頁÷4 ＝24頁	平均每天讀　96÷4 ＝24（頁）
例(4) 故事書96頁，每天讀4頁，多少天讀完？	
式一　共需　96頁÷4頁 ＝24（天） 或共需　96頁÷4頁 ＝24天 式二　共需　（96÷4）天 ＝24天	共需　96÷4＝24(天)

例(5)　長方形長 8cm，闊 5 cm，面積是多少？

<u>式一</u>　面積：$(8 \times 5)\ cm^2$ $= 40\ cm^2$ <u>式二</u>　面積：$8\ cm \times 5\ cm$ $= 40\ cm^2$	面積：$8 \times 5 = 40\ (cm^2)$

例(6)　圖書 460 本，按照人數分配給五年級甲、乙班，甲班有 43 人，乙班有 49 人，兩班各應分得圖書多少本？

兩班共有　43 人＋49 人 ＝92 人 甲班分得　460 本×$\frac{43}{92}$＝215本 乙班分得　460 本×$\frac{49}{92}$＝245 本	兩班共有　43＋49＝92(人) 甲班分得　460×$\frac{43}{92}$＝215（本） 乙班分得　460×$\frac{49}{92}$＝245（本）

從以上的比較中，可提出幾點來討論：

一、以上的簡化寫法是配合香港沿用寫法加以簡化而成。有些地方的寫法以例(1)來看是寫成：

43＋2＝45（人）

答：共有 45 人

換言之，在算式之前不加任何文字，而在最末以文字將答案敘述一次。

一些歐美國家採用的算式則往往更爲簡單、隨便。

(二) 簡化寫法無疑可以或多或少地減省學生書寫的功夫，從而也減低了出錯的機會，這在除法算式中尤爲明顯。沿用的除法算式，把等分除法（見例(3)）和包含除法（見例(4)）作不同的處理，而包含除法也有兩種不同的格式。學生在初學時往往感到眼花撩亂，無從入手；其後在列式時也往往把不同格式混淆。在簡化的算式中，等分除法與包含除法在格式上並無分別，答數的單位則可憑題意作定奪，這就大大提高了列式的準確性。

趁這機會一提：在傳統的包含除法算式中，<u>式一</u>即所謂「名數除名數得異名數」，是沒有甚麼說服力的，而且書寫繁複，因此不是一個好格式。

(三) 名數的加、減書寫比較簡單，名數的乘、除算式較爲複雜，值得細味。

首先看乘法：例(2)以「每株花有 5 朵，8 株花共有多少朵？」來列式，假如比較嚴謹的話，可以寫成：

5 朵／株 × 8 株 = 40 朵

同理，這一題的逆運算（求每株有花多少朵的除法）也可以寫成：

40 朵 ÷ 8 株 = 5 朵／株

這與由距離與時間求速率的算式相類似：

300 公里 ÷ 6 小時 = 50 公里／小時

例(3)因此也可寫成：96 頁 ÷ 4 天 = 24 頁／天

不過，這種 5 朵／株或 24 頁／天的寫法，較爲累贅，不易被一般人接受，低年級學生尤然。

在例(2)的沿用算式內，兩個名數「5 朵」與「8 株」之中，爲甚麼「8 株」會變成不名數而列成「5 朵 × 8 = 40 朵」呢？

這是由於解題的對象是「朶」而不是「株」，因此在算式中保留「朶」這個單位；8株樹的花朶數目應是1株樹花朶數目的8倍，是一個倍數，因此「8株」便降格成爲「8」(由量變爲數）。這種做法可視爲數學上的一種「約定」(Convention)——一種訂定下來讓大家遵守的規則。

有些人認爲簡化寫法 5 × 8 = 40（朶）不合數理，因爲 5 × 8 只可以等於40，事實上這個簡化寫法也是另一種「約定」，這與 5 朶 × 8 = 40 朶相去只是「一百步」與「五十步」而已。

四 例(5)求長方形面積的兩個沿用格式中，以式一較爲可取，式二似是將兩個量 8 cm 與 5 cm 相乘，這在概念上已與求長方形面積的方法違背，一個長 8 cm 闊 5 cm 的長方形可以分割成（8 × 5）個 面積是 1 cm^2 的小正方形，因此面積是 $(8 \times 5) cm^2$，我們是不能把兩個長度相乘的。

（事實上我們應該明白：在任何應用題的運算中，我們是將量中的數抽離作運算的——上例是將 8 乘以 5，例(1)是將 43 加以 2，……——而並非直接將量作運算。）

結語：爲了減輕學生的負擔而相對地提高學生學習的質素，簡化名數的書寫是值得鼓勵的。

應用題的處理

林秉明

常常聽到老師們感慨地說：「我們怎樣才能夠使小學生在認識基本運算後，就會運用他們的數學知識去解決應用題？」所謂「應用問題」就是課本內的文字題。教導兒童處理文字題，可說是數學老師的一大負擔。

其實人類天生就有解決問題的本能，而老師的責任就是要引導兒童充份發揮他們的本能，使他們能運用所學的知識去加强解決問題的能力。

要幫助及引導兒童解決問題，可循下列幾方面着手：

(甲) 多提問一些富啓發性的問題，但這些問題並不一定有固定答案的。教師可着兒童從活動中找出答案。例如：教授一年級學生時，老師可指導兒童通過砌積木找出「5」的組合，並要求他們將不同的結果記下。如：

1 加 4 是 5

2 加 3 是 5

3 加 2 是 5

4 加 1 是 5

這就是他們解決問題的記錄。通過這類練習，兒童會發現有時同一問題是可能有多於一個答案的。

(乙) 加强兒童對數學語言的認識——兒童在處理文字題時常常因爲對數學語言認識不足，以致不能達到解決問題的目的。因此，老師應加强兒童對常用數學名詞的認識，例如：

…比…重
…比…輕
…比…多
…比…少
…比…長
…比…短
一樣重
一樣多
共有、合計
還有、餘下
較多、較少
每份有多少？可分多少份？
兩倍、一倍
·
·
·

（見小學數學課程綱要詞彙舉例）。

要使兒童能眞正了解這些數學名詞，有效的做法就是要兒童通過活動眞正去體驗一下。例如：

(一) 著兒童嘗試類似下列的文字題：

「小明有鉛筆 6 枝，小珍有鉛筆 4 枝，問小明的鉛筆比小珍的多幾枝？」

教師可以安排兩個學生作比較，通過實際的活動使兒童徹底明白題意。

(二) 當兒童有足夠的解題練習後，教師可以先列出算式，如：

4 + 2 = 6

然後着兒童配合算式自擬故事口述出來，這樣做法可以使他們能夠經過思考後，靈活地運用數學語言。數學科新課程綱要中亦屢次提出讓兒童就他們熟悉的事例自編應用題，就是這個道理。

(三) 在兒童編製簡單象形圖後，讓兒童進行口頭讀圖報告，也是提供訓練兒童運用數學語言的好機會。

(丙) 對兒童的不同答案加以認許和讚賞——兒童在解決問題時，思考過程往往與老師不同。作爲老師的應小心分析兒童的答案，了解兒童對該問題的思維過程，就兒童錯誤的地方加以指導。當老師發現兒童用不同的計算方法算出正確的答案時，老師應把握這個機會對兒童的做法加以讚許，並指出解決問題並非只得一個途徑的。藉此加强兒童對解決應用問題的信心。如果老師爲了節省時間，而制止兒童嘗試不同的計算方法，這樣不單錯失了一個輔導學生弱點的機會，而且更會打擊兒童的學習信心。許多時，兒童往往比成人們更富想像力，只要我們耐心聆聽，細意分析，我們就更容易看到兒童的內心世界，從而找尋到一些有效的途徑，領導他們進入數學的美麗領域。何況，兒童的答案有時也可以加深我們對數學的體會，使我們對某些數學觀念有更加深刻的認識和理解。

(T) 在學習初期，對兒童的計算格式，不宜太過苛求。培養兒童良好的計算習慣和精確的表達方式，固然是數學教學的一大目標，不過對於初入學的兒童來說，如果在他們的學習初期，就過份強求他們用嚴謹的格式來表達，是很容易令兒童失去學習興趣的。所以低年級老師在處理應用題時，可容許兒童只填上答案，無須詳列敘述句和算式。例如：

我有糖 5 粒，吃去 3 粒，還有糖______粒。

兒童只需在答案綫上填上「2」字便可。待兒童初步建立了處理文字題的信心後，再循序漸進地要求他們寫好格式，千萬不要把一堂好好的數學課變成只訓練書寫格式而缺乏數學內容的一課。筆者也曾遇到一些兒童在學習方程式應用題時，思考及計算方面，本來是毫無困難的，但他們不肯嘗試解決問題。原因是不懂得怎樣運用文字去清楚地列出假設和敘述句。這可能由於在他們的學習過程中，縱使計算正確，但格式不對，也不會獲得任何分數，導致他們每當使用文字敘述遇到困難時，就乾脆放棄。可見老師對兒童表達方式的要求過於嚴格時，是會打擊兒童的學習信心的。

上述各點，純粹是筆者個人的一些意見，希望其他對數學教學有經驗的老師多抒發他們的寶貴意見，俾能使從事數學教學的同工們能引領兒童愉快地進入奇妙的數學王國。

收穫

芬

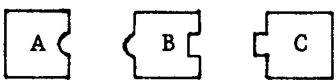
去年，得到有關方面的提名，我被委任爲一個與數學教學有關的委員會作委員，實在感到十分高興。從會議中，吸取那些資深的數學教育工作者寶貴的教學經驗和意見，使我獲益不少。他們提出的意見往往就是經驗淺薄的我所不知曉的，有些更是我從未想過的一些教學途徑；如此眞豐富了我的教學方法，使我對今後的教學更充滿信心。現在就讓我列擧一些例子來和大家分享吧。

加法結合性質

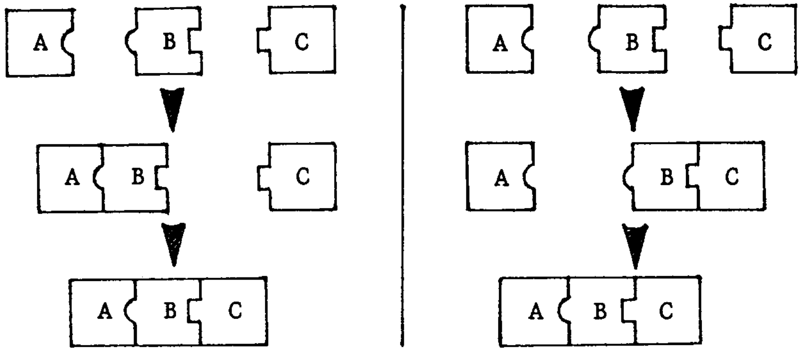
有些老師在介紹加法結合性質時，往往不自覺地同時將加法交換性質一起應用，因而未能將加法結合性質的意義清楚地說明。例：

$$
\begin{aligned}
& 4+5+6 \\
&= (5+4)+6 \quad \text{（加法交換性質）} \\
&= 5+(4+6) \quad \text{（加法結合性質）} \\
&= 5+10 \\
&= 15
\end{aligned}
$$

若要學生了解加法結合性質，老師宜簡化例子，利用拼圖板的原理去解說：



我們可先拼合A和B，然後才拼合C；又或者先將B和C拼合，然後將A拼合上。兩個方法拼合的結果是一樣的。



例題：

5 + 4 + 6	5 + 4 + 6
=（ 5 + 4 ）+ 6（連加的意義）	= 5 +（ 4 + 6 ）（加法結合性質）
= 9 + 6	= 5 + 10
= 15	= 15

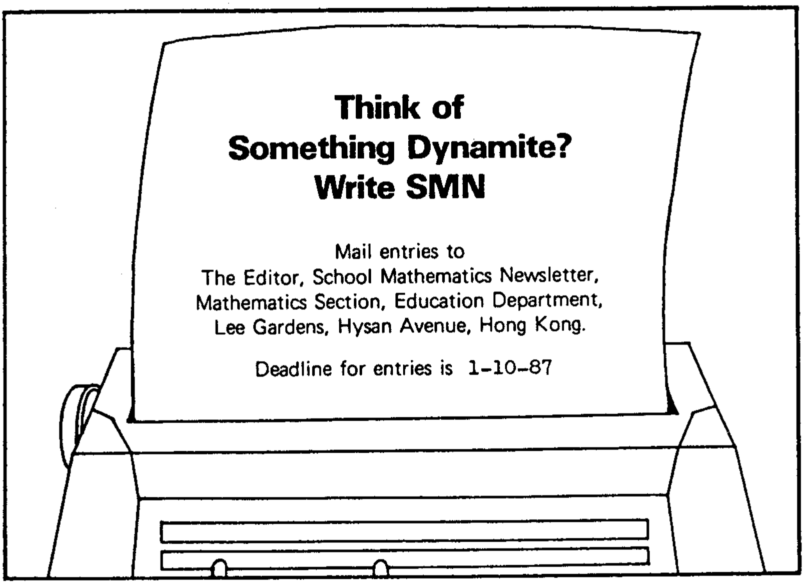
各數在式中的位置沒有變動，只是數的運算次序不同而已。如果數的位置有所變換，那麽便應用了加法交換性質了。

用拼圖板的例子去解釋加法結合性質，你說是不是很清楚及明白呢？

公倍數

有一位老師提出用十行表來教公倍數的效果雖然好，但是會缺乏新鮮感，學生可能會因此而覺得沉悶，提不起勁去學習。於是她在處理這一課題時，便先安排學生依照他們的班中編號入座，這樣，課室的座位便成爲一張活動的十行表而學生便是活動的數了。例如：要找2和3的公倍數，老師首先請那些編號是2的倍數（即：2、4、6、8、10……等）的學生站起來，然後坐下；其次，老師再請那些編號是3的倍數（即：3、6、9、12……等）的學生站起來，然後坐下。最後，老師便請那些曾經站起來兩次的學生再度站起，依次報出他們的編號。於是便可找到2和3的公倍數：6、12、18、24………更可加深學生對公倍數的認識，而最小公倍數也由此而找到是6了。

你們認爲這方法可行嗎？不妨試一試。



科任教師的編排對數學教學的影響

詩

科任教師對數學教學其實是有着深遠的影响，所以於每學年的開始，校長在編排科任教師工作分配的時候，必先要詳細考慮個別教師任教數學的能力及對數學的興趣，千萬不可視數學科爲非常淺易的科目，認爲在教學上只要懂得怎樣計算加減乘除數便可以勝任，在批改作業上，只要對對答案，便可把練習簿一一批改好。否則，用這樣的心態去分工，數學科便可能變成「豬肉」一般的被分派給那些既沒有選修數學而又對數學沒有興趣的教師身上了。倘若對數學沒有興趣而數學修養又不好的教師眞的被指派任教數學時，他們必定感到十分困擾，而學生亦會因缺乏教師積極和悉心的指導而提不起興趣去學習，影响他們學習數學的效果，須知道數學的主要教學目標之一，就是要引起學生對數學學習的興趣，啓發他們的數學思考，及培養他們的創造能力。教師本身若對數學沒有興趣或甚至厭惡數學的話，試問又怎可以達到引起學生對數學的學習興趣呢？故此，要奠定學生的數學基礎，科任教師的適當安排實不容忽視。

Models in Mathematics

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Introduction

When you read 'mathematical model' what do you think of? Very likely a nicely made and painted card dodecahedron; or a skeleton of PVC rods glued together to show the important angles that relate to an inclined plane; or may be a piece of curve stitching (a favourite in primary schools) showing a pleasing geometrical pattern; or a polystyrene cone carefully sliced to reveal ellipse, parabola and hyperbola. The list is endless - but they are all models in that they are crafted by hand (like model aircraft), and mathematical in that their construction requires and illustrates some mathematical concept. They can be invaluable teaching aids, and the basis for splendid activities in your mathematics club. The classic book on such mathematical models was written by Cundy and Rollett (1961), but their ideas have been used in many school textbooks since then.

But a model is not just something made by hand. A knitted sweater may be made by hand, but we do not call it a model. Why? - because it's the real thing. When we say model we have the idea of something very much like the real thing, but in other respects quite different. Think again of the model aircraft. It might have the exact proportions of a Jumbo, and painted in the colours of Cathay Pacific - but on the other hand it is only 50 cm long, made of balsa wood, and cannot fly! A model is a representation of reality, but it is not reality.

Do we have this sense of representation when we talk of mathematical models? Certainly. When you use a card tetrahedron in your lesson on pyramids it is of course an imperfect tetrahedron - its edges are not sharp, the sides are not absolutely straight. The model pyramid is only an approximation to a geometrical pyramid. But we can go further than this and talk about mathematical models which are primarily representations, and which may have little or no physical reality. Also we do this in two ways. First, we make models of mathematics - the card tetrahedron comes into this category. Second we make models using mathematics - this is the essential activity of applying mathematics.

A model for directed numbers

Let us consider in more detail models of mathematics, and we will begin with an example. Miss Wong is teaching Form One how to add and subtract directed numbers. She begins with the model of temperature - the temperatures on the previous day in Beijing had ranged from $-12^\circ C$ to $-3^\circ C$, so this provided a good example. She then switched to bank balances as a model - about which even Form One students seemed quite knowledgeable. Both these examples modelled the same thing, namely the number line, and with the model the students were able to decide which of -12 or -3 was the greater. We see the essential feature of this type of model - it is a familiar, real situation, which matches the abstract concept, and enables the student to decide on the answers to questions about the abstract idea.

With this preliminary work Miss Wong decides that the number line is now sufficiently clear to act in turn as a model for the more abstract idea of directed numbers and their addition and subtraction. To make the model work (that is, answer questions for us) we must have a clear correspondence rules between the abstract concepts and the model. In this case we need to be able to represent in the model both directed numbers and the operations addition and subtraction. Miss Wong uses the following rules. Positive numbers are represented by arrows, of appropriate length, pointing to the right; negative numbers by arrows to the left. Addition is carried out by putting the arrows end on and the result is given by an arrow from the origin to the end of the second arrow. Subtraction is performed by drawing two arrows from the origin, with the answer being represented by an arrow that joins the end of the second arrow to the end of the first. Of course, Miss Wong does not state these rules formally in the kind of language just used, but explains them by drawing them on the board for the cases $4+3 = 7$ and $6-2 = 4$. Thus she draws something like Fig. 1.

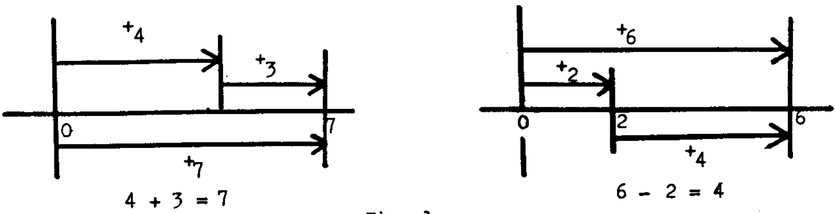
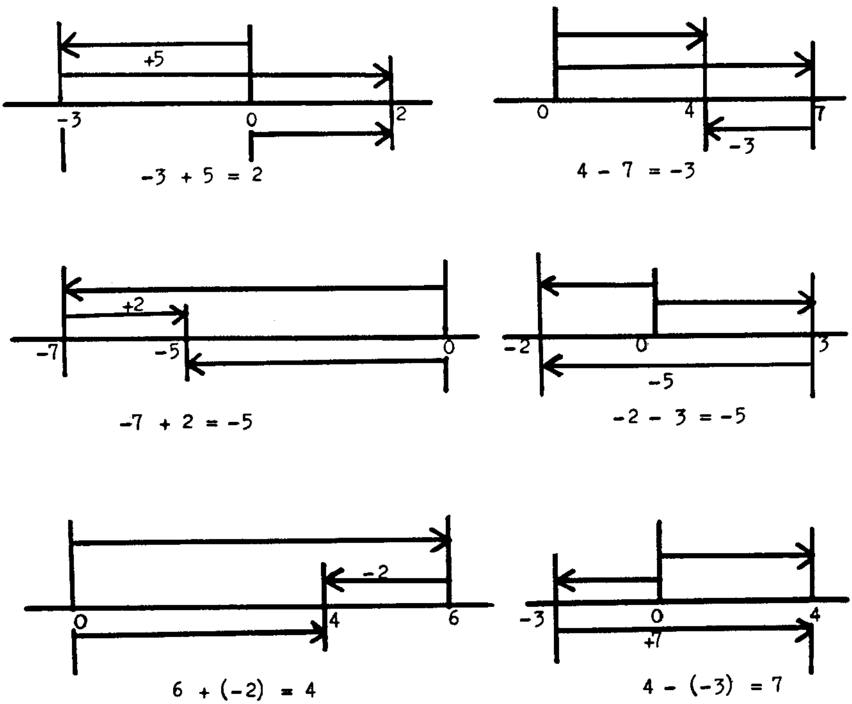


Fig. 1

The students see from these diagrams that they match their ideas of addition and subtraction (thinking of this as complementary addition) for the already familiar whole numbers. The model is thus justified for the area that is known. It is then extended to explore the unknown region. Miss Wong guides her class to the production of the diagrams shown in Fig. 2.



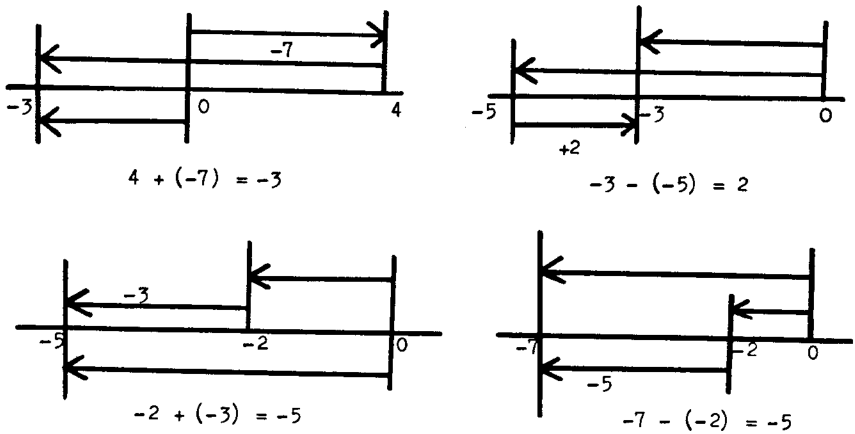


Fig. 2

You will notice that Miss Wong has been careful to draw different diagrams for (for example) 4-7 and 4+(-7). After all the first is the subtraction of a positive number and the second is the addition of a negative number, and this difference is faithfully modelled by the use of different diagrams. But what the diagrams reveal is that the <u>result</u> is the same; and so by this, and similar examples, the equality of a-b and a+(-b) is established. Miss Wong could have been even more emphatic on this point by insisting on the notation $^{+}4+^{-}7=^{-}3$, and for English speaking students, for whom there is an ambiguity about the words <u>plus</u> and <u>minus</u>, this could be advisable. Chinese has the advantage of the unambiguous words 加 and 正 , 減 and 負 . Miss Wong's students were therefore less likely to be confused by the ambiguity of the standard notation.

<u>A different model for directed numbers</u>

This is not the only model for addition and subtraction of directed numbers. Mr. Lam is using a different approach with his class. A number of his students like football and have their favourite teams in the English league. Mr. Lam studied in England so he knows how the fans there give the results of matches. When Jenny tells Peter that the match score at Anfield was 'two, three', Peter knows that it was the home side, Liverpool, that scored two goals, and the away side scored three. So Liverpool lost this match by one goal. Mr. Lam explains this to his class and introduces the obvious notation (2,3) for the match score. He then introduces (in effect defines) positive and negative numbers as the result of a match. A win by two goals is $^{+}2$, a loss by one goal is $^{-}1$, and a draw is 0. Thus :

$$(4,3) = (3,2) = (2,1) = (1,0) = {}^{+}1$$
$$(2,4) = (1,3) = (0,2) = {}^{-}2$$
$$(2,2) = (1,1) = (0,0) = 0$$

To complete the model addition is defined as the process of combining the score at half time, with the score for the second half of the match to get the match score. So if Liverpool scored two goals to Manchester United's one in the first half, but with a change of fortune Manchester scored twice on the second half, the match story would read:

$$(2,1) + (0,2) = (2,3)$$

This then is a model for the directed number statement :

$${}^{+}1 + {}^{-}2 = {}^{-}1$$

Similarly subtraction answers the question, 'What happened in the second half if the final score was (4,3) after a half time score of (1,2)?' In this way a set of footballing stories, which the students can answer by thinking about the football, are equivalent to directed number questions.

A third model for directed numbers

Miss Kwan is working with her Primary Six class. She shows them some electrical plugs. There are red ones with a pin, and black ones with a socket, so that a red and black can join as a pair. This is shown in Fig. 3.

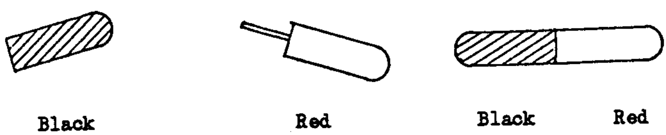


Fig. 3

She then asks the class how many plugs she has grouped together. They count, and come up with the answer 3 black and 5 red. She then asks how many are there if we disregard all those that can be joined as pairs; and they give the answer 2 red. Using this idea that a joined pair 'counts-for-nothing', plugs can be added or subtracted to any given collection. For example Miss Kwan gives a child 2 black plugs and asks him to subtract 3 reds. Fig. 4 illustrates the student's action.

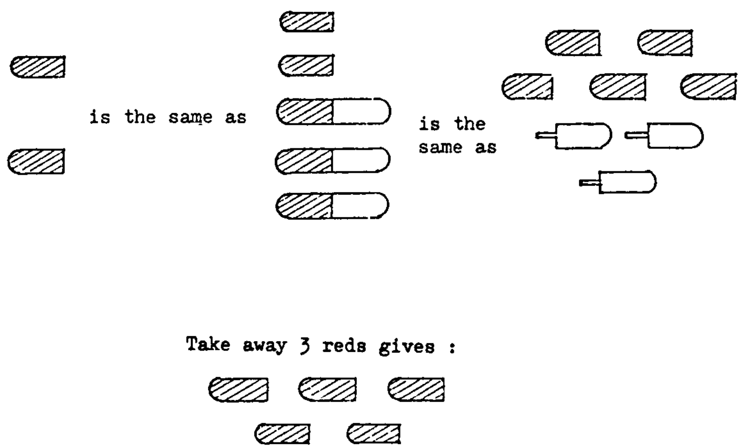


Fig. 4

Five blacks is the answer to this particular game. It only needs the identification of red with positive and black with negative (Miss Kwan follows the electrical convention; accountants might prefer it the other way round) for this game to be an acting out of the statement $-2-(+3)= -5$.

Arrow diagrams

We see that there is not just one model for a mathematical concept. Miss Wong, Mr. Lam and Miss Kwan have each used different models for the same concept. However with any of these models it is possible for the student to use familiar ideas, and think about them in a commonsense way, to solve problems equivalent to the addition and subtraction of directed numbers, which (initially at least) they knew nothing about. The model provides them with, in Seymour Papert's (1980) phrase, an 'object-to-think-with'. This key function of models in the teaching of mathematics is illustrated in Fig. 5, in a diagram that I will call an arrow diagram.

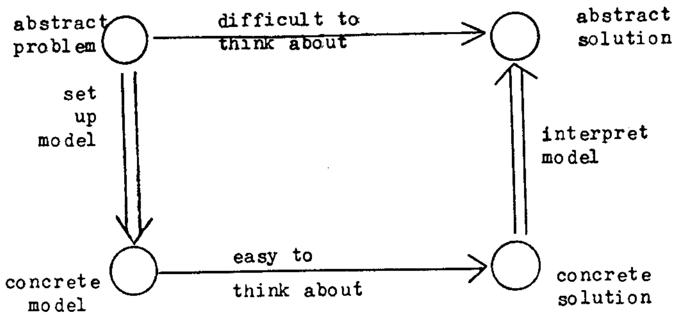


Fig. 5

Bruner's modes of representation

The three models just discussed have their differences. Miss Kwan's electric plugs allow the students to answer -7-(-2) = ? in a very 'hands-on' sort of way. The 'easy-to-think-about' stage is a question of manipulation, perhaps, more than thought. Miss Wong's arrows provide a clear picture of what is going on, and the use of them certainly requires thought on the part of the student. Mr. Lam's football scores, whilst allowing the students to think about a familiar situation, do seem somewhat more abstract. The idea that (4,3) = (3,2) etc. may seem rather artificial to the student, and the model stresses how an algebra of directed numbers can be built up out of an algebra of whole numbers. In a rough way these models illustrate the three well-known stages of concept development proposed by Bruner (1960). The plugs provide an enactive representation: the arrows an iconic representation: the football scores a symbolic representation (though the full symbolic form is our teaching goal, namely statements like -7-(-2) = -5). Bruner argued that students need to go through each of these stages in acquiring a concept. Hence his call for a spiral curriculum in which a concept is developed on separate occasions, each time at a higher level of abstraction.

Dienes principles

The presentation of the model by the teacher and getting the students to use it is not the end of the story. After all we do not expect students to solve their directed number problems for ever using a collection of red and black plugs. At some stage we expect the student to give up the model, and Bruner's spiral approach is intended to help achieve that goal. Z.P. Dienes (1960) a collaborator of Bruner, proposed a number of principles which he regarded as important in this respect. He used another word for model. He called them 'embodiments' in that they embody, or hold, the mathematical principle being taught. So that a student does not get fixed to a particular model (and so unable to transfer to the abstract level). He suggested that the student should see the concept in as many embodiments as possible. He called this the principle of multiple embodiments. In order that the student should be able to make the transition from the plugs, arrows, and football scores to directed numbers, each embodiment should look as different as possible from the others. This he called the principle of perceptual variability. I am sure we recognize a certain truth in these ideas from our own teaching. When a student says that he does not understand, repetition of what we have just said is usually insufficient - we have to think of another picture, story, model, embodiment (I don't mind which word you prefer). There is no best embodiment for teaching a particular concept - what appeals to one student may not work with another. When it comes to models what the teacher needs is variety - or, using a colloquial phrase, 'the more the merrier'.

The limitations of models

A further, important, point to note about our plug, arrow and football score models is that they are models in the same way that the model aircraft is a model of a real aircraft. The model represents the real thing in some respects, but it is quite different in others. Our mathematical models represent the addition and subtraction of integers. Only the arrow model can be extended to non-integral values, and none of them can represent even multiplication of integers (let alone division). To complete our teaching of the arithmetic operations on real numbers we will need yet further models. What model do you use to teach $(-7)x(-3) = 21$?

A model for factorization

Let me give a further example, taken from Bruner (1966), to illustrate the use of this type of model in teaching. You have prepared some 1 cm squares of thick card, some strips of card 1 cm x 7.4 cm, and some large squares 7.4 cm x 7.4 cm. (In fact, the 7.4 could be any figure, but there is an advantage in it being non-integral). You then give the student 6 small squares, 5 strips and 1 large square (Fig. 6).

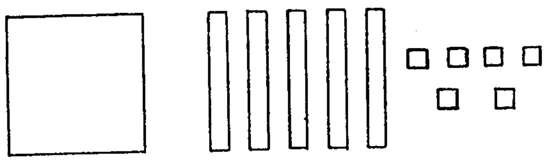


Fig. 6

Your instruction to the student is to arrange the pieces of card to form a rectangle. (If the strip had been 6 cm long there would have been an obvious solution; and not the one we want - hence the reason for the 7.4 cm.) I do not think it would be long before the student (Form Two) would produce Fig. 7 (or something like it).

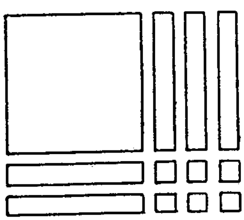


Fig. 7

It only remains for the strip length to be labelled x and we have a model for the factorization

$$x^2 + 5x + 6 = (x + 3)(x + 2)$$

Notice that this is very much an <u>enactive</u> model. The student is able to identify the LH-side of the above equation as the given material and the RH-side as the rearrangement, but the student moves from the LH to RH-side not by algebraic thinking but by geometric manipulation. However there is more to the model then this - it does not only provide a concrete way of getting the right answer. An examination of the final arrangement (and similar examples) reveals the necessary connection between the number of strips and the number of small squares (i.e. the squares must be able to be written as a product for which the factors sum to the number of strips). The foundation has thus been laid for teaching an algebraic method of factorization.

The model has its limitations. It can be extended to examples of the type ax^2+bx+c which have factors with integral coefficients, provided a, b and c are positive integers (and, for convenience of manipulation, not too large). To cater for negative and rational integers would probably stretch the model beyond credulity and hence usefulness. On the other hand it can be used (perhaps in Form Four) to illustrate why x^2+x+1 cannot be factorized with real factors, whereas x^2+6x+6 can, even though this latter cannot be done with integers. In fact, and I leave you to think this out, you can see from this model why for x^2+bx+c to have real factors it is necessary for $b^2 \geqslant 4c$.

<u>Transformations in mathematics</u>

The models <u>of</u> mathematics considered so far model an abstract mathematical concept with something more pictorial or concrete. There are also models of mathematics which are themselves mathematics: there are situation where we map mathematics into mathematics. Why should we want to do that? For the same reason as before - to make the solution of a problem easier. A representative, and striking, example of this technique is the use of logarithms to aid calculations. Here, of course, we model numbers with their logarithms and the operation of multiplication is replaced with the simpler operation of addition. In cases like this we tend to use the word <u>transform</u> rather than <u>model</u> (there is a rather more exact correspondence than we would expect from a model). However the process can be illustrated with an arrow diagram almost identical to Fig. 5 - as shown in Fig. 8.

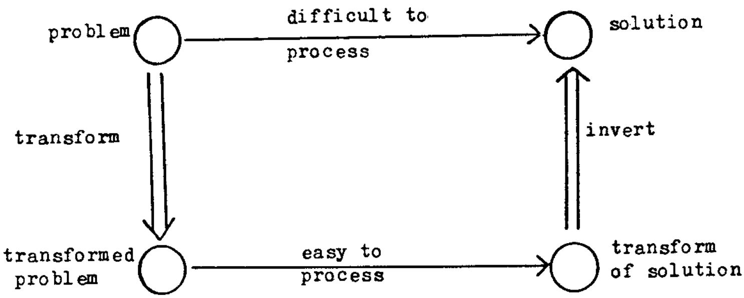


Fig. 8

Taking logarithms is but one example of this transformation idea. The transformation of geometrical problems into algebraic ones (and vice-versa) is the great achievement of co-ordinate geometry. Integration by substitution, conformal transformations, Laplace and Fourier transforms are examples from the higher reaches of mathematics. Transformations make up a thread that runs right through mathematics, not only making life easier in the solution of specific problems but binding together what otherwise seem disparate branches of the subject.

Applied mathematics

We now come to models _using_ mathematics, which as I remarked earlier is the backbone of applied mathematics. This time the mathematics is the model of the real situation. The direction of modelling, from concrete to abstract, is reversed; but the reason for modelling is the same - to make life simpler. The difficulty is that the real world is too complex, so in the modelling process we make simplifications and assumptions (which may not be justified) in order to transform our real problem (which we understand, but cannot solve) into a mathematical problem which we can solve. However, because of our simplifications the predicted result may or may not agree with reality. The completion of the modelling cycle is the comparison between our predictions and the empirical reality. If these are insufficiently in agreement then we may have to revise our simplifications and repeat the modelling cycle. The cycle can be illustrated (Fig. 9) with an arrow diagram as before.

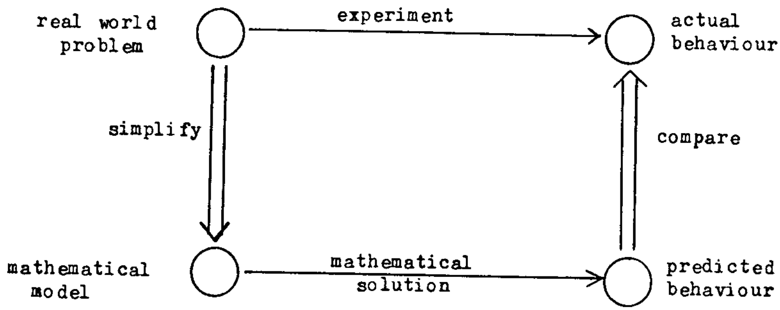


Fig. 9

The simple pendulum will serve as an example. The real pendulum has an elastic support, a finite sized bob, is resisted by the air etc. But in order to predict the period of swing for a given length we simplify by making assumptions on all these points. This allows us to set up a mathematical model. It turns out that this is soluble by rather advanced techniques. So we make a further assumption (that the angle of swing is small) to get an equation that can be dealt with by more elementary means. We are then able to derive the well known period formula, and a comparison with experiment will tell us if our swinging bob behaves like a mathematical simple pendulum.

The modelling process as a means of solving real problems has attracted considerable interest in recent years, with many feeling that it has been a neglected part of the curriculum at all levels. Too often applied mathematics has only been concerned with the lower arrow of Fig. 9. There is a growing literature on modelling in the secondary school of which Burkhardt (1981), Spode Group (1981) and the journal Teaching Mathematics and its Applications are representative.

Using mathematics

I would like to point out that the arrow diagram of Fig. 9 is not confined to those problems we traditionally associate with applied mathematics, nor to the more recent innovations just referred to. It has its place right from the first years of primary school.

Mr. Leung is talking to two of his primary one students. He notices that Mei Ling has collected 4 bricks while Siu Ming has collected 3. He asks them how many they would have altogether. The groups are joined together and recounted giving the answer 7. We are back to our first arrow diagram - a concrete model of a mathematical statement - with the student working enactively. At this stage the students are not familiar with the number bond 3+4 = 7: this little exercise is to help towards that end.

We have no intention of the students staying at this stage. At some time Mr. Leung will hope to be able to ask Mei Ling how many bricks would there be altogether if she had 17 and Siu Ming had 25, and get the answer 42 without Mei Ling having to arrange piles of 17 and 25 bricks and recounting the combined pile. By now Mei Ling will have learnt an efficient algorithm for addition, and the word problem is translated to the more abstract mathematical statement and solved using that algorithm. The arrow diagrams for the two stages are shown in Fig. 10.

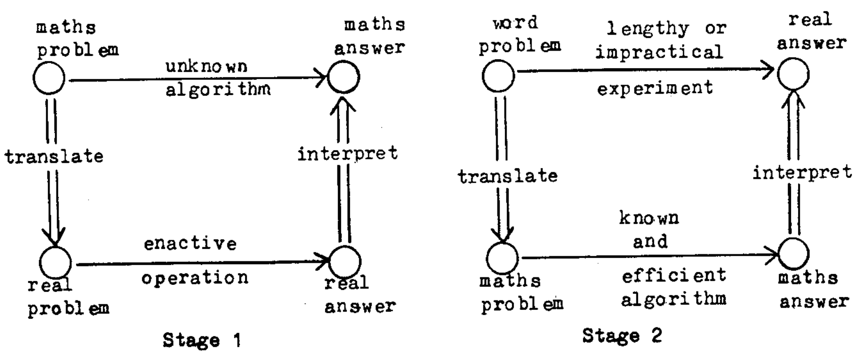


Fig. 10

We see that stage 1 is a model of mathematics. Stage 2 is a model using mathematics, and contains all the elements of the applied mathematics cycle, albeit in simplified form.

Problems for teachers

We can see here a far reaching problem for teachers. If learning involves going from a stage 1 to a stage 2, at least in the learning of computational algorithms, how is the transition from one stage to the next made, and what can the teacher do to help this? The writings of Dienes suggest that the student will stay in stage 1 only as long as they need to, that students will only continue to solve problems enactively while their understanding of the algorithm is unsure. Repeated examples of the stage 1 type (and remember he suggested in as varied forms as possible) will be the way to help the transition. Gagne (1983) has strongly disputed this. He suggests that stage 1, should only be used as an introduction to topic, and that the teacher's efforts should be spent on the direct instruction of the efficient algorithm.

In my view Gagne's position is invalid. There is evidence - many of you may know this from your own experience - that the successful teaching of an algorithm does not ensure the successful solving of problems that require that algorithm. Mei Ling may be able to compute correctly 27x7 using the multiplication algorithm that Mr. Leung has taught. But when he asks her the question, 'You buy 16 apples every day for 5 days. How many apples do you buy altogether?', she writes down 16+16+16+16+16, and finds the answer by addition. When Mr. Leung gives the problem, 'Pizzas are made in 4 sizes, and you can choose any of 9 flavours. How many different Pizzas can you choose?', Mei Ling is unable to solve the problem. (See Hart (1981) and Booth (1981)). Mei Ling is able to carry out a multiplication when it is given as a mathematics problem. But when given a word problem the difficulties of translating it into the appropriate mathematics problem are such that she uses a less efficient algorithm - repeated addition - which matches the wording of the problem - buying apples every day. The problem with the Pizzas requires seeing sizes and flavours as a 4 by 9 array, and from this deducing the need for multiplication. This, at this stage, proves too much for Mei Ling.

My opinion is that the relationship between stage 1 and stage 2 is quite complex, and that we operate in both ways throughout our mathematical life. A difficult problem may be resolved by turning it into a cubic equation - provided we are happy with algebraic representation and cubic equations hold no special fears for us. This is stage 2 thinking. The definition of a limit is notoriously difficult for students. They may be able to understand all the notation of the definition, but without it making sense. Some exercises with a calculator looking at the behaviour of such functions as $(1 - \cos x)/x^2$ might help. This is stage 1 thinking.

Although I agree with Gagne that instruction in computational algorithms is important, it seems to me that the difficult task for the teacher is how to help the students make the vertical links on the arrow diagrams - which I have labelled variously 'model', 'interpret', 'transform', 'invert', 'simplify', 'compare', 'translate'.

Conclusion

I have in this article perhaps extended the meaning of the word model to its limits. My aim has been to show that mathematical thinking is not simply a linear path from problem to solution, but that it is often characterized by different levels of thinking. These levels may be abstract/concrete, familiar/unfamiliar, simple/complex. We can represent the thinking process by what I have called an arrow diagram, which has the general feature that we travel round 3 sides of a rectangle to cross the gap of one side - and we do that because it is an easier route. My diagrams may look familiar to some as the commutative diagrams much used in modern algebra texts - but where the journey is usually from one corner to its opposite. An article on their use in teaching by Flectcher (1971) has been published in Mathematics Teaching.

The task of the teacher is to use these different levels effectively; but how to do this is imperfectly understood. To deny that modelling, in all the ways that can be understood, is important in mathematics teaching is to fail to grasp the complex nature of mathematical thinking.

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What is an answer ?

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Every good question deserves an answer - but what do we mean by an answer? When you ask your students "What is .....", have you ever queried yourself what is your "What is" ?

Well, let us make things simple by considering an example. When you ask your students "what is the portion each of 6 persons can get when dividing $4 among them equally", would you expect the answer 4/6? Or, when you ask about the locus of a point at unit distance from (2,3), would you accept the answer of

$$\sqrt{(x-2)^2 + (y-3)^2} = 1 .$$

Well, you say that they are not expressed in the simplest form, yet, you cannot deny that they are, indeed, correct answers, and there is no definition of "simplicity" whatsoever.

Which of the following is simpler :

$$\sin\theta \tan\theta \quad \text{or} \quad \frac{\sin^2\theta}{\cos\theta} \quad ?$$

$$x-2y = 3 \quad \text{or} \quad x = 2y + 3 \quad ?$$

It really depends on how you are going to use them.

Now, if a student presents his solution y by saying that it satisfies

$$xy + y + 1 = x \qquad (*)$$

surely, you would not be happy with it. Though for every given x, we can find the corresponding y, we would rather prefer the answer to be expressed explicitly as

$$y = \frac{x-1}{x+1} \quad ,$$

and when we want to find y', we pick up our (*) again and differentiate, to get

$$xy' + y + y' = 1 \qquad (1)$$

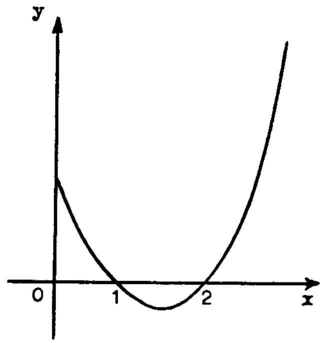
One of the questions the students are fond of asking is that of solving $2^x = 2x$ (or the like). In general, this kind of equation is not "solvable". However, one may argue that it has two "obvious" solutions, 1 and 2. As a matter of fact, for $y = 2^x - 2x$, we have

$$y' = \log_e 2 \cdot 2^x - 2$$

$$y' = 0 \Rightarrow x = 1 - \log_2(\log_e 2)$$

and

$$1 < 1 - \log_2(\log_e 2) < 2 \quad ,$$



hence y has a minimum with a negative value.

From the graph, one can see that 1 and 2 are the only solutions. For other more complicated equations, such as $2^x = 3x$, in which there are no more "obvious" solutions, the above method of proving the existence of solution in a certain range can still be employed. The value of the solution can only be estimated numerically.

So, what we actually mean by "solving" is to "deduce" the solution step by step from the given equation. In other words, what we have in mind is an algebraic solution. It can be proved that $2^x = 2x$ doesn't have one. Quite similarly, for polynomial equations with degree $\geqslant 5$, though Gauss had proved the existence of their solutions (in $\mathbb{C}$), Galios did also prove that their algebraic solutions do not exist. Numerical solutions, though we think "inexact", is still possible.

And so, we would not accept approximate solutions by pressing the calculator buttons. $\sqrt{2}$ is $\sqrt{2}$, it is different from 1.414 ..., since the "..." never ends. The situation would be similar for

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

while answers like $\sqrt{2}$, $\sin(2^\circ)$, $\log 2$, e^2 are acceptable. However , what is your beautifully dressed e? In actual fact, it is nothing but

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots,$$

another series without an end. Similarly, $\sqrt{2}$, $\sin(2^\circ)$, $\log 2$, ... cannot be defined without using infinite series or even more complicated methods. In this aspect, e^2, $\sqrt{2}$ provides no better answer than $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ nor 1.414... In actual practice, what we mean by an answer is one expressed in the simplest algebraic functions. However, controversies did arise among mathematicians about what should be taken as an (mathematical) answer.

There was one school (intuitionism) which only accept the natural numbers (1,2,3,...) together with any other thing expressed in the natural numbers by a finite number of steps (e.g. the fractions). So, they did not accept the irrational numbers as a mathematical existence, since they can only be defined in the natural numbers through an infinite procedure (by taking limits, for example). And so, to them, $\sqrt{2}$, e^2, ... are all undefined objects (may be that's the reason why there are not many intuitionists nowadays). When Lindemann proved that π is a transcendental number (hence an irrational number), Kronecker wrote him "Of what use is your beautiful investigation regarding π ? Why study such problems, since irrational numbers do not exist."

While another school (Formalism) holds that logic (and the language of sets) is the only foundation of mathematics. All of mathematics should be laid upon sets. Thence, an ordered pair (x,y) is defined as

$$\{\{x,y\},\{x\}\},$$

and 0,1,2, ... as the sets $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, ...

Next time, when you ask a question, make sure what answer you expect!

Basic Numeracy of Form Three Pupils

A Survey undertaken by the Mathematics Department, Grantham College of Education

I. Introduction

In April 1984, in the course of teaching practice, the Mathematics Department of Grantham College of Education conducted a survey of basic numeracy of Form 3 pupils in Hong Kong. A total of 1417 pupils from 39 government/aided secondary and technical schools on Hong Kong Island (HK), in Kowloon (Kln) and the New Territories (NT) participated in a numeracy test which formed the basis of the survey. This report presents the results of the survey, the purpose of which is to provide quantitative evidence on the level of attainments in simple arithmetical skills of pupils at the end of nine years of compulsory education. Some of these pupils will take up employment or vocational training whereas the majority will proceed to senior secondary education. In both cases, the basic arithmetical skills they have acquired will be needed in their further studies or everyday life.

II. Purpose and objectives of the survey

(1) Purpose

The system of nine years of compulsory education (from Primary 1 to Form 3) was implemented in 1978, setting up a milestone in education in Hong Kong. At the end of the 9th year, approximately 60% of the pupils will be allocated subsidized places in Form 4 in grammar or technical schools to continue their study while the remaining 40% will enter into various jobs to earn their living and will need the basic arithmetical skills relevant to everyday life.

One of the aims of the mathematics department in a college of education in Hong Kong is to train students to be competent mathematics teachers in primary and lower secondary classes. It is hoped that the results of the test will reflect the basic numeracy of Form 3 pupils and the survey may serve as guidelines for mathematics curriculum planning in future.

(2) Objectives

The objectives are: firstly, to show the likely performance of basic numeracy of Form 3 pupils in Hong Kong and secondly, to compare the results of performance on the first six questions of Form 3 pupils in Hong Kong with those of Form 5 pupils in the U.K.

III. Research Methodology

(1) The Sample

The sample of 1417 pupils who sat for the test represents approximately 2 percent of the total Form 3 population in Hong Kong (see Tables I, II and III). The pupils in the various areas are selected at random and it is reasonable to assume that they are representative groups from schools in both urban and rural areas.

Table I

Distribution of Form 3 pupils in Hong Kong in 1983

Area	Boys	Girls	Total
HK	9511	9783	19294
KLN	20053	19957	40010
NT	9819	10456	20275
TOTAL	39383	40196	79579

Table II

Distribution of participating pupils

Area	Boys	Girls	Total
HK	150	181	331
KLN	380	384	764
NT	155	167	322
TOTAL	685	732	1417

Table III

Distribution of participating schools

		HK	KLN	NT
Technical or Prevocational Schools	English :	3	1	1
	Chinese :	0	1	0
Grammar Schools	English :	3	17	7
	Chinese :	3	2	1
Total		9	21	9

(2) The Test Paper

The test paper, shown in Appendix I, is more or less adopted from "A Pilot Test of Basic Numeracy of Fourth-year and Fifth-year Secondary Pupils undertaken by the Institute of Mathematics and Its Applications" published in the U.K. in April 1978 devised by a small committee of experienced school teachers of Mathematics. The first six questions, each consisting of two parts, are the same as that of the U.K. test. In the mechanical manipulation, only the simplest arithmetical skills are involved with which the pupils would have been familiar in primary schools. With regard to the rest of the questions in the U.K. test, those relevant to the experience of Hong Kong pupils are retained while the others are reframed so as to apply simple arithmetical skills to everyday life situations in Hong Kong.

It can be assumed that basically Form 3 pupils who are about to complete nine years of compulsory education should be able to tackle such simple problems. It is also realized that the pupils are liable to make occasional slips in arithmetical calculations even though a question is fully understood and since the test is only concerned with the most basic skills, an average pupil is expected to obtain scores in the 80-100 percent range.

It is worth mentioning that the questions are set to measure how far the overall objectives of the two Mathematics syllabuses, A and B for the first three forms, recommended by the Curriculum Development Committee in Hong Kong in 1982, are being achieved. These objectives are :

(a) To continue the development of numeracy begun in the primary school - this includes much of the number work and ability to cope with approximations, percentages, rates and ratios and simple mensuration.

(b) To prepare students to understand everyday applications outside the classroom - for example, by teaching the fundamentals of statistics and probability.

The questions are weighted according to their level of difficulty as shown in Appendix II. The maximum time allowed for the test is 40 minutes.

IV. Results

The test was held on 13th March 1984 in a Mathematics lesson arranged by the participating schools. The results obtained are shown in the following tables :

Table IV

Number and percentage of pupils scoring (a) 100 percent on whole paper and (b) 100 percent on questions 1 to 6

	(a) Full marks on whole paper						(b) Questions 1-6 all correct					
	Boys		Girls		Total		Boys		Girls		Total	
	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%
HK	0	0	1	0.6	1	0.3	112	75	133	73	245	74
KLN	4	1.1	8	2.1	12	1.6	295	78	312	81	607	79
NT	0	0	1	0.6	1	0.3	118	76	136	81	254	79
TOTAL	4	0.7	10	1.4	14	1.0	525	77	581	79	1106	78

Table V

Number and percentage of pupils answering all questions correctly excluding question 17

	No. of pupils	%
HK	45	14
KLN	82	11
NT	25	8
TOTAL	152	11

Table VI

Percentage of pupils scoring (a) 90% or better and (b) 80% or better

	(a) 90% - 100%			(b) 80% - 100%		
	Boys %	Girls %	Total %	Boys %	Girls %	Total %
HK	41	44	43	83	71	77
KLN	47	39	43	77	69	73
NT	37	31	34	86	64	75
TOTAL	44	39	41	80	68	74

Table VII

Percentage of wrong answers to selected questions

Questions		HK %	KLN %	NT %	Total %
7b	percentage into decimal	13	12	12	12
10	writing a number in figures	13	20	18	18
11	conversion of units	27	32	37	32
13	train timetable	56	53	63	56
14	wages	33	37	36	36
17	postage	97	93	96	94
18a	currency change	16	12	14	14

Table VIII

Number and percentage answering questions 1 to 6 all correctly in Hong Kong and in the U.K. (Total no. of pupils tested were 8427)

Places	Boys		Girls		Total	
	No.	%	No.	%	No.	%
HK	525	77	581	79	1106	78
*UK	1377	38	1359	40	2736	38

* see references

V. Discussion and Conclusion

(1) The results shown in Table IV part (b), Tables V and VI are quite satisfactory. In Table IV part (a), the difference between the percentage of boys and girls scoring full marks is of little significance. A high percentage of full marks is not expected in this criterion-referenced test.

(2) Table IV part (b) reveals that the total percentages of pupils who have attempted questions 1-6 all correctly in the three areas are quite homogeneous. The figures also suggest that in general the ability of mechanical calculations of girls in Form 3 in Hong Kong is very close to that of boys. The difference in performance is statistically insignificant.

(3) It is interesting to note that the percentage of pupils answering all questions correctly, excluding Question 17, totals 11 percent as shown in Table V.

(4) The figures in Table VI reflect a tendency that boys perform better than girls in applying basic arithmetical skills in everyday situations. This is possibly explained by the sex difference in Mathematics learning. The total percentage of pupils scoring 80% or better once again gives strong support that the results of the test of the pupils from the three areas are homogeneous. It indicates that the selection of pupils from these three areas bears little significance. The participating pupils are thus seen to form a representative sample of Form 3 pupils in Hong Kong.

(5) It is encouraging to note that approximately three quarters of Form 3 pupils who took part in the survey scored over 80%.

(6) Table VII shows the percentage (considerably greater than 10%) of wrong answers to selected questions. It reflects the following :

(a) It is easier for pupils to express a decimal into percentage than the other way round and it reveals that the concept of percentage may not be well understood. However, such conversion which facilitates calculations has now become unnecessary with the introduction of pocket-size calculators.

(b) Some pupils have difficulty in solving Questions 11 and 18a. The likely explanation is that, in spite of the large amount of exercises in conversions in textbooks, practical examples from everyday applications in Hong Kong are still not sufficient.

(c) It is seen from Question 10 that Form 3 pupils are not familiar with writing numbers in figures. Of course, they seldom deal with the writing of cheques.

(d) In Question 13, it shows that Form 3 pupils are possibly not so strong in "train time-table reading". Such skill is not often practised in Hong Kong. The small area and its well-developed traffic system have rendered such skill non-essential.

(e) Posters of employment with conditions similar to Question 14 are often seen on factory buildings, but they hardly attract the attention of Form 3 pupils as factory work is not within their experience. Examples or exercises of this kind are seldom found in the textbooks.

(f) Question 17 resembles a real-life situation. It is a pity that the difference in postage is so small that pupils tend to overlook their values. In fact, the aim of setting such a question is not to test whether the pupils know how to post a pile of posters if they are given sufficient money, but to find out whether the pupils can read and interpret a question carefully. Reading a question carelessly, thus overlooking key words, is a common mistake made by many pupils especially when the question is placed among seemingly easier ones. Whether Question 17 is a straight forward question that requires only table-reading skill or one involving some analytical thinking in order to get to the solution depends very much on whether or not the pupils can catch the key word 'least'. Furthermore, the wording in Question 17 is too long. Pupils sitting the test also lack the incentive to attain full marks.

(7) In comparing the results on Questions 1 to 6 in the two tests, one held in Hong Kong and the other in the U.K., the figures in Table VIII show that 78 percent of Form 3 pupils of age 15 in Hong Kong obtain full marks as against 38 percent of Form 5 pupils of age 16 in the U.K. with the same score in 1977. This provides some interest. The plausible explanation of the marked difference may be that Form 3 pupils in Hong Kong are working harder so as to fare better in the very competitive Junior Secondary Education Assessment in order to be allocated subsidized Form 4 places.

VI. Recommendations

On the whole, the pupils find it difficult to deal with questions which appear unfamiliar to them although the numeracy skill required is only basic. Many of them are capable of solving more difficult problems in textbooks but are at a loss when they come across easier ones with which they are not familiar. It indicates that they are equipped with basic manipulation techniques but rather weak in application. Thus, teachers should not only drill pupils basic arithmetical skills but also pay more attention to their applications.

It is recommended that in the curriculum for training mathematics teachers, more emphasis be laid on "Process-oriented Learning" in the hope that they will in turn help their pupils develop analytic power and logical reasoning, thus enabling the pupils to apply mechanical calculations to solve problems in everyday situations.

The appropriate applications of "Learning Theories" in mathematics teaching are essential. Successful learning depends very much on applying a concept or a technique to deal with circumstances in everyday life. It is hoped that from time to time, relevant surveys should be conducted to ascertain the likely performance of basic numeracy of pupils. The quantitative evidence may guide the curriculum planners, setters of public examination papers, textbook writers and teachers to take into consideration new materials in the ever changing world.

Acknowledgement

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Appendix I

Please put down all answers in the answer boxes. Thank you for your cooperation.

請將答案填在空格內。多謝合作。

1. (a) 14 + 35 = Ans:

   (b) 43 + 282 = Ans:

2. (a) 77 − 53 = Ans:

   (b) 911 − 102 = Ans:

3. (a) 7 x 8 = Ans:

   (b) 6 x 79 = Ans:

4. (a) 24 ÷ 6 = Ans:

   (b) 243 ÷ 9 = Ans:

5. (a) 13.3 + 2.8 = Ans:

   (b) 79.3 − 8.1 = Ans:

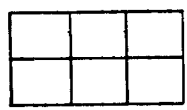
6. (a) 3 x 42.5 = Ans:

(b) 13.5 ÷ 5 = Ans:

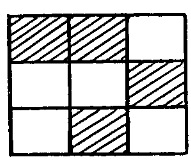
7. (a) Write $\frac{1}{4}$ as a percentage. Ans: %
將 ¼ 寫成百分數

(b) Write 40% as a decimal. Ans:
將 40% 寫成小數

8. (a) Shade in 2/3 of this diagram.
將右圖的⅔加上陰影



(b) What fraction of this diagram is shaded?
下圖的陰影部份佔全圖的幾分之幾？



Ans:

9. A reduction of 20% is given on sale items. What is the sale price of a shirt marked at $80?
一件標價 80 元的恤衫以八折出售，問售價是多少？

Ans: $

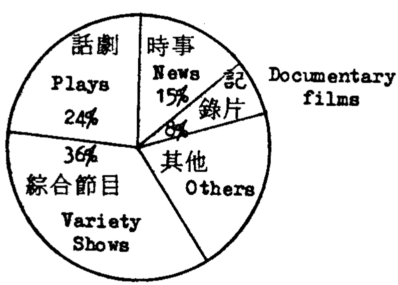
10. A cheque is made out for one hundred and forty-nine dollars and ninety cents. Write this amount in figures.
支票一張，面額壹佰肆拾玖圓玖角正，請以阿拉伯數字寫出該款項。

Ans: $

11. The former price of beef was $13 per lb. Now it is $30 per kg. What is the increase in price in dollars per kg?
(Take 1 kg = 2.2 lb)
牛肉以前一磅售價 13 元，現每千克售價 30 元。問牛肉現在每千克的售價比以前貴多少元？（設 1 千克 = 2.2 磅）

Ans: $

12. The following diagram shows the result of an opinion poll on the most favourite programmes on a TV channel.
下圖是某電視台各類節目受歡迎程度的統計圖



(a) What percentage is the "others"?
問「其他」節目佔百分之幾？

Ans: %

(b) If 4000 people were consulted altogether, how many chose "variety shows" as their most favourite programme?
如統計資料是由 4000 人提供，問多少人選擇「綜合節目」爲最受歡迎的節目？

Ans:

13. The timetable at Hunghom Railway Station in one afternoon is as follows :
某下午紅磡火車站的行車時間表如下：

Terminus 終站		Departure time 開出時間
Shatin 沙田	all stations 停各站	1400
Sheung Shui 上水	all stations 停各站	1410
Lo Wu 羅湖	——	1420
Shatin 沙田	all stations 停各站	1430
Tai Po Market 大埔墟	all stations 停各站	1440

A group of students have booked the 3 p.m. tennis court at Jubilee Stadium in Shatin. They set out at Hunghom Railway Station. If it takes approximately 35 minutes to reach the Jubilee Stadium from Hunghom Railway Station, what is the latest train that they can take?
一班學生訂了沙田銀禧體育館下午三時的網球場時間，並相約在紅磡火車站集合•如從紅磡火車站至銀禧體育館需時約 35 分鐘，問他們最遲可乘那一班車？

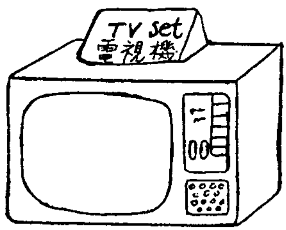
Ans :

14. The daily wage of a worker is $100. A bonus of $30 will be awarded to anyone who works 5 days or more per week. A sum of $25 per hour is paid for overtime work. If a worker's record in a certain month is as shown, how much will he receive at the end of the month?
工人日薪 $100，如一星期工作五天或以上者則獲 $30 勤工獎。
超時工作以每小時 $25 計算•
若一工人某月的工作紀錄如下，問他該月應得薪金多少？

Record of Work 工作紀錄		
	Working days 工作(日)	Overtime hours 超時(小時)
First Week 第一週	5	—
Second Week 第二週	4	—
Third Week 第三週	5	2
Fourth Week 第四週	6	4

Ans : $

15. A TV set is on sale under the following conditions.
一部電視機以下列方式出售。



Cash Price $2000
現金支付

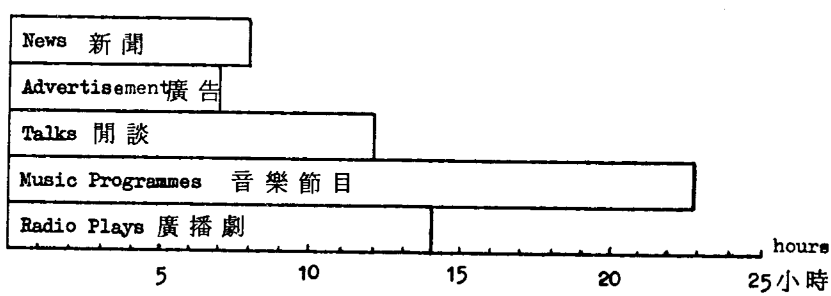
Deposit $—
免首期

Superterms $220 12 months
特惠分期 十二個月

How much more does one pay by instalments than by cash?
問分期付款比現金購買需多付幾元？

Ans: $

16. The number of hours given to various types of programmes in one week, on a radio channel, is as follows :
某電台在一星期內編排的節目時間如下：



(a) How many hours are given to music programmes?
音樂節目佔幾多小時？

Ans: hours 小時

(b) To what type of programme is the least time devoted?
那項節目佔最少時間？

Ans:

(c) What is the total number of broadcasting hours, in one week, of that radio channel?
該電台一星期的廣播時間共有幾多小時？

Ans: [hours 小時]

17. Postage Rates for Surface Mail 平郵收費表

	letters & postcards 信件及明信片		printed papers 印刷品	
weight not over 重量不超過	China 中國 Macau 澳門 Taiwan 台灣 $	other coun- tries 其他國家	China 中國 Macau 澳門 Taiwan 台灣 $	other coun- tries 其他國家
20 g 二十克	0.50	1.00	0.40	0.90
50 g 五十克	1.00	1.50	0.60	1.30
100 g 一百克	1.90	2.40	1.00	2.00
250 g 二百五十克	4.30	4.80	1.80	3.60
500 g 五百克	8.40	9.20	3.00	6.50
1 kg 一千克	14.40	16.00	5.40	10.80
2 kg 二千克	20.40	26.00	7.20	15.00
each additional kg (Books up to 5 kg) 每加一千克（只限書籍，重量限制：五千克）	—	—	3.60	7.50

A pile of posters weighing almost 700 grams is sent to China after suitable packing. What is the least delivery charge?
將總重量接近 700 克的海報適當包裝後寄去中國，問最低郵費是多少？

Ans: [$]

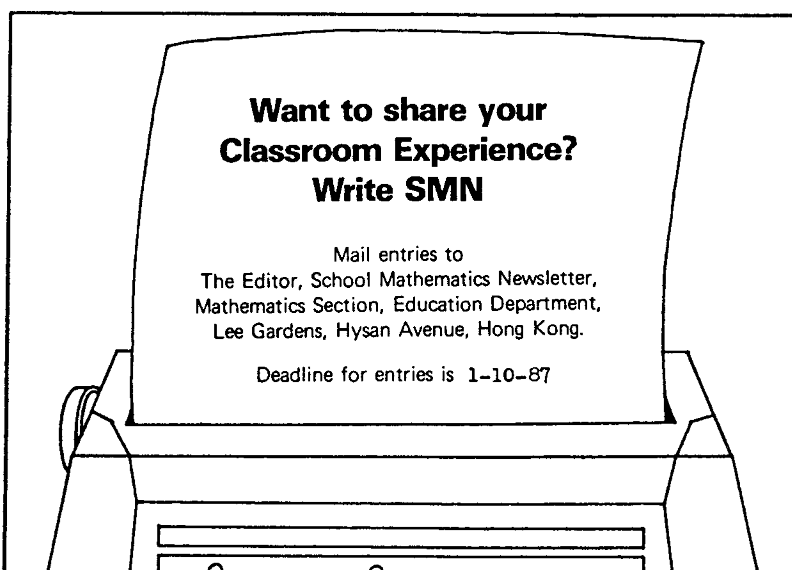
18. The exchange rate of HK dollars vs. US dollars on a certain day is HK$ 100 to US$ 13.
某日港幣兌美元的價格爲 100 港元兌 13 美元。

(a) What is the equivalent sum in US$ for HK$2500?
2500 元港幣可換多少美元？

Ans: US$ ______

(b) What is the equivalent sum in HK$ for US$390?
390 美元相當於多少港元？

Ans: HK$ ______



Appendix II

Answers and allocation of marks

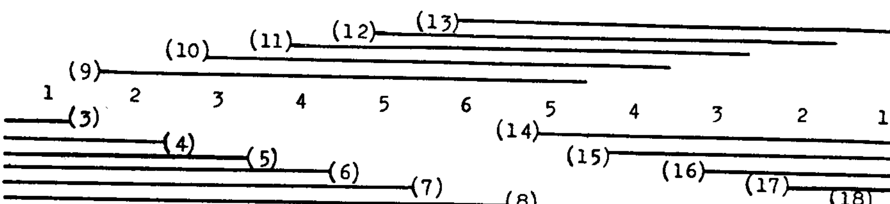
Item No.	Answers	Marks Awarded
1a	49	1
1b	325	1
2a	24	1
2b	809	1
3a	56	1
3b	474	1
4a	4	1
4b	27	1
5a	16.1	1
5b	71.2	1
6a	127.5	1
6b	2.7	1
7a	25	1
7b	0.4	1
8a	any correct answer	1
8b	4/9	1
9	64	2
10	149.90	1
11	1.4	2
12a	17	2
12b	1440	2
13	1410 or Sheung Shui	1
14	2240	2
15	640	2
16a	23	1
16b	advertisement	1
16c	64	2
17	4.80	2
18a	325	2
18b	3000	2
	Total =	40

Method for finding the probability of a certain sum in a throw of three dice

George Lui

Chan Shu Kui Memorial School

Rule and examples



Rule. If it is required to find the probability of obtaining a certain sum X in a throw of 3 dice, just find the line marked (X) and add up all the numbers above or below that line. Then divide the sum by 216, which is the total number of possible outcomes in a throw of 3 dice. The result will be the required probability.

Example 1. Find the probability of obtaining a sum of 10 in a throw of 3 dice.

Method. Sum of all numbers below line (10) = 3+4+5+6+5+4 = 27

Therefore, $P_{(\text{sum} = 10)} = \frac{27}{216} = \frac{1}{8}$

Example 2. What is the probability of obtaining a sum greater than 15 in a throw of 3 dice?

Method. Total number of cases in which the event of getting a sum greater than 15 occurs

= (3+2+1) —— numbers above line (16)

+ (2+1) —— numbers above line (17)

+ (1) —— number above line (18)

= 10

Therefore, $P_{(\text{sum greater than 15 in a throw of 3 dice})} = \frac{10}{216} = \frac{5}{108}$

This rule can be extended for finding the probability of obtaining a certain sum in a throw of 4 dice.

Example 3. Find $P_{(\text{sum} = 15 \text{ in a throw of 4 dice})}$

Method.

If the number cast out in a certain die is	then the total sum of the other 3 dice is	Total number of possible cases
1	15-1 = 14	5+4+3+2+1 = 15
2	15-2 = 13	6+5+4+3+2+1 = 21
3	15-3 = 12	5+6+5+4+3+2 = 25
4	15-4 = 11	4+5+6+5+4+3 = 27
5	15-5 = 10	3+4+5+6+5+4 = 27
6	15-6 = 9	2+3+4+5+6+5 = 25

Therefore, $P_{(\text{sum} = 15 \text{ in a throw of 4 dice})} = \frac{15+21+25+27+27+25}{6x6x6x6}$

$= \frac{35}{324}$

Example 4. Find $P_{(\text{sum} = 20 \text{ in a throw of 4 dice})}$

Method.

If the number cast out in a certain die is	then the total sum of the other 3 dice is	Total number of possible cases
1	20-1 = 19 impossible	———
2	20-2 = 18	1
3	20-3 = 17	2+1 = 3
4	20-4 = 16	3+2+1 = 6
5	20-5 = 15	4+3+2+1 = 10
6	20-6 = 14	5+4+3+2+1 = 15

Therefore, $P_{(\text{sum} = 20 \text{ in a throw of } 4 \text{ dice})} = \frac{1+3+6+10+15}{6x6x6x6} = \frac{35}{1296}$

<u>The discovery and derivation of the rule</u>

no. on 2nd die

no. on 1st die	sum	1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

L

In the case of 2 dice, we can easily find the probability of obtaining a certain sum in a throw by means of the adjoining table. For example, the probability of obtaining a sum of 4 in a throw of 2 dice $= \frac{3}{36} = \frac{1}{12}$ where 3 is the number of cases in which a sum of 4 is obtained and 36 is the total number of possible outcomes.
In the table, we notice an outstanding feature. All the cases in which the sum (e.g. 8) is the same lie on the same sloping line (e.g. L). This inspiration brings up a riddle. Will there be a similar feature in the case of 3 dice? A close exploration gives the answer 'yes'. If three mutually perpendicular lines OA, OB and OC are taken to represent the numbers on the 3 dice, and the points in space (on or within the

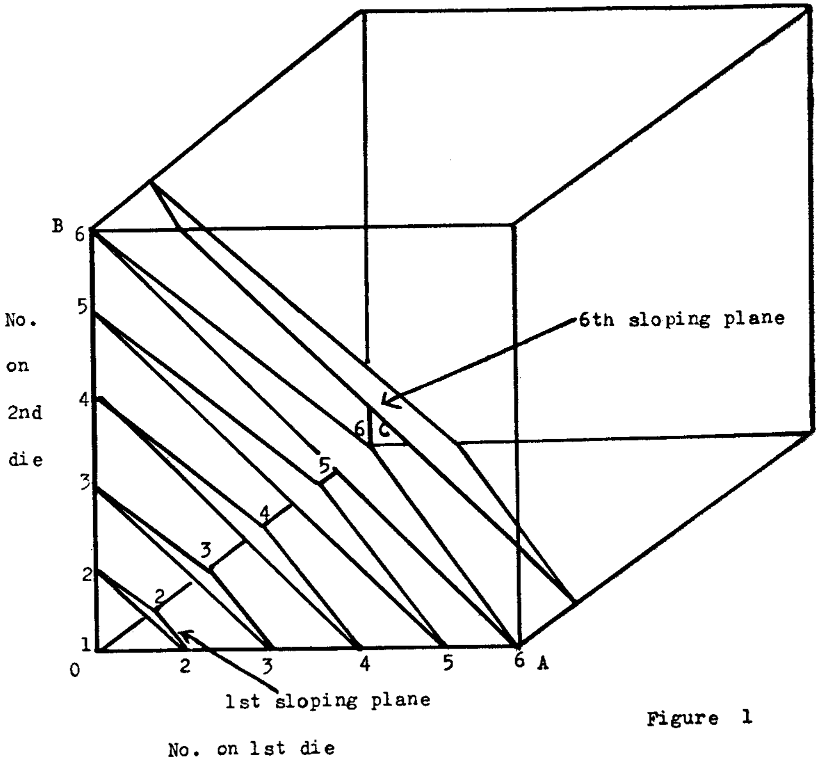


Figure 1

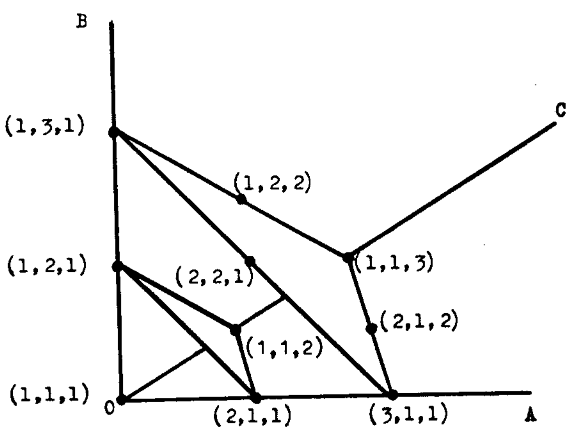
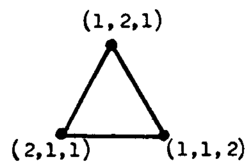


Figure 2

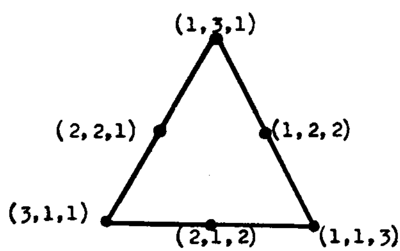
cube), to represent the different possible outcomes, then it can easily be seen that all the outcomes (or rather the points) which bear the same sum lie on the same sloping plane (Figure 1). Figure 2 is the enlargement of a part of Figure 1.

The possible outcomes in the sloping planes display the following patterns.

a point — No. of outcomes = 1 (sum=3)

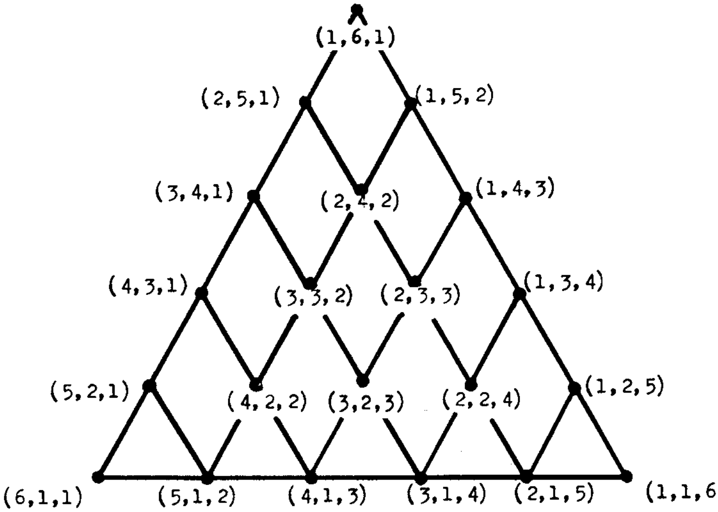


1st sloping plane — No. of outcomes = 1+2 = 3 (sum=4)

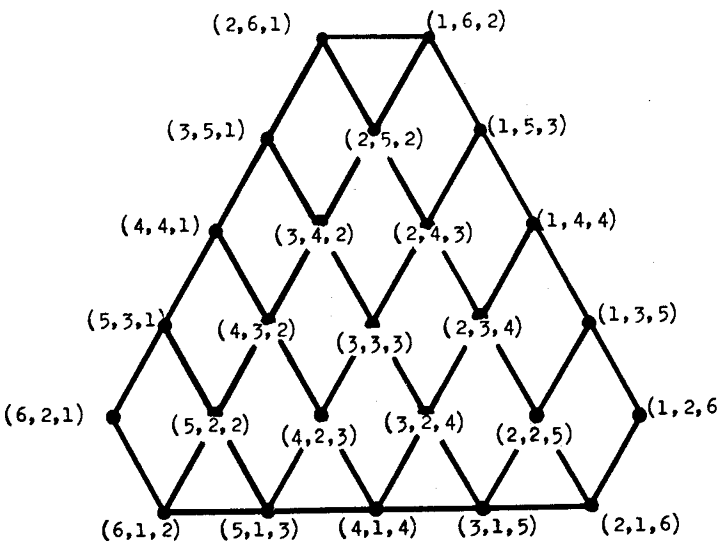


2nd sloping plane — No. of outcomes = 1+2+3 = 6 (sum=5)

⋮



5th sloping plane   No. of outcomes (sum=8)
= 1+2+3+4+5+6 = 21



6th sloping plane   No. of outcomes (sum=9)
= 2+3+4+5+6+5 = 25

and so on.

All these results conform to the rule and that is how the rule was derived.

The fact that all the points representing the possible outcomes (in which the sum is the same) lie in the same sloping plane can easily be verified as follows :

Take 216 cubes and mark them (1,1,1), (1,1,2), (1,1,3),...., (1,2,1), (1,2,2),..., (6,6,6). Then try to build up the big cube in Figure 3 step by step as shown in Figure 4. It will be seen that the cubes with the same total lie in the same sloping plane.

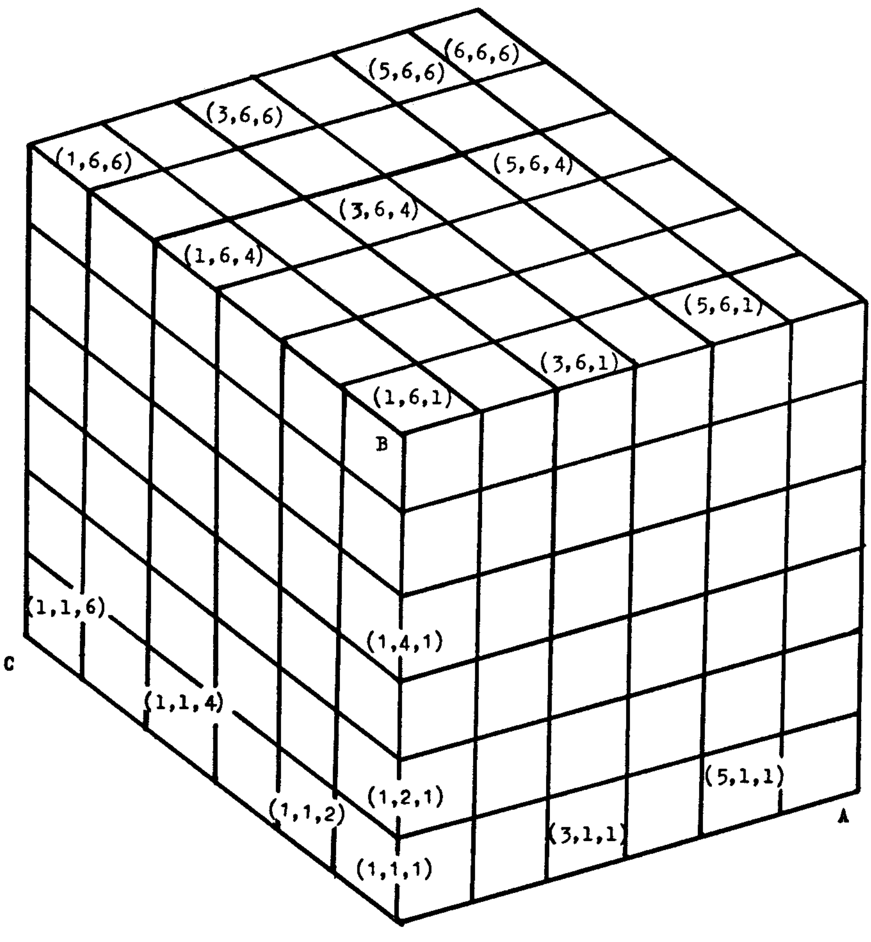
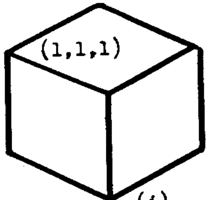
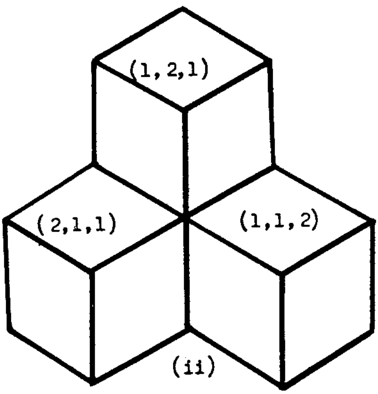


Figure 3 (For simplicity, only some of the small cubes are labelled.)

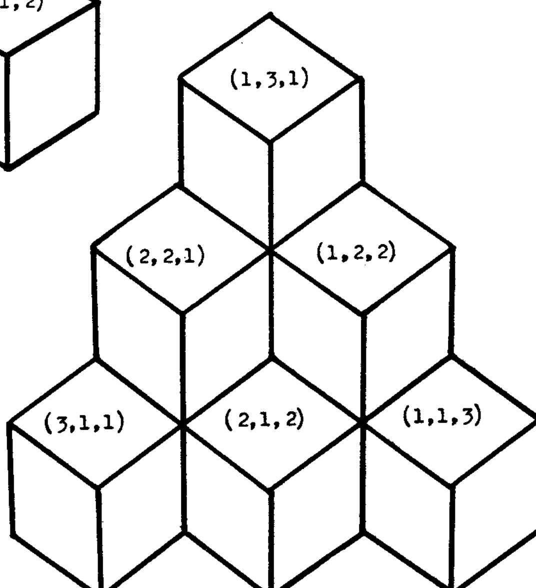
Figure 4



(i)



(ii)



(iii)

Some notes on School Statistics

Wong Ngai Ying

Wong Tai Shan Memorial College

Discrete, Grouped and Continuous Data

The distinction between discrete and continuous distributions is very important in statistics. Since the concept of continuous frequency distribution is not introduced in secondary school statistics, what replaces is the notion of "grouped data". Let us look at the following two examples.

Example 1.

The number of patients suffering from a certain disease is recorded among 20 countries. The data is collected in the following distribution table.

No. of patients	No. of countries having that no. of patients
0 - 9	1
10 - 19	4
20 - 29	7
30 - 39	5
40 - 49	3

Example 2.

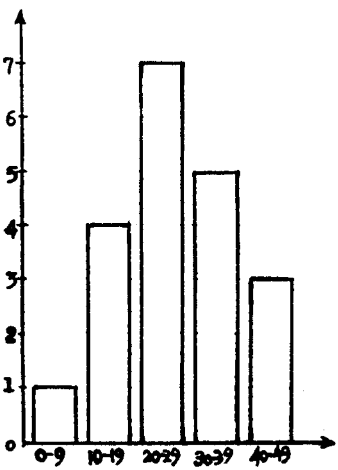
The height of the students in a class of 40 is recorded as follows :

Height (cm)	No. of students
150 - 154	2
155 - 159	6
160 - 165	9
165 - 169	12
170 - 174	8
175 - 179	3

If one looks at the examples in detail, one can see that there is actually a difference between these two sets of data. Obviously, the second is a continuous one since the heights are already corrected to the nearest cm and the class 150 - 154, for instance, actually includes heights from 149.5 to 154.5. That's why, when we are going to represent it in a histogram, we have to convert the above table into another one first.

Height (cm)	No. of students
149.5 - 154.5	2
154.5 - 159.5	6
159.5 - 164.5	9
164.5 - 169.5	12
169.5 - 174.5	8
174.5 - 179.5	3

For the first example, though the data are grouped, they are actually discrete. There cannot be a country having, say, 9.5 patients. It is therefore, unsuitable to put the data in a histogram, frequency polygon or cumulative frequency polygon. A bar chart is more appropriate.



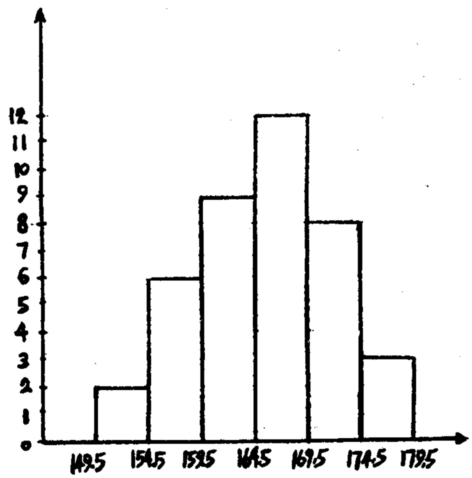


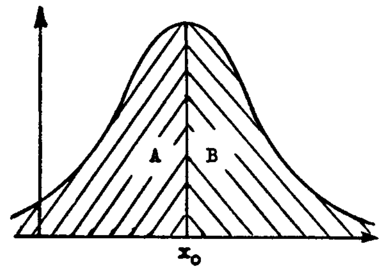
Fig. 1 A bar chart is more suitable for example 1, since there is no real meaning for data between 9 - 10, etc.; whereas a histogram is more suitable for example 2.

N/2 or (N+1)/2 for Medians

Since there is a distinction between discrete and continuous data, there is also a difference in calculating the medians of these two types of data. For discrete data, naturally, the median is the $(N+1)/2$th datum, and for continuous distribution, the median is the reading x_0 so that the areas on the two sides of x_0 are equal, but for grouped data, such as the one in

Example 2, it is generally agreed that the median should be the $\frac{N}{2}$ th datum. There are many interpretations to this choice, anyway, it is not due to "convenience" as said in some books. Let us investigate the case in this way. In order to find the median of the distribution in example 2, we consider the histogram in Fig. 1, and pick out the so-called 20 th datum. There are 17 data from 149.5 to 164.5, so, we want three more from 164.5 - 169.5. Hence, the median is

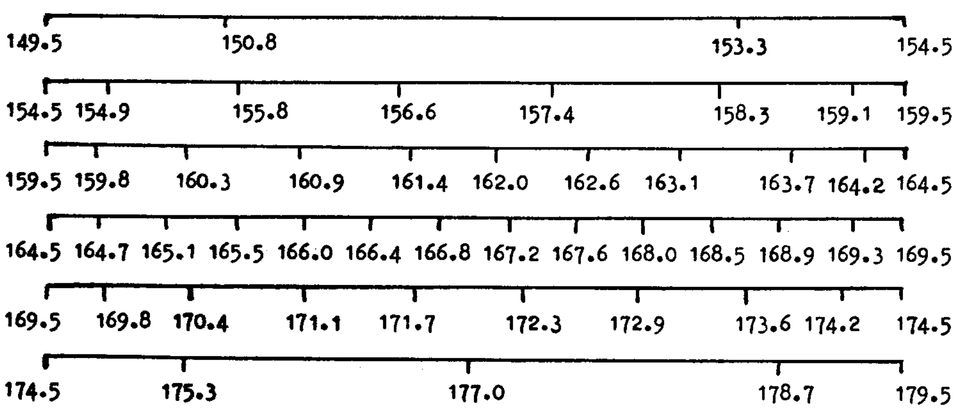
Fig. 2(a)



$$164.5 + \frac{3}{12}(169.5 - 164.5) = 165.75 \text{ cm}$$

Fig. 2(b)

In this method, we look at the histogram as a continuous distribution. However, we can look at it in another way. We can split the data into a discrete pattern, since the number of data is, in fact, finite. Let us assume that the first two data are evenly distributed in the interval 149.5 - 154.5, the next 6 data evenly distributed in 154.5 - 159.5, etc., i.e.



Since the data are now discrete, the median is the (40+1)/2 th one, i.e.

$$\frac{165.5 + 166.0}{2}$$

$$= 165.75 \text{ cm}$$

In general, for the following distribution

class	frequency
a — a + d	n_1
a + d — a + 2d	n_2
⋮	⋮
a+(m-1)d — a + md	n_m

with $N = \sum_{r=1}^{m} n_r$, if the median happens in the $(a+pd, a+\overline{p+1}d)$ class, then

$$\sum_{r=1}^{p} n_r \leqslant N/2 < \sum_{r=1}^{p+1} n_r .$$

If we let $k = N/2 - \sum_{r=1}^{p} n_r$, the median would be, by our usual method of using a histogram,

$$a + pd + \frac{kd}{n_{p+1}} .$$

If we split the data in the $(a+pd, a+\overline{p+1}d)$ class, assuming that they lie evenly in the interval, they are :

$$a+pd + \frac{1}{2}\frac{d}{n_{p+1}},\ a+pd+\frac{3d}{2n_{p+1}},\ \ldots,$$

the rth term being $a + pd + \frac{(2r-1)d}{2n_{p+1}}$ & the median being the $(k+\frac{1}{2})$ th term from a+pd, hence, it is

$$a+pd+\frac{(2(k+\frac{1}{2})-1)d}{2n_{p+1}} ,$$

coinciding with the above one.

From this proof, one can see that, in this view-point, the so-called $\frac{N}{2}$ th term for the median in grouped data is actually the $\frac{N+1}{2}$ th term, coinciding with what we have learned for discrete cases.

PASTIMES

1. By adding one letter to each of the following words, it can be changed into a word of mathematical significance. What are the words? CUE, FIST, PINT, QUALITY, RAGE.

2. There is a cake which is square and iced across the top and down the sides. It is to be divided in such a way that each person gets exactly the same amount of cake and icing. Now there are nine persons, how should we divide the cake?

3. Solve the following systems of equations and use the given graph to help you decode the name of a mathematican who lived from 1777 to 1855 :

<u>(1)</u> <u>(2)</u> <u>(3)</u> <u>(4)</u> <u>(5)</u> <u>(6)</u> <u>(7)</u> <u>(8)</u> <u>(9)</u> <u>(10)</u>

(Any point on the graph that has not been assigned a letter represents a blank.)

(1) $\begin{cases} x + y = 1 \\ 2x - y = -7 \end{cases}$

(2) $\begin{cases} 3x - y = 5 \\ 2x + 2y = 6 \end{cases}$

(3) $\begin{cases} 2x - 3y = 3 \\ x + 2y = 5 \end{cases}$

(4) $\begin{cases} 4x + 2y = 0 \\ x + y = -1 \end{cases}$

(5) $\begin{cases} x - 2y = -1 \\ x + y = 2 \end{cases}$

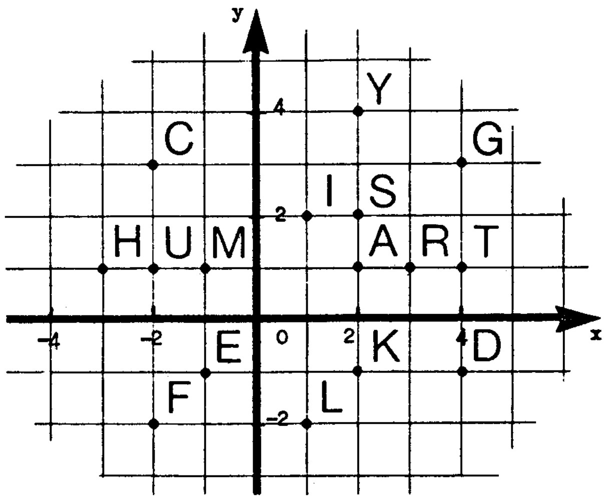
(6) $\begin{cases} x - 2y = -2 \\ 2x - y = 5 \end{cases}$

(7) $\begin{cases} 5x - y = 9 \\ 2x - y = 3 \end{cases}$

(8) $\begin{cases} 2x + 3y = -1 \\ x + 2y = 0 \end{cases}$

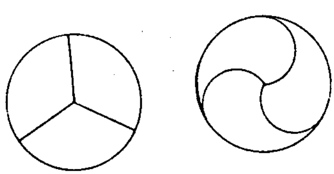
(9) $\begin{cases} x + y = 4 \\ 2x + y = 6 \end{cases}$

(10) $\begin{cases} 2x + 3y = 10 \\ x + 2y = 6 \end{cases}$



4. If $\textcircled{3} = 47$, $\textcircled{10} = 138$ and $\textcircled{1} = 39$, and if $\boxed{1} = 5$, $\boxed{20} = 43$ and $\boxed{99} = 201$, then find n so that $\boxed{\textcircled{n}} = \textcircled{\boxed{n}}$.

5.



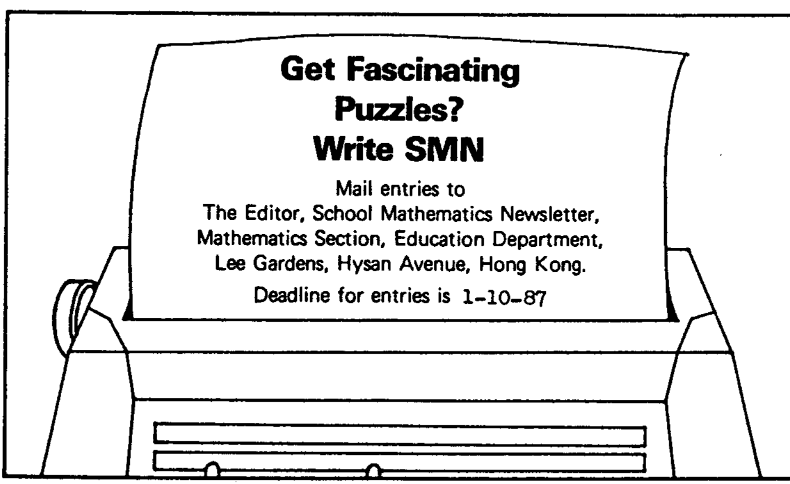
The above diagrams show two methods of dividing a circle into three equal parts, with the same perimeter and the same area; the boundary lines being 3 radii and 3 semicircles respectively. Can you trisect a circle so that each part has the same area and a perimeter equal to the circumference of the circle?

6. Work out $7^1, 7^2, 7^3, 7^4, 7^5, 7^6, \ldots$

(a) What are the patterns of the last digits and the last but one digits of these numbers?

(b) Do the patterns continue if higher powers of 7 are calculated?

(c) If calculator is used in the calculation, what is the constraint in preserving the above patterns?



ANSWERS TO PASTIMES

1. CUBE, FIRST, POINT, EQUALITY, RANGE

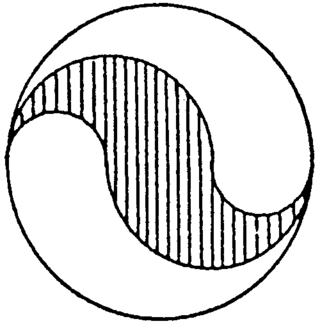
2. The square iced cake can be equitably shared among nine persons by marking nine points, separated by equal distances along its perimeter, and slicing from the points toward the centre.

3. Solving $\begin{cases} x + y = 1 \\ 2x - y = -7 \end{cases}$, the solution is (-2, 3)

which corresponds to the letter C. Similarly, by solving the simultaneous equations, the name of the mathematican is CARL GAUSS.

4. $\textcircled{n} = n^2 + 38$, $\boxed{n} = 2n + 3$ $\therefore$ $n = -8$ or 2

5. Divide a diameter into 3 equal parts and draw semicircles.



6. (a) 7931, 4400

(b) Yes

(c) Not to exceed the number of digits that the calculator can handle.

DO YOU KNOW ?

CDC Syllabus for Mathematics (Forms I-V), 1985

This syllabus is an amalgamation of, and replaces, the two mathematics syllabuses A and B issued to schools in 1982 and has been implemented since September 1986 starting with Forms I and IV. It serves as a teaching guide which will lead to the Hong Kong Certificate of Education mathematics examination syllabus for 1988 and onwards issued by the Hong Kong Examinations Authority.

Glossary of Terms Commonly Used in Secondary School Mathematics

This glossary provides Chinese translations of those English terms commonly used in the teaching of Mathematics in secondary schools and is intended to facilitate the wider use of Chinese as the medium of instruction. The first draft of this glossary was issued to all secondary schools in November 1986.

Hong Kong Mathematics Olympiad

The Heat Event of the Fourth Hong Kong Mathematics Olympiad was held on 20 December 1986. Hundred and eighty-three secondary schools participated in the competition and the results were encouraging. The Final Event was scheduled to take place on 14 February 1987. Also, a poster design competition would be launched to promote the Fifth Hong Kong Mathematics Olympiad. Secondary schools interested in the Fourth Hong Kong Mathematics Olympiad would be invited to participate in the poster design competition in March 1987.

The International Mathematical Olympiad in Australia, July 1988

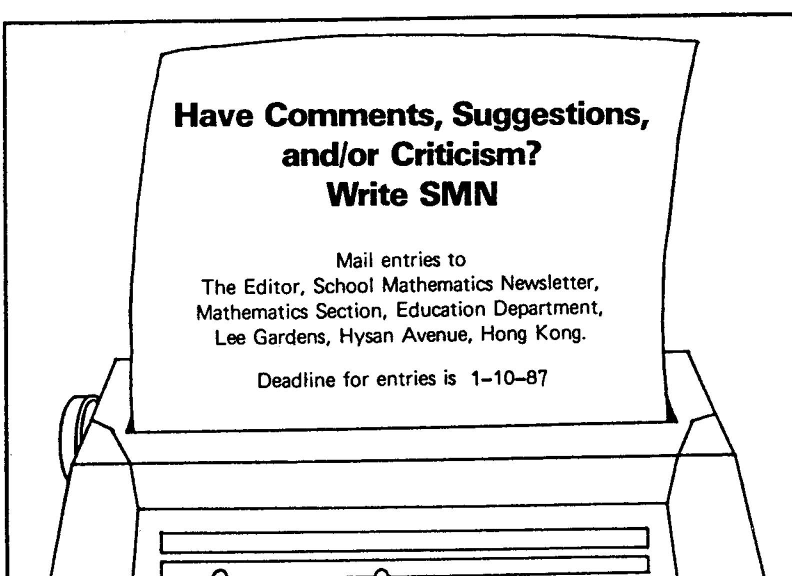
The Hong Kong Mathematical Society, in order to select students to represent Hong Kong in the captioned competition, is organizing an International Mathematical Olympiad Selection Contest to be held in July 1987. The contest, besides identifying students of outstanding mathematical abilities, will help promote students' interest in mathematics. A circular about the Selection Contest was issued to all secondary schools in January 1987.

FROM THE EDITOR

I wish to express my sincere thanks to those who have contributed articles and also to those who have helped in the preparation of this issue of SMN.

Readers are cordially invited to send in articles, puzzles, games, cartoons, etc. for the next issue. Your contribution is essential for prompt issue of the SMN. Please write : The Editor, School Mathematics Newsletter, Mathematics Section, Advisory Inspectorate, Education Department, Lee Gardens, Hysan Avenue, Hong Kong.

For information or verbal comments and suggestions, please contact the editor on 5-614364.



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