SCHOOL MATHEMATICS NEWSLETTER

8

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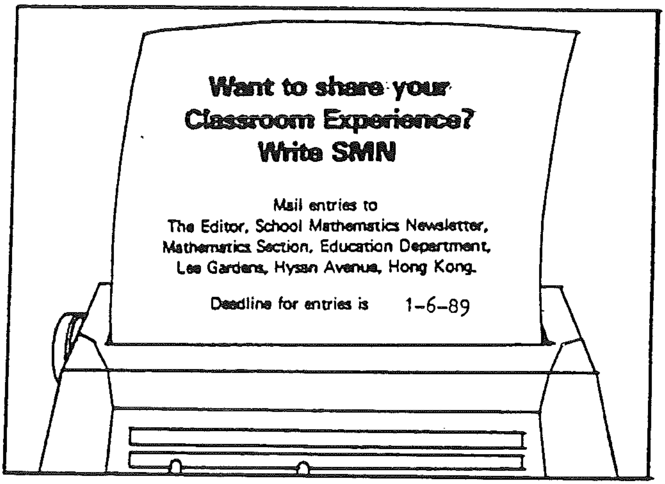
Mathematics Section, Advisory Inspectorate
Education Department, Hong Kong

Please ensure that every member of your mathematics staff has an opportunity to read this Newsletter.

The views expressed in the articles in this Newsletter are not necessarily those of the Education Department, Hong Kong.

IN THIS ISSUE



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The Editor, School Mathematics Newsletter,
Mathematics Section, Education Department,
Lee Gardens, Hysan Avenue, Hong Kong.
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FOREWORD

When the 7th issue of the School Mathematics Newsletter (SMN) was issued in February 1987, it was published in a format different from all the previous issues, i.e. its physical size was reduced from A4 to A5 size and it was produced by reprography instead of by cyclostyle. Experience has shown that the new format is more handy and makes reading easier. The present issue is, therefore, produced in the same format.

In this issue, we have included some extracts from the printed materials distributed in two of our teachers' seminars on Activities for Mathematics in Secondary Schools and on Internal Assessment for Mathematics in Secondary Schools. We have also reproduced an article on Problem Solving as an Art Form from the Graduate magazine of the University of Toronto. And, as usual, some contributions from the mathematics teachers are included, together with some mathematical puzzles. We hope you will again find the contents both interesting and useful.

We are thankful to all those who have contributed to this issue of the SMN. We would like to have even more contributions from the classroom teachers whose support is of vital importance; without their support the SMN will not be as useful as we would like it to be.

Please send your contributions directly to the Mathematics Section of the Advisory Inspectorate. Suggestions and comments on both the layout and contents of the Newsletter are also welcome.

I would like to thank my colleagues who have spent their valuable time to help produce this newsletter. Their efforts and contributions are much appreciated.

C. P. Poon

Principal Inspector (Mathematics)

圍攻集合論

黃毅英　黃棣珊紀念中學

引言

邏輯、集合、關係和映射均在不少地區的大學入學試範圍內。事實上，這些概念都是介乎初等數學與大學數學的一種橋樑，而這亦正是預科學生最感困難之處。

首先，教材之鋪排，已有些大學的味道，內容不只較爲抽象和符號化，又動輒用公設引入概念，而不是以一種數學的語言導出，這都是令學生大惑不解的。

至於參考書籍方面，又大多只屬兩類。一類是普及讀物。集合嘛：一群羊、一班學生、一碗米，這便是集合；父子、朋友就是關係；每人的高度、年齡等便是映射。這種描述，對吸引初學者和給出一個粗略的觀念不無幫助。但集合運算的規率和證明呢？就不容易作較嚴謹的處理了。

至於另一類是大學用的參考書籍。在裡面、等價關係、等價類（equivalence class）、商集、一一映像都有，而且十分嚴謹，但是每個概念大抵一兩句就交代過了。大家在看過一個證明或推論後想找類似的問題演習，數目就差幾等於零。這種情況顯然是初等數學和大學數學間的一道空隙。

事實，集合論的發明是本世紀初的事，而它的打進大學以下的教程則是十數年的事，這一道空隙是需要急切填補的。

反機械式的處理

在初等數學裡，大部份的內容都對學生並不陌生的。比如三角與幾何，因有圖像的關係，學生會覺得是可以看得見的。至於代數，雖然有較多的符號，但大部份是基於數字的四則運算，可以計，故亦不會帶來多大的恐懼感。但集合論就不同了，於是不少同學就迴避瞭解個中的意義，而索性作機械性的代入。這種做法可能是源於低班之不積極瞭解和思考。就如學過畢氏定理後，便只知找出所謂的 a、b 和 c 以求代入，甚至不可考慮「鄰邊」、「對邊」和「斜邊」有甚麼意義，甚至連三角形是否直角也懶得仔細的去核對了。

到學習較抽象的集合論時，由於全面瞭解遭遇阻力，機械性代入的吸引力更爲強烈，對概念的眞義似通非通。這種做法初時會拖得過去，但這種做法，以長遠計是危險的。

其實所謂瞭解一個概念的含意，未必單指智性地去分析其定義。我們可以從多個角度去探索其定義帶來的基本特性。比如學過了集合、元素和子集後，我們可以問：ϕ 與 $\{\phi\}$ 是否相等？它們各有多少元素？我們發覺 ϕ 沒有元素、$\{\phi\}$ 有一個元素，故此不等，那麼 $\{\phi\}$、$\{\{\phi\}\}$ 各有一個元素，又是否相等。$\phi \cup \{\phi\} = \{\phi\}$，$\{\phi\}$ 又是否等於 $\{\phi, \{\phi\}\}$ …等。

當我們面對一條等式時，更可探究各集合裡面的是些甚麼形式的元素，是數字呢、序偶呢、還是集合（比如冪集裡的元素便是集合）。又如商集的元素便是形如 a/R 的等價類。

我們還可追索各元素的來源。譬如 R 是由 A 到 B 的關係，S 是由 B 到 C 的關係，$S \circ R$ 裡面的元素便是（a,c），a 存於 A，c 存於 C[(1)]。比如要證明 $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ 時，則知應由 $(c,a) \in (S \circ R)^{-1}$ 證起。又如 $f: A \to B$，$g: B \to C$，$h: C \to A$，則 $h \circ g \circ f\,(x)$ 是 A 內的東西，$h^{-1} f^{-1} g^{-1}[X]$ 則是 C 的子集，而 X 亦應是 C 的子集。

就是這樣多方面的探討，我們對每一集合有更深的瞭解，在推算的過程中用這種方法探討，亦有助於核對步驟上有否出錯。若證明裡左右方集合內的元素不吻合時，我們便知道出錯了。

對付抽象

抽象是數學一大特色，亦是其強處。不過數學的抽象，也是不少同學感到恐懼者。對付抽象的概念，首先要多點用圖像表示，而且要回歸到其實例的根。因爲抽象概念就是由這些實例昇華出來的。

在較基礎的參考書裡，我們找到不少的實例。以等價關係而言：數字上的同餘、同奇偶；幾何上的全等、有同一投影；序偶（a,b），（c,d）之適合 ad = bc（分數）等等一大堆。但對於較深的概念如等價類和商集等，則大多數的參考書再「不屑」再舉太多的實例了。但我們仍可充份的利用上面的實例。比如問：對於同餘這等價關係，1 的等價類是甚麼呢？它的商集共有多少元素呢？又如在證明過商集是一分類（partition）後，我們亦可將實例套入反覆研究。就以同餘來說，我們可以問：$1/R$ 和 $2/R$ 何以不相交呢？以模 3 的同餘而言，$1/R$ 與 $4/R$ 爲何有相交呢？

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(1) 一些書的關係的定義爲（A,B,G），$G \subset A \times B$。

在證明數題時，我們若將之化作較簡單的等價命題時，抽絲剝繭，則有助於瞭解個中原委。比如當證明 $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ 時，我們知道只須證明 $x\in A\cup(B\cap C)\Longleftrightarrow x\in(A\cup B)\cap(A\cup C)$ 就夠了。這便化成了邏輯運算中之分配律而可以真值表證得。如此集合論的分配律，通過建立等價命題的方式，可借助邏輯的分配律證得了。

又如證明對於 A 的等價關係 R，商集 A/R 是一分類時，我們細意分析，就知必須證明兩件事：一所有 A 內的元素必須存於某一 a/R 內；二 a/R 與 b/R 兩兩不相交。對於第一件事，任一元素 a 就存於自己的等價類 a/R 內，這是由於自反性之故。至於第二件事，若 $a/R\cap b/R\neq\phi$，我們須證明 $a/R=b/R$（現正利用等價命題之方式）。亦卽當有 $x\in a/R\cap b/R$ 時，要證明 aRb。但 $x\in a/R\cap b/R$ 卽 aRx 和 xRb，由傳遞性得 aRb，證畢。

如此，要證明命題甲，原來只須證較簡單的命題乙；而證乙卽要證丙等……層層剖釋，結果會得到（等價於）一條很簡單的命題；如是逆推，便可得出原來的命題甲了。

再舉一例：對於 $f:A\longrightarrow B$ 及 $X\subset A$，證明 $X\subset f^{-1}[f[X]]$。我們先以多角度的分析：X 是 A 的子集，$f[X]$ 是 B 的子集，$f^{-1}[f[X]]$ 則再是 A 的子集。要證明 $X\subset f^{-1}[f[X]]$，卽在 X 內取任意的元素 x，看是否必在 $f^{-1}[f[X]]$。建立等價命題：要證 $x\in f^{-1}[f[X]]$，$f(x)$ 必須存於 $f[X]$ 中。$f(x)$ 存於 $f[X]$ 中，卽 $f(x)$ 等於某些 $f(y)$，$y\in X$。那麼只須將 y 揀作 x 就可以了。逆推之便是：$x\in X\Rightarrow f(x)\in f[X]$，$x\in X\Rightarrow x\in f^{-1}[f[X]]$，證畢。

集合恒等式的證明

以上陳述關於集合的證明，裡面的集合是有特定意思的。比如 f〔X〕不是一般的集合，而是 $\{ f(x) \in B : x \in X \}$，他如 A/R，a/R 等都是特定的集合，但一些集合上的證明是一般的恒等式，如在

$$(A \triangle B) \setminus (A \cap C) = ((A \cup B) \setminus (A \cap B)) \setminus ((A \cap C) \setminus B)$$

中，A、B、C是任意的集合。一些書籍是用維恩圖(Venn diagram)顯示的，不過一般人認為這只算是一種示意圖，不算得上一個嚴格的證明。另一些則以分配律、第摩根律從左手邊逐步推到右邊。但這種做法較為繁複。

在邏輯的命題運算裡，我們用來證明恒等式的工具是眞値表，這種做法十分方便。以下介紹的證明集合恒等式方法是借助了眞値表；而所謂用眞値表，細心分析下，亦即將集合分成區域（regions）去考慮，亦即用了維恩圖的精神。

首先，我們先用定義去除△及\。左方即

$$((A \cap B') \cup (B \cap A')) \cap (A \cap C)',$$ (2)

右方則為

$$((A \cup B) \cap (A \cap B)') \cap ((A \cap C) \cap B')' 。$$

要證左方＝右方，即需證 $x \in$ 左方 $\Leftrightarrow$ $x \in$ 右方。若以 p，q，r 表以下的命題

$p \equiv x \in A$

$q \equiv x \in B$

$r \equiv x \in C$

即需證

$$((p \wedge \sim q) \vee (q \wedge \sim p)) \wedge \sim (p \wedge r)$$

$$\equiv ((p \vee q) \wedge \sim (p \wedge q)) \wedge \sim ((p \wedge r) \wedge \sim q)$$

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(2) 這裡假設 $A, B, C \subset X$，而 M' 表 $X \setminus M$。

而此為一命題的恒等式，可用真值表證得。

p q r	$((p\wedge\sim q)\vee(q\wedge\sim p))\wedge\sim(p\wedge r)$	$((p\vee q)\wedge\sim(p\wedge q))\wedge\sim((p\wedge r)\wedge\sim q)$
T T T	F F F F F F F T	T F F T F T T F F
T T F	F F F F F F T F	T F F T F T F F F
T F T	T T T F F F F T	T T T F F F T T T
T F F	T T T F F T T F	T T T F T T F F T
F T T	F F T T T T T F	T T T F T T F F F
F T F	F F T T T T T F	T T T F T T F F F
F F T	F T F F T F T F	F F T F F T F F T
F F F	F T F F T F T F	F F T F F T F F T
步驟	2 1 3 2 1 4 2 1	1 3 2 1 4 3 1 2 1

這種技巧不只對等式之證明適用，亦可借用到不少集合論的問題上。再舉一例如下，以 x_A 表A的特徵函數，定義 $x_H * x_K(a) = x_H(a) + x_K(a) - 2\,x_H(a)\,x_K(a)$，證明 $x_H * x_K = x_A$，A為某些關於H，K的式子，製造真值表如下：

x_H	x_K	$x_H * x_K$
1	1	0
1	0	1
0	1	1
0	0	0

故 $x_H * x_K = x_{H\Delta K}$。

邏輯的展述

命題的證明全部可用真值表證得，但從邏輯學裡，除學了一大堆的符號外，最主要是能幫助我們達至一個合情理的推演和展述。所謂合情理的展述，包括推演是邏輯上有效的，在適當的地方加上解釋（引述定理），亦包括在句法上是讀得通的。

譬如說，

$\sin X = \frac{1}{2} = 30^\circ$

便不通了，又例如

$$\begin{cases} x + y + z = 4 \\ 2x + 3y + z = 6 \\ x - y + 3z = 7 \end{cases}$$

$$\therefore \quad 2x + 4y = 5$$

便是不清楚的展述。第三者難以明白 $2x + 4y = 5$ 是如何得出來的。應寫明從 3 ×① －③得出。一般來說，我們得將前後的兩件物件連貫，對於數字、集合等，我們可用 $=$，$\leq$，$\subset$ 等連貫之，如

$$\begin{aligned} & (x+1)^2 + 8x + 23 \\ = \; & x^2 + 2x + 1 + 8x + 23 \\ = \; & x^2 + 10x + 24 \end{aligned}$$

但對於命題（如等式、不等式等有真值之命題），則須用 $\Rightarrow$ 或 $\Leftrightarrow$ 連貫，如

$$\begin{aligned} & (x+1)^2 + 8x + 23 = 0 \\ \Leftrightarrow \; & x^2 + 2x + 1 + 8x + 23 = 0 \\ \Leftrightarrow \; & x^2 + 10x + 24 = 0 \\ \Leftrightarrow \; & x = -4 \text{ or } -6 \end{aligned}$$

兩者是不可混淆的。再者，我們應在適當的地方講述意圖，將過程交代。就如上面的證明

$$(A\triangle B)\setminus(A\cap C) = ((A\cup B)\setminus(A\cap B))\setminus((A\cap C)\setminus B) ,$$

我們首先說，

$$左方 = ((A\cap B')\cup(B\cap A'))\cap(A\cap C)'$$

$$右方 = ((A\cup B)\cap(A\cap B)')\cap((A\cap C)\cap B')'$$

若設

$$p \equiv x\in A \quad , \quad q \equiv x\in B \quad , \quad r \equiv x\in C$$

則只需證明

$$((p\wedge\sim q)\vee(q\wedge\sim p))\wedge\sim(p\wedge r) \equiv ((p\vee q)\wedge\sim(p\wedge q))\wedge\sim((p\wedge r)\wedge\sim q)$$

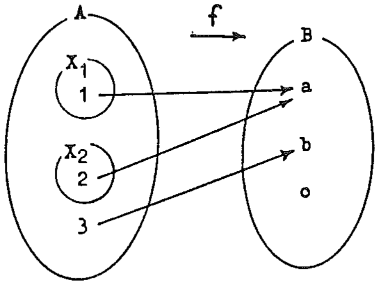
就可以了，現在，我們欲以眞值表以達成……。

在代數裡，我們若想試驗推演有否出錯，我們可將特別的數字代入。如

	代 n = 1
$((n+1)^2 + 3(n-2)^2)^2$	49
$= (n^2+2n+1+3n^2-12n+12)^2$	49
$= (4n^2-10n+13)^2$	49
$= 16n^4+100n^2+169-80n^3+260n+104n^2$	569
$= 16n^4-80n^3+204n^2+260n+169$	569

故此，我們知道，第三步錯了。這種方法一樣適用於集合論。對於命題或集合之推算，我們可以眞值表或維恩圖去驗證；而對於映射之推算，我們則可代以特別的映射以測試。由於雙射（ bijective ）是比較「完美」的，我們可以代入一些非內射（ not injective ）或非滿射（ not surjective ）的映射以驗證。茲再舉一例如下：

「證明」對於 $f: A \to B$ 及 X_1，$X_2 \subset A$，$f[X_1] \cap f[X_2] \subset f[X_1 \cap X_2]$。我們考慮 $A = \{1, 2, 3\}$，$B = \{a, b, c\}$，$f: A \to B$ 爲非內射 $f(1) = f(2) = a$，$f(3) = b$，又設 $X_1 = \{1\}$，$X_2 = \{2\}$，從圖中可見 $f[X_1 \cap X_2] = f[\phi] = \phi$；$f[X_1] \cap f[X_2] = \{a\} \cap \{a\} = \{a\}$，故式子肯定是錯的了，但錯在那裡呢？讓我們仔細的分析以下的「證明」：



	用以上之例
$y \in f[X_1] \cap f[X_2]$	$y = a$
$\Rightarrow y \in f[X_1] \wedge y \in f[X_2]$	$y = a$
$\Rightarrow y = f(x), x \in X_1 \wedge y = f(x), x \in X_2$	$y = f(1), y = f(2), 1 \neq 2$
$\Rightarrow y = f(x), x \in X_1 \cap X_2$	
$\Rightarrow y \in f[X_1 \cap X_2]$	

由此可知錯在第二步，我們只爲推出

$$y = f(x),\ x \in X_1 \wedge y = f(x'),\ x' \in X_2,$$

而不能斷定 $x = x'$。

這種做法不只能測出邏輯上的謬誤，亦是製造反例的最佳方法。以上的 f 便是對於錯誤句子 $f[X_1] \cap f[X_2] \subset f[X_1 \cap X_2]$ 的現成反例。

<u>完整定義問題</u>

在集合論的題目常有要求證明某映射（或集合）是完整的被定義（well-defined）。這本身是個極有趣的問題。比如說，定義 $f(x)$ 作 x^2+1，$f(x)$ 又怎會有不定義之理呢？其實問題主要是出自定義是用隱（implicit）的方法。例如

1. 已知 f ： $A \longrightarrow B$ ，對於任一 $b \in B$ ，定義 $g(b) = a$ ，其中 $f(a) = b$ 。這裡，若 f 非滿射，即有些 b 不等於任何 f(a) ，而 g(b) 就沒有定義了。

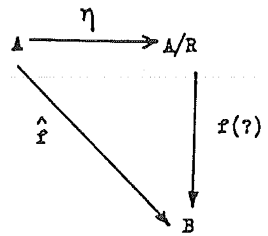
2. 定義 $y = f(x)$ 如下：

$$\cos(xy) + \sin(x+y) = 3 \text{ 。}$$

這裡，左邊最大是 2 ，不可能等於 3 。故根本找不到任何 x、y 適合上式。

3. 對於兩已知集 A、B ，取 B 中之任意元素 b_0 而定義 f ： $A \longrightarrow B$ 作恒常映射（ constant mapping ）： $f(a) = b_0$ ，$\forall a \in A$ 。但若 $B \neq \phi$ 時，則此映射便沒有定義了。

另一類需要處理完整定義的是涉及等價關係的商集。若 R 是集 A 中的等價關係。在定義 f ： $A/R \longrightarrow B$ 時，我們須測看 f 與 R 是否協調（ compatible ）。

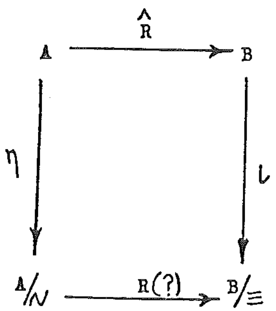


比如對於模 3 的等價關係，定義 $f(a/R) = a^2 + 1$ 便非完整定義了。因為 $1/R = 4/R$ ，但 $f(1/R)$ 和 $f(4/R)$ 却分別為 2 和 17 。這就好像定義一個班的高度。對於每一班，我們任意抽出一位同學量一量其高度，便叫這高度作為全班的高度。現小明與大明同於中三甲班，他們的高度分別為 1.5 米和 1.7 米。那末，中三甲班的高度應為 1.5 米呢、還是 1.7 米呢？

這問題就等於已有 $\hat{f}$ ： $A \longrightarrow B$ ，問是否存在 f ： $A/R \rightarrow B$ 使得 $f \circ \eta = \hat{f}$ ，η 即為自然滿射（ natural surjection ）。充要條件為

$$aRb \Rightarrow \hat{f}(a) = \hat{f}(b) \text{ 。}$$

將這問題稍爲轉換便可成爲下圖：已有等價關係 $\hat{R}$ 與自然滿射 η ：$A \to A/\sim$ 及 ι ：$B \to B/\equiv$，其中 $\eta(a) = a/\sim$，$\iota(b) = b/\equiv$。這就好像定義「友班」的問題。若兩班各能找出一位同學爲好朋友，則該兩班稱爲友班，若找到兩人不友好，則稱之爲「非友班」。今大明與小明同於中三甲班、大華與小華則在三乙班，大明與大華是好友，小明與小華則不友好，那末三甲與三乙究竟是友班還是非友班呢？所以，這種「友班」、



「友校」、「友國」等之能有完整定義，則同班、同校、同國的意見必須一致，即

$$a \sim a',\ b \equiv b' \wedge a\ \hat{R}\ b \Rightarrow a'\ \hat{R}\ b'$$ 。

今以一例再作說明：

集 A 內的一關係 R 若適合自反性（ reflexive ）與傳遞性則稱爲前序（ preorder ）。設 R 爲 A 之一前序 。於 A 定義 ～ 如下：

$x \sim y$ 當且僅當 $x R y$ 及 $y R x$ ，

證明～爲等價關係。

再於 $A/\sim$ 定義 R 如下：

$(x/\sim)\ \hat{R}\ (y/\sim)$ 當且僅當 $x R y$ ，

證明 $\hat{R}$ 是完整定義且爲一次序[3] 。

(3) 即前序加上

$x\ \hat{R}\ y \wedge y\ \hat{R}\ x \Rightarrow x = y$。

要證明 $\hat{R}$ 是完整的被定義即要證

$$x \sim x' , \quad y \sim y' \wedge xRy \Rightarrow x' R y' 。$$

根據定義

$$x \sim x' , \quad y \sim y' , \quad xRy$$

即

$$xRx' , \quad x' Rx , yRy' , y' Ry , xRy$$

$\therefore$ $x' Rx \wedge xRy \wedge yRy'$ $\qquad$ ($p \wedge q \Rightarrow p$)

$\therefore$ $x' Ry'$ $\qquad$ (傳遞性)

結語

一個較佳的學習方案可能是先於較低班時有一個第一次的接觸，先對集合論的基本概念和其發展歷史有所認識，到預科時才作較嚴格的學習（螺旋式教學）。由於學生學習上的困難亦是當初數學發展時所遇到的困難，數學史家指出：透過學科發展的重演（對發展史的學習）有助於瞭解爲何要建立集合論、其應用及如何將之一步步納入公設系統等問題。由這方面的瞭解，我們可以明白爲何要將一切以集合形式寫成，甚至將序偶（ a,b ）也要硬寫作 $\{\{a\},\{a,b\}\}$ ，$f:A \rightarrow B$ 定義成 (A,B,G) ，0 定義作 ϕ 、 1 定義作 $\{\phi\}$ ， 2 定義作 $\{\phi,\{\phi\}\}$ ……，以及 $Z = \{(0,x):x \in Z^+\} \cup \{\phi\} \cup \{(1,x):x \in Z^+\}$ 、 $Q = (Z \times Z \setminus \{0\}) / R$ 其中 $(a,b)\ R\ (c,d)$ 當且僅當 $ad=bc$ 等。其中的作用都是要保證不會有「宇集」等詭論。事實上，集合論的發展和第三次數學危機等本身就是極有趣味的歷史盛事。

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教室拾貝

逸

在我的腦海中，孩子們的歡笑聲及笑臉通常是出現於遊樂場中，海灘上和山野間，而課室中的孩子，往往總是循規蹈矩，誠惶誠恐，難得一展笑顏。上數學課時，更是如臨大敵，小腦袋裏滿是四則運算和各類圖形的面積公式，試問那能輕鬆起來呢！我曾在小兒與同學交談間，知道他們稱數學老師為「搾汁機」，稱數學課為「搾汁堂」，眞使天下數學老師「同聲一哭」。其實，身為數學老師的，有誰不希望學生能在上課時，帶著輕鬆的心情學習數學呢？但有時卻礙於很難找到有趣而又適合數學課的題材，以致沒法在學生感覺沉悶時，提出來改善課室氣氛。筆者偶在閱讀刊物時，發現一些有趣的數學題材，因此特別拿出來與老師們分享，更希望能藉以拋磚引玉，使老師們能提出更多適合的資料，為課室帶來歡樂的氣氛。

×××

小明發覺他的新校服褲子過長，於是請求大姐說：「姐姐，可否替我把褲子改短五厘米呢？因為它實在太長了。」

大姐回答說：「我現在很忙，改天再說吧！」於是小明便跑到二姐跟前，請二姐替他把褲子改短五厘米，但二姐也說工作太忙，不能幫他修改，於是小明只好悶悶不樂的去睡覺。當大姐工作完畢，覺得剛才拒絕了小明的要求，實在不對，於是靜靜地替小明把褲子改短了五厘米。而當二姐做完工作後，也覺得應該幫弟弟的忙，於是也靜靜地把褲子改短了五厘米，小明第二天清早起來，穿上褲子的模樣，你能想像嗎？

筆者按：當老師教授厘米加減時，不妨以這類笑話來博學生一笑。

×××

小珍的媽媽吩咐小珍在信封上貼上一個五角郵票，小珍卻把一枚一元郵票貼在信封上，媽媽很生氣，小珍卻說：「不要緊，我可以在旁邊再貼上一個五角郵票，中間加上減號，這樣，一元減五角，不是等於五角嗎？」

筆者按：「元、角、分」這課題在小學課程中常常出現，當課室氣氛沉悶時，可作調劑的用途。

×××

有一次，一位財主請中國詩人倫文敍為他新購的一幅百鳥圖題詩一首，倫文敍只看了那圖畫一眼，便於其上疾書：「天生一隻又一隻，三四五六七八隻。」寫完後一言不發便走了。財主莫名其妙，你呢？

筆者按：其實這打油詩暗含一題四則運算，可把它列成（1＋1＋3×4＋5×6＋7×8）隻，如此不剛好是一百隻鳥嗎？後來財主請倫文敍吃飯，他才解明其中含意，並補上下列兩句，成爲一首七言詩。「鳳凰何少鳥何多，啄盡人間千萬石。」用以諷刺時弊。

× × ×

一間糖果店慶祝一週年紀念，定下一個很特別的贈送辦法，就是憑兩張包某種糖果的糖紙，可再換領該糖一粒。但當小强到達時，店內已擠滿了小朋友，他們一邊吃一邊計算，如何用糖紙換糖，才能吃到最多的糖果。小强首先買了十粒糖，吃完後也和小朋友一樣，把兩張糖紙換取一粒糖果，他是一個非常聰明的小朋友，你猜猜他共可吃糖多少粒？十五粒？錯了！十九粒？仍舊不對！

筆者按：小强總共可吃二十粒糖，因他用兩張糖紙換回一粒糖的方法，可吃十九粒糖，還剩一張糖紙，但小强可向店內一位小朋友借用糖紙一張，用兩張糖紙換回一粒糖果，吃完後把糖紙歸還小朋友便成了。

× × ×

以上的例子，只想說出老師上數學課時，可藉著一些與課題有關的趣味性資料，使課室充滿輕鬆的氣氛，或許有時未能找到適合的有趣題材，但只要老師本人表現出對數學的興趣，言語間帶著幽默感，一句合宜的說話，一個關懷的眼神，也能爲課室帶來一抹的彩虹，一絲的陽光。

計算機與小學數學的學習

濬

談及計算機，很多人不期然就想起算術的四則運算。不錯，計算機無疑是現代科技的輝煌成果之一，而且由於設計愈來愈精巧，加上大量生產而導致價錢愈來愈便宜，所以香港絕大部分的家庭都能擁有一部或多部。普通的四則運算，可藉著它的幫助而得到「快而準」的效果，它的出現更令中國傳統算盤的地位日益降低，因為人們只需按動幾個鍵子，答案就很快地出現在計算機的顯示屏上，眞是省時省力，何樂而不為呢？

由於計算機的普及，有人就提出數學的學習歷程亦應作相當的改變。基本的四則運算，不論是心算或是筆算，已被認為毋需强調，甚至連乘數表也毋需記憶，因為計算機已可提供答案，學生所需學習的反而是計算機的操作方法。當然，對這種論調，大部分數學教師都認為過於極端，而且沒有充足理論根據。不過近年來世界各地都紛紛因計算機的普及而提出數學教學的新路向。例如美國數學教師聯會（NCTM）為計算機在數學課的應用而建議「在所有班級，無論堂課、家課或對學習的評估，均應將計算機結合在數學課程中」（註一）。在英國，檢討全國學校數學教學路向委員會在它的報告書中，對計算機的應用，提供兩個考慮的方向：其一是如何應用計算機使數學教學方法得以改善，其二是計算機的應用如何改變教學內容(註二)。

「藉著計算機的應用，兒童可學習數學概念和解決數學問題，他們毋需耗費過多的時間進行繁複及較為困難的運算。」

「計算機可用作進行數學發現和探究工作的工具。」

以上都是近年來數學教育工作者對計算機在課室內應用的一般理論，他們當中，不少為上述理論細緻地提供兒童應用計算機的機會和例子。

對於香港小學數學課程來說，計算機的適當應用，肯定可以提高兒童學習數學的興趣。例如，在六年級學習循環小數的時候，學生可利用計算機來取得下列結果：

$\frac{1}{3} = 0.333333333$

$\frac{1}{9} = 0.111111111$

$\frac{2}{9} = 0.222222222$

$\frac{5}{6} = 0.833333333$

$\frac{6}{11} = 0.545454545$

……

由此他們可以體會到循環小數的產生和使用循環節來表示循環小數的需要。應用計算機的結果，一方面減少了學生的繁複而單調的運算，另一方面對學生也有較大的說服力。同時學生更可用這個方法來探究一系列分數和它們的循環小數之間的規律（例如：$\frac{1}{9}$，$\frac{2}{9}$，$\frac{3}{9}$，$\frac{4}{9}$，…，$\frac{8}{9}$；$\frac{1}{11}$，$\frac{2}{11}$，…，$\frac{10}{11}$），因而對循環小數的了解會更為深刻。他們並可從實例體會有限小數和循環小數不同的地方。對學習能力較高的學生來說，他們更可憑藉計算機的幫助，對於那一類分數的分母可得到有限小數的規律有更深的體驗。

又例如在探究正方形數的規律時，學生可利用計算機逐步計算

$$1，1 + 3，1 + 3 + 5，1 + 3 + 5 + 7，\cdots$$

從而發現各正方形數與連續奇數之和的關係。這是通過圖形來觀察數型以外的另一手法。這兩個不同的方法可互相配合來達至更佳的效果。

從上面的兩個例子，有人可能會說計算機只適用於高年級課題，對低年級並不適用。這種看法有以偏蓋全之弊。無疑在高年級課題，如數型、幻方等，應用計算機的機會較多，但在低年級加法中，計算機也可用來探究 10 + 10，10 + 20，20 + 30，…的結果和他們的已有知識（1 + 1，1 + 2，2 + 3，…）有些甚麼關係。由此加深對位值的認識。這個例子更可推廣至學習三位數或四位數加法時的 100 + 100，200 + 300，1000 + 1000，2000 + 3000，…。

此外，通過計算機來計算97＋1，98＋1，99＋1所得的結果，更可給予學生由兩位數進至三位數的體驗。至於位値的探索活動，學生又可藉著將〃1988〃錯按爲〃1688〃時，由計算知道，所差的〃3〃實際上是〃300〃。此外，利用同數連加的計算（卽利用大多數計算機均具備的〃田田〃常數加法特性），讓學生取得如〃每兩個一數〃或〃每5個一數〃等結果，這些結果可鞏固學生在利用實物數數中所得的經驗，從而爲他們的乘法學習作充足的準備。

以上一系列低年級通過計算機的應用來進行數學活動的例子，完全配合香港數學課程內申言的「…教師應盡量讓學生有機會自己進行探索及發現」（註三），對達到課程的一些目的（「引起兒童對數學學習的興趣」和「誘導兒童對數和圖形的規律和結構的欣賞」），甚有幫助。

不過，相信大部分數學教師都同意，切勿視計算機爲可以取代學習基本運算技巧的工具。基本四則運算技巧仍應被看作不可缺少、也是無可取代的數學知識，因此計算機在小學階段不宜被過度强調。此外，記錄計算機活動結果仍是不可缺少的步驟，它可把探索活動的資料有系統地組織起來，並可鞏固學生的概念，在教學過程中具極大意義。

註一：National Council of Teachers of Mathematics, A Position Statement on Calculators in the Mathematics Classroom. April, 1986.

註二：Mathematics Counts : Report of the Committee of Inquiry into the Teaching of Mathematics in Schools under the Chairmanship of Dr. W.H.Cockcroft, H.M.S.O., 1982, pp. 109.

註三：香港課程發展委員會：小學課程綱要──數學科，1983，第三頁。

π的近似值

陳偉仲

利用數列或級數，我們可以找到很多求取圓周率 π 的近似值的計算方法。但是要找出一個可以使初中學生徹底明白的方法，却不容易。現在試推薦一個方法給同工參考，它具備以下幾個特點：

1. 祇包括四則、平方、開方、畢氏定理及相似三角形等基本數學知識。
2. 不需依賴任何計算工具。
3. 可以求取至任何準確程度，並可確保所得的近似值的準確程度。

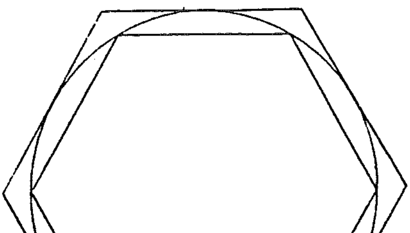


圖 1a

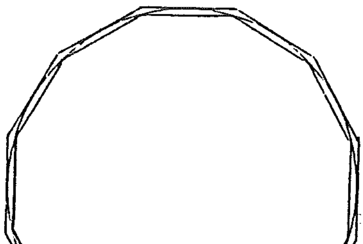


圖 1b

首先在半徑等於 1 的圓內外分別作一正 6 邊形（圖 1a），然後不斷將邊數加倍，變成 12、24、48………。學生很易理解到當邊數不斷增加時，兩個正多邊形的周界也就愈來愈接近圓周。不過，圓內的那些正多邊形的周界必較圓周爲小，而圓外的却必較圓周爲大。

假設 n_k 爲正多邊形的邊數，s_k 爲圓內的正 n_k 邊形的邊長。首先我們有

$$n_o = 6,\ n_k = 2\, n_{k-1}\ ,\ k = 1, 2, 3, \ldots$$

$$s_o = 1$$

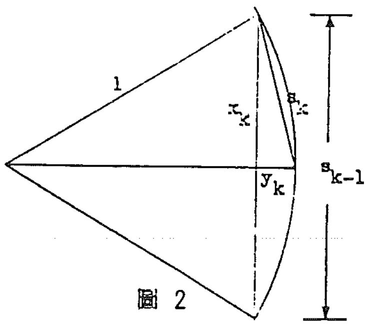


圖 2

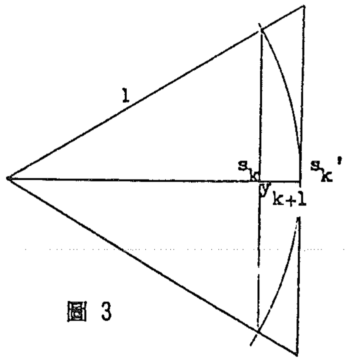


圖 3

從圖 2 可知

$$\begin{cases} x_k = \frac{1}{2}\, s_{k-1} \\ y_k = 1 - \sqrt{1 - x_k^2} \\ s_k^2 = x_k^2 + y_k^2 \end{cases} \qquad k = 1, 2, 3, \ldots$$

假設 s_k' 爲圓外的正 n_k 邊形的邊長。從圖 3 可知

$$\frac{s_k'}{s_k} = \frac{1}{1 - y_{k+1}}\ , \qquad k = 0, 1, 2, \ldots$$

再假設 p_k 、 p_k' 分別爲圓內外正 n_k 邊形的周界。很容易知道，

$$p_k = n_k s_k \text{ , } p_k' = n_k s_k' \text{ ,}$$

$$\frac{p_k}{2} < \pi < \frac{p_k'}{2}, \qquad k = 0, 1, 2, \ldots$$

k	n_k	x_k	y_k	s_k	s_k'
0	6			1	1.1547005383
1	12	0.5	0.13397459622	0.51763809022	0.53589838484
2	24	0.25881904511	0.03407417372	0.26105238442	0.26330499516
3	48	0.13052619221	0.00855513863	0.13080625846	0.13108692562
4	96	0.065403129233	0.00214107676	0.065438165644	0.065473220826
5	192	0.032719082822	0.00053541253	0.032723463250	0.032727844268
6	384	0.016361731625	0.00013386209	0.016362279209	0.016362826807
7	768	0.008181139605	0.00003346608	0.008181208053	0.008181276500
8	1536	0.004090604026	0.00000836656	0.004090612582	0.004090621138
9	3072	0.002045306291	0.00000209164	0.002045307360	0.002045308430
10	6144	0.001022653680	0.00000052292	0.001022653814	0.001022653948
11	12288	0.000511326907	0.00000013073	0.0005113269237	0.0005113269404
12	24576	0.0002556634619	0.00000003269	0.0002556634639	0.0002556634660
13	49152	0.0001278317320	0.00000000818	0.0001278317322	0.0001278317325
14	98304	0.00006391586612	0.00000000205		

表格 1

k	n_k	$\frac{p_k}{2}$	$\frac{p_k'}{2}$	π 的近似值	準確度（有效數字）
0	6	3	3.464101615	3	1
1	12	3.105828541	3.215390309	3	1
2	24	3.132628613	3.159659942	3.1	2
3	48	3.139350203	3.146086215	3.1	2
4	96	3.141031951	3.142714600	3.14	3
5	192	3.141452472	3.141873050	3.14	3
6	384	3.141557608	3.141662747	3.142	4
7	768	3.141583892	3.141610176	3.1416	5
8	1536	3.141590463	3.141597034	3.1416	5
9	3072	3.141592105	3.141593748	3.14159	6
10	6144	3.141592517	3.141592927	3.141593	7
11	12288	3.141592619	3.141592722	3.141593	7
12	24576	3.141592645	3.141592670	3.141593	7
13	49152	3.141592652	3.141592658	3.1415927	8

表格 2

現在我們祇要增加每次計算的準確度和把邊數再增加，便可以求到更準確的 π 的近似值。

最後一提，上述的方法可以編寫成很簡單的電腦程式。有興趣的同工不妨一試，這裡不贅述了。

Problem-solving as an Art Form

Ed Barbeau

REPRODUCED WITH PERMISSION FROM GRADUATE, THE UNIVERSITY OF TORONTO ALUMNI MAGAZINE.

WHEREIN LOGIC VIES WITH INTUITION, AND CALCULUS MAY NOT HELP

When I was in Hangzhou, China, Last June, I had the opportunity of watching a Chinese brush artist at work. After his assistant anchored the corners of a blank page with slate bars, the artist began. Dipping his brush first into the ink, then into a water dish, he made a number of apparently random smears. However, with deft and delicate strokes, he incorporated them into hills, temples and trees. A middle band across the page which he assiduously avoided proved to be mist which divided peaks from lowlands. In three-quarters of an hour, he was done. I thought as I saw him working how much like doing mathematics his task was. One begins with a few ideas whose form may be unclear, and by a combination of judgement and technique, binds them together into a whole which is coherent and even beautiful to contemplate.

Ideas and solid reasoning are the brush and ink of mathematics. Just as the artist begins with an image or concept and with skill and economy makes it live for his viewer, so the mathematician starts with a conjecture or a problem and with skilful and economical juxtaposition of ideas arrives at a convicing and satisfying solution. I will illustrate with some problems.

What was on the turtle's back

Since we started in China, let us remain there for a while. The ancient Chinese were fascinated with numbers. There is a legend that about 4,000 years ago, the Chinese Emperor Yu encountered, on the banks of the

Yellow River, a divine tortoise. On its back were the numbers from 1 to 9 inclusive in a three by three square array such that the sums of the digits in each row, each column and each diagonal were the same (a "magic" square).

Can we reconstruct this array?

First, let us make clear what the problem is. If we denote the numbers by letters, we have the array

A	B	C
D	E	F
G	H	I

for which all the sums: A+B+C, D+E+F, G+H+I, A+D+G, B+E+H, C+F+I, A+E+I and C+E+G ought to be equal with the appropriate substitution of numbers for letters. There is an obvious way to solve the problem : simply try all 362,880 ways of filling in the numbers and chuck out those which do not work. However, this pedestrian approach seems unattractive. We have to be more selective, finesse away the unpromising avenues of approach and go to the core of the matter. Where should we focus our attention?

Perhaps we should go after the centre number : E. Why? It is unique. There are four corner numbers, four numbers in the middle of an edge but only one in the centre. And, it is involved in more sums (four) than any other number. This is vague, but gives us a place to start.

Since we will make use of the sums involving the centre number, what will the sum be? The sum of each row must be one-third of the sum of all the figures in the array. This latter sum we know : it is 45, the sum of all the digits from 1 to 9 inclusive. So the answer is 15.

The remaining numbers can be divided into four pairs and, since the sums of all rows are equal, the sum of each pair must be equal. In the sums involving E, this gives us A+I = B+H = C+G = D+F. Therefore, the sum of the eight numbers not in the middle, which is 45 - E, must be divisible by 4. Thus, E is 1, 5 or 9. If E is 1, the sum of the other eight numbers is 44 so the sum for the four pairs, A+I and the others, must be 11. But this gives the wrong sum for A+E+I which is supposed to be 15. Thus E cannot be 1. Neither can E be 9 for the same reason. The only possibility is that E is 5.

To narrow down the possibilities for the remaining numbers, it is often useful in such situations to make a parity determination, that is, to see which numbers can be even and which odd. Can any number in the middle of an edge (B, D, F, H) be even? Suppose B is even. Since all sums are 15, an odd number, if B is even, H must also be even. One of A and C must be odd and the other even. Because A and C are in symmetrical positions, we might as well assume that it is A which is odd and C which is even. We are led to the following arrangement

odd	even	even
even	odd	even
even	even	odd

Thus the array has to have six even and three odd numbers. Unfortunately, we were provided with four even and five odd numbers, so we have to conclude that none of the numbers in the middle of an edge can be even. All must be odd. Since we have four odd numbers to place (1, 3, 7, 9) they have to be placed in the middle of the edges. By the symmetry of the array, we might as well take B to be 1, in which case H is 9. If we take D to be 3, then F must be 7. This gives us

A	1	C
3	5	7
G	9	I

A cannot be 2, 4 or 6. (Why? See what these possibilities force on C and G.) Hence A must be 8 and we are left with

8	1	6
3	5	7
4	9	2

which works. This is the only possibility in the sense that the edges must contain those numbers in those relationships whether horizontally or vertically.

If you arrived at a solution on your own, your reasoning might have been quite different. In fact, you might have been able to narrow down the possibilities much more efficiently. I would be interested to see these more elegant solutions. Of course, mathematicians have not been content to leave the matter at this point. The situation can be generalized in many different ways. For example, we could increase the size of the square, and ask how to arrange the numbers from 1 to 16 in a four by four array, from 1 to 25 in a five by five array, and so on, so that all row and column (and, perhaps, diagonal) sums are equal. In fact, there are general methods which will enable a solution with equal row and column sums to be found regardless of the size of the square array. Here is a solution to the eight by eight problem due to Karl Friedrich von Jänisch in 1859 which will be of particular interest to chess players.

50	11	24	63	14	37	26	35
23	62	51	12	25	34	15	38
10	49	64	21	40	13	36	27
61	22	9	52	33	28	39	16
48	7	60	1	20	41	54	29
59	4	45	8	53	32	17	42
6	47	2	57	44	19	30	55
3	58	5	46	31	56	43	18

The sum of each row and each column is 260. If we number the squares of a chessboard in this magic way and place a knight on the square marked 1, then a succession of legal moves will carry him to each number in order and finally back to the square he started on.

You could also try your hand at creating your own problem. Like the writer of a good detective novel, the creator should try to find one for which some basic reasoning will go a long way towards the denouement. Here are two such problems.

An H of a problem

Fill in the numbers from 1 to 7 inclusive so that the two sides and the bar of the H all have the same sum.

```
J       K
L   M   N
P       Q
```

The twenty trinity

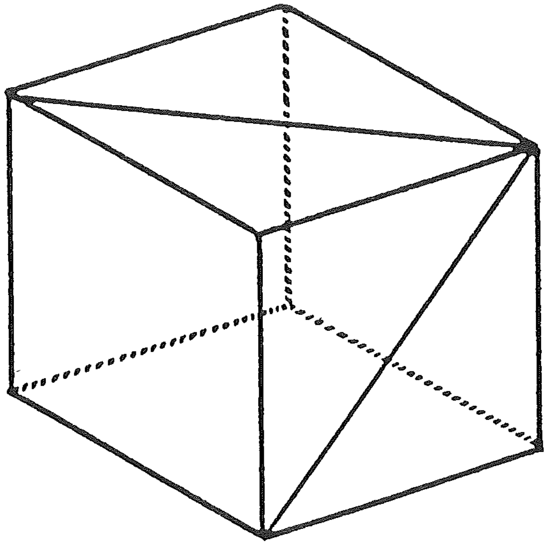
Fill in the numbers from 1 to 9 inclusive to make each of the three sides of the triangle add up to 20.

```
      R
    S   T
  U       V
W   X   Y   Z
```

An open-ended problem

In the situations discussed until now, a flash of insight may help to cut down the amount of work needed. One of my favourite problems which admits a quick solution if you have the right inspiration is the following.

Two face diagonals of a cube meet at a common vertex. They determine an angle. What is the angle?



If you give this problem to a first year university mathematics student, he may try to apply some sophisticated geometry or vector techniques. But you do not need any of this high-falutin' stuff; all you have to know is that the sum of the angles in any triangle is 180 degrees.

What is this lying on the surface?

Another problem of the same ilk is attributed to Samuel Beatty, for many years head of mathematics and dean of arts at the U of T. The centre of gravity of an empty bear mug lies above the bottom of the mug. Thus, when you start to fill it, the beer will go below this point and the centre of gravity of the mug and beer together will descend. However, as the mug fills up, the centre of gravity will eventually rise. Argue that when the centre of gravity is at its lowest point, it is resting on the surface of the beer. (This is a fiendish question for first-year calculus students; they will often get tied up in high powered techniques rather than use some basic reasoning. One of the most important things to learn about calculus is when not to use it.)

A classical feast

We began with a turtle in ancient China. Let us change the setting to an ancient Roman coliseum and an animal of another sort. The stadium surrounds a circular field from which there is no exit. Onto the field come a ferocious lion and a hapless gladiator. Both are tireless, agile and intelligent. (In mathematical problems, we pick only the best.) The lion is also hungry and wishes to meet the gladiator for lunch, while the latter bends every effort to avoid leo's clutches.

Let us suppose that the top speed of the lion exceeds that of the gladiator. What will happen? Obviously (you say), the lion will catch the gladiator and seal his doom. But is this so clear? Certainly, if both were out in the open, the gladiator would run like sixty and the lion would take after him and eventually catch him. But it is conceivable that, in a confined space, the gladiator could twist and dodge enough to avoid his fate.

How can we resolve this element of doubt? Surely, the best the lion can do is run at top speed, keeping his quarry in front of him at all times. Actually to show that the lion will succeed, we have to set up and solve what is called a pursuit problem which requires some fairly advanced mathematics. But there is a simpler solution. We can prescribe for the lion a strategy which is probably not as good as the one just described but which can be justified much more easily. The lion should track the gladiator. He should first run to a point that the gladiator has passed and then follow his exact trail. Since the beast can run faster, eventually he will catch up.

There are variants to the problem : what if the gladiator runs faster than the lion? and both use their best strategies? What if the best speed of both is the same?

I will end with two problems for which solutions are invited. There is an appendix with some comments to get you started.

Love of five oranges

You have five oranges, which look identical in every respect. However, they all have different weights. Using only an equal-arm balance, arrange the oranges in increasing order of weight. (An equal-arm balance permits only a rough comparison of the masses placed in the two pans; you can tell whether both have the same weight or else which one is heavier. You cannot determine the actual weight in grams.) One way to solve the problem is to compare all possible pairs of oranges; this will require ten weighings. Reduce the number of weighings required. What is the least number of weighings that may be necessary to sort the oranges?

Problem of the dancing pairs

A line of six dancers is arranged so that three ladies are to the left and three gentlemen are to the right : L L L G G G. When the music begins, two dancers, keeping the same order, move to a new position

not occupied by any of the others. Sometimes the pair that moves will be two ladies, sometimes two gentlemen, sometimes a lady and a gentleman. As each pair reaches its new position, a new pair sets off. Arrange a dancing sequence so that, at its conclusion, ladies and gentlemen alternate with no gaps : G L G L G L or L G L G L G.

Here is one solution (the pair about to move is underlined) :

$\underline{L}$	$\underline{L}$	L	G	G	G		
		L	G	G	$\underline{G}$	$\underline{L}$	L
G	L	$\underline{L}$	$\underline{G}$	G			
G	L			G	L	G	$\underline{L}$
G	L	G	L	G	L		

Is this the solution which requires the fewest moves?

Solve the same problem with four dancers of each sex. See how high you can raise the number of dancers and how small you can make the number of moves.

When doing these problems one tends to think on different levels. Initially there is a period of familiarization which might involve some experimentation, trying out special cases or trying to falsify the assertion. Gradually an intuition develops about what might be required to solve the problem. Finally, one's thinking becomes more formal as one analyzes the possibilities and arrives at a solution which is definitive.

APPENDIX

The twenty trinity

If we total the totals of the three sides, we get 60. The corner numbers R, W, Z each appear twice, one for each side to which it belongs. Therefore, since the sum of the digits is 45, R+W+Z must be 15. Now it is not hard to argue that 5 must be at a corner : if 5 is within an edge, say S=5, then the sum of the rest of the numbers in that edge, R+U+W, must be 15; but then, R+U+W = R+Z+W, so U=Z, which is not true since all the numbers are to be distinct.

An open-ended problem

Close the open ends by joining the other ends of the face diagonals to produce an equilateral triangle. The correct answer is 60 degrees.

The beer and mug problem

The argument is reduction ad absurdum. We argue that the centre of gravity can be neither above nor below the beer at its lowest point, and therefore must be on the beer. Suppose, if possible, it is above the surface of the beer; then any further beer would initially land below the centre of gravity, and the centre of gravity of the new system would have to be even lower. If the position of the centre of gravity is below the surface of the beer, then we could make it even lower by removing a little of the beer.

A classical feast

If the gladiator is the faster of the two, he should run to the wall and around the edge at top speed. The lion, to catch his prey, must attain both the same distance from the centre and the same direction as the gladiator. Any attempt to do one of these means that he loses ground on the other. If both run at the same speed, the situation is more complex. The gladiator must avoid going to the edge; if he stays on the edge and the lion starts at the centre, the lion will catch him in the time taken to run a quarter of the way around. It can be shown that if both gladiator and lion act wisely, the lion can come arbitrarily close to the gladiator but will never actually catch him.

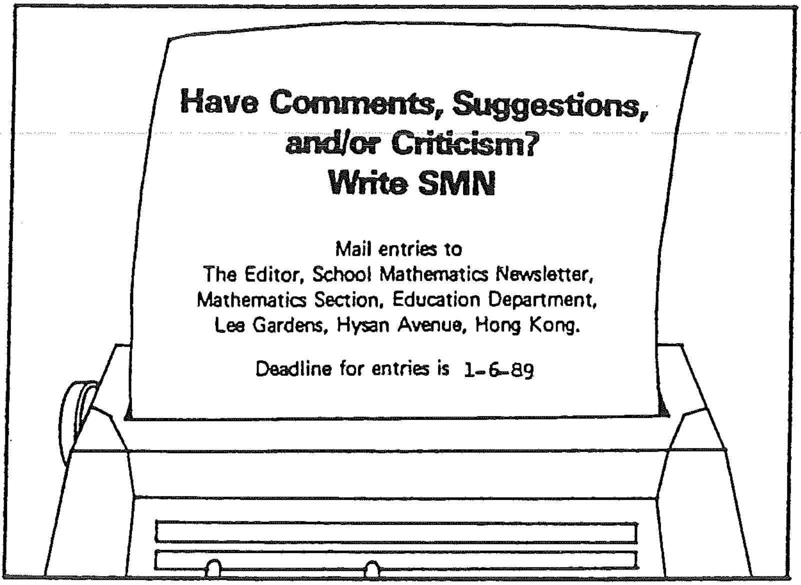
Love of five oranges

There are 120 possible ways in which the oranges might be ordered; your task is to arrange the weighings so that all possibilities are eliminated except the one which actually exists. Each result on the balance will eliminate some of the possibilities while being in accord with others. The strategy to follow is to try to pick a pair to compare at each stage so that whatever happens, the number of possibilities to check is as small as possible. For example, the first application of the balance should be such that each outcome leaves you with only 60 possibilities to check. The second application reduces the number of outstanding possibilities to 30, and so on.

To get a feel for the situation, you may wish to do the problem with three (three weighings required) or four oranges (five weighings required). How many weighings are needed for six?

Problem of the dancing pairs

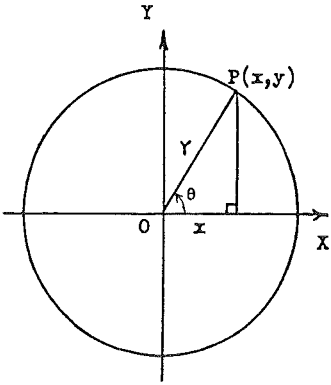
This one is quite time-consuming. There is a better solution for three dancers. Keep plugging away to try to get the absolute best solution for four and five dancers; you may then be able find a pattern for dealing with higher numbers of dancers. Be warned that this is tough.



Solving Trigonometric Equations of the Type

$a\cos\theta + b\sin\theta = c$

George Lui   Chan Shu Kui Memorial School



By definition, $\tan\theta = \dfrac{y\text{ coordinate}}{x\text{ coordinate}}$

when $\theta = 90^0$, $y = r$, $x = 0$

$\therefore \tan 90^0 = \dfrac{r}{0}$ is undefined .

So in solving $a\cos\theta + b\sin\theta = c$ by using the substitutions $\tan\theta = \dfrac{2t}{1-t^2}$, $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$ where $t = \tan\dfrac{\theta}{2}$, special care should be taken.

Although $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$, yet $\sin 180^0 = \dfrac{2\tan 90^0}{1+\tan^2 90^0}$ is absurd,

because $\tan 90^0$ is undefined.

So if $a\cos\theta + b\sin\theta = c$ is solved by using such substitutions, it is necessary to test whether $\theta = 180^0$ is a solution of the equation or not.

Example. Find the general solution of $\cos\theta + \sin\theta = -1$.

Method 1. $1\cos\theta + 1\sin\theta = -1$

$\therefore \sqrt{1^2+1^2}\cos(\theta-\alpha) = -1$

where $\tan\alpha = \dfrac{1}{1} = 1$

i.e. $\alpha = 45^0$

$\therefore \cos(\theta-45^0) = \dfrac{-1}{\sqrt{2}}$

$\theta - 45^0 = 360n^0 \pm 135^0$ where n is an integer.

$\theta = 360n^0 \pm 135^0 + 45^0$

$= \underline{\underline{360n^0 + 180^0}}$ or $\underline{\underline{360n^0 - 90^0}}$

Method 2.

By using the substitutions

$\tan\frac{\theta}{2} = t$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\tan\theta = \frac{2t}{1-t^2}$

$$\cos\theta + \sin\theta = -1$$

$$\therefore \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = -1$$

$$\therefore 1-t^2+2t = -1-t^2$$

$$2t = -2$$

$$t = -1$$

$$\therefore \tan\frac{\theta}{2} = -1$$

$$\frac{\theta}{2} = 180n^\circ - 45^\circ$$

$$\theta = \underline{\underline{360n^\circ - 90^\circ}}$$

where n is an integer.

From the above example, it is seen that if the second method is employed, only the solution $\theta = 360n^0 - 90^0$ is obtained, while the other solution $\theta = 360n^0 + 180^0$ is missing, So the complete work should be as follows:

Method 2.

By using the substitutions

$\tan\frac{\theta}{2} = t$

$\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$\tan\theta = \frac{2t}{1-t^2}$

$$\cos\theta + \sin\theta = -1$$

$$\therefore \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = -1$$

$$\therefore 1-t^2+2t = -1-t^2$$

$$2t = -2$$

$$t = -1$$

$$\therefore \tan\frac{\theta}{2} = -1$$

$$\frac{\theta}{2} = 180n^0 - 45^0$$

$\theta = 360n^0 - 90^0$ is one solution.

When $\theta = 180^0$, LHS $= \cos 180^0 + \sin 180^0 = -1$

RHS $= -1$

$\therefore$ $\theta = 360n^0 + 180^0$ is another solution

$\therefore$ the general solution is $\theta = 360n^0 - 90^0$

or $\theta = 360n^0 + 180^0$

where n is an integer.

Solving Quadratic Inequalities in One Variable

Tse Ping Nam
Ng Wah College

If students are asked to solve an inequality such as

$$3x^2 - x - 2 \geqslant 0 \text{ (or } \leqslant 0)$$

within 15 seconds or so, most of them can hardly find out the solutions without memorizing some properties or theorems, e.g. if $(x-a)(x-b) < 0$ and $a < b$, then the solution is given by $a < x < b$. This article tries to exploit a rather tricky but reliable technique which enables us to solve a standard quadratic inequality in a few seconds. Also, this technique will be found useful in graph sketching as well as tackling multiple choice problems.

Let us illustrate the idea by looking at the above inequality. Traditionally, there are three ways to solve

$$3x^2 - x - 2 \geqslant 0 .$$

Method 1 (Inequality Property)

$$3x^2 - x - 2 \geqslant 0$$

$$(3x + 2)(x - 1) \geqslant 0$$

Either $\begin{cases} 3x + 2 \leqslant 0 \\ x - 1 \leqslant 0 \end{cases}$ or $\begin{cases} 3x + 2 \geqslant 0 \\ x - 1 \geqslant 0 \end{cases}$

Either $\begin{cases} x \leqslant -2/3 \\ x \leqslant 1 \end{cases}$ or $\begin{cases} x \geqslant -2/3 \\ x \geqslant 1 \end{cases}$

The solution is $x \leqslant -2/3$ or $x \geqslant 1$

Method 2 (Theorem)

For $(x - a)(x - b) \geqslant 0$ and if $a < b$, then solution is $x \leqslant a$ or $x \geqslant b$.

Thus $3x^2 - x - 2 \geqslant 0$

$(3x + 2)(x - 1) \geqslant 0$

$[x - (-2/3)]\ (x - 1) \geqslant 0$

The solution is $x \leqslant -2/3$ or $x \geqslant 1$.

Method 3 (Tabulation)

$3x^2 - x - 2 \geqslant 0$

$(3x + 2)(x - 1) \geqslant 0$

	$x < -2/3$	$-2/3 < x < 1$	$x > 1$
$3x + 2$	−	+	+
$x - 1$	−	−	+
$(3x + 2)(x - 1)$	+	−	+

Table 1

The solution is $x \leqslant -2/3$ or $x \geqslant 1$.

The method of tabulation can be put in another way by sketching the quadratic function

$$y = 3x^2 - x - 2$$

as shown in Figure 1. The graph is divided into 3 regions and the signs of y in various regions are tabulated in Table 2, which conforms with that shown in the last row of Table 1.

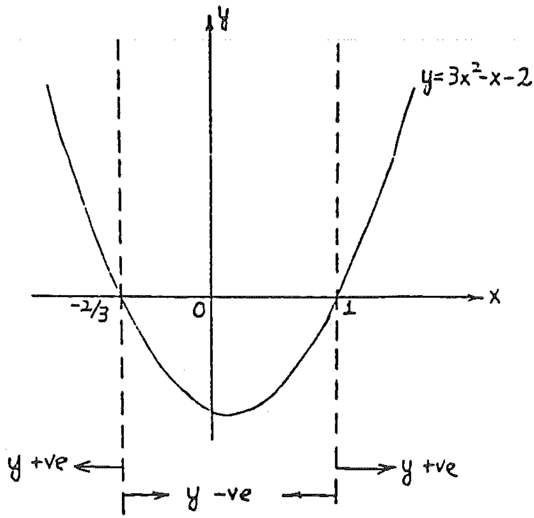


Figure 1

Sign of $y = 3x^2 - x - 2$	Region
+	$x < -2/3$
−	$-2/3 < x < 1$
+	$x > 1$

Table 2

Thus, from the above discussion, if the sign of $y = 3x^2 - x - 2$ is known for a particular region, say $x > 1$, then its nearby region must be of opposite sign, i.e. the sign of y in $-2/3 < x < 1$ is negative and hence y must have a positive sign in $x < -2/3$. This is summarized in Figure 2, from which it can be seen that not only does it provide an answer for $3x^2 - x - 2 \geqslant 0$ but also solution to $3x^2 - x - 2 \leqslant 0$, whose solution is given by the middle region $-2/3 \leqslant x \leqslant 1$.

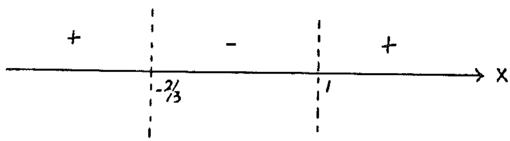


Figure 2

We are now in a position to put the above idea into practice.

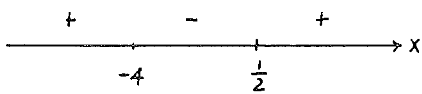
Example 1 Solve the inequality $2x^2 + 7x - 4 < 0$.

Solution Step 1 For $2x^2 + 7x - 4 = 0$

$$(2x - 1)(x + 4) = 0$$

$$x = \tfrac{1}{2} \text{ or } -4$$

Step 2 Since, for $x > \frac{1}{2}$ (e.g. $x = 100$), the sign of $(2x - 1)(x + 4)$ is positive, thus we have



The solution is $-4 \leqslant x \leqslant \frac{1}{2}$.

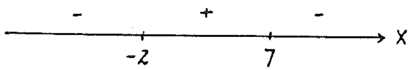
<u>Example 2</u> Solve the inequality $14 + 5x - x^2 > 0$.

Solution <u>Step 1</u> For $14 + 5x - x^2 = 0$

$$(7 - x)(2 + x) = 0$$

we have $x = -2$ or 7

<u>Step 2</u> For $x > 7$ (say $X = 100$), $(7 - x)(2 + x)$ is negative. Thus



The solution is $-2 < x < 7$.

In this example we have " − + − " instead of " + − + " . This occurs simply because of the sign change in the coefficient of x^2 − term. If we sketch the graph of $y = 14 + 5x - x^2$ as in Figure 3, it can be seen that the sign of y follows the pattern of " − + − ".

In fact one can always obtain the " + − + " pattern if the above inequality is modified into another equivalant system with positive coefficient in x^2 − term by multiplying the original system by −1, together with the necessary change of the inequality sign. This is illustrated as follows.

$$14 + 5x - x^2 > 0$$

and its equivalent form is

$$x^2 - 5x - 14 < 0$$

$$(x + 2)(x - 7) < 0$$

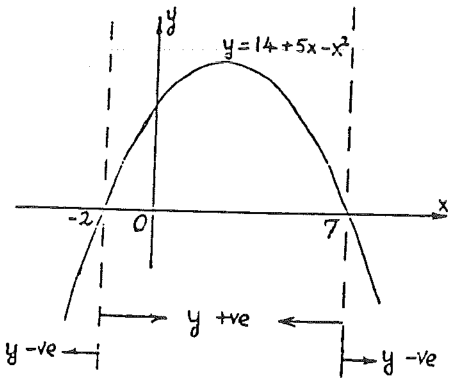


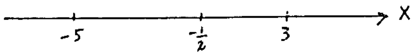
Figure 3

∴ The solution is again $-2 < x < 7$.

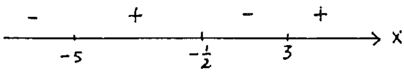
Example 3 Solve the inequality $(2x + 1)(x - 3)(x + 5) < 0$.

Solution The method introduced in this article will be found useful and convenient in solving standard inequalities of higher degrees.

Step 1 For $(2x + 1)(x - 3)(x + 5) = 0$
we have $x = -\frac{1}{2}$, 3 or -5



Step 2 For $x > 3$ (say $x = 100$), $(2x + 1)(x - 3)(x + 5)$ has a positive sign. Thus



$\therefore$ The solution is $x < -5$ or $-\frac{1}{2} < x < 3$.

A rough sketch of $y = (2x + 1)(x - 3)(x + 5)$ is then obtained as shown in Figure 4.

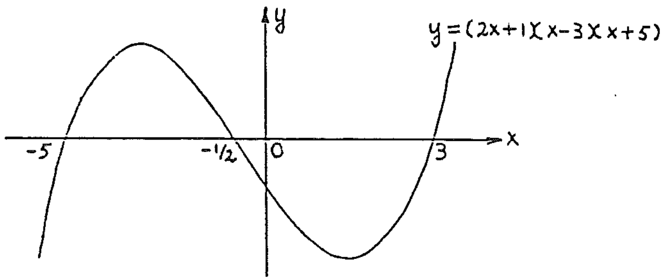


Figure 4

Extracts from Seminar Handouts on Internal Assessment for Mathematics in Secondary Schools

Mathematics Section

Advisory Inspectorate E. D.

Ia. Introduction

1. The most common types of internal assessments for Mathematics are tests and examinations.

   Tests :

   - to be administered at the conclusion of a unit or a number of units.
   - to determine whether the pupil has mastered the material and is ready to proceed to the next unit of instruction.
   - to provide information to the teacher to guide him in his instructional program.

   Examination :

   - to be administered at the end of a school term.
   - to provide an overall assessment of the pupils' achievement.

2. Procedure of test construction

   - defining test objectives
   - specifying test content
   - preparing a test blueprint
   - writing the items
   - administering the test
   - scoring the answers
   - analyzing test results

Ib. Bibliography

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2. Measurement and Evaluation in Teaching

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3. Measuring Educational Achievement

William J. Micheels &
M. Ray Karnes

4. Assessment in Schools

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6. Techniques and Problems of Assessment

H. G. Macintosh

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Secondary School Examinations
Council Examination Bulletin
No. 1

Ic. Notes on how to make a better assessment

1. Allow adequate time for preparation

Designing an assessment should not be rushed. Teachers should ensure that there is sufficient time to spend on the preparation.

2. Prepare an assessment blueprint or specification grid

The skills, abilities or knowledge which should have been acquired or developed during the period to be assessed must be clearly established. These can be tabulated and questions are to be set for each category to reflect the balance of the teaching.

Alternatively, an analysis of the assessment may be made after the questions have been written, to check that the syllabus coverage is adequate. Ideally, both procedures should be carried out.

3. Ensure a wide and balanced coverage, also an adequate variety in types and skills

Questions should be set to cover all essential areas of the teaching syllabus. This is particularly important in annual examinations. A good variety of questions can definitely help to achieve a better coverage of the syllabus. While deciding on varieties of questions, aims should be set to test on pupils' understanding of concepts and mastery of problem solving skills rather than rote memory or mechanical computations.

A possible composition of a test/examination paper is to have 30% MC items, 30% short questions and 40% long questions. In any case, the weighting for multiple-choice items should not be greater than 40%.

4. Prepare more test material than you think you will need

At a later stage, you may reject some questions and it is easier just to eliminate them rather than having to think up new ones to fill the gaps.

5. Avoid allowing a choice of questions

Allowing a choice of questions complicates everything. Pupils have to make the choice and it is often only too apparent that some choose the wrong ones and fail to do themselves justice. Moreover, when the pupils have chosen different questions, they are in effect doing different assessments and it is a very difficult task to compare the performance of one pupil with another. For a choice to be valid the questions should either all be of equal difficulty or the harder ones should carry more marks.

6. Set the assessment at a reasonable level of difficulty for the pupils

The level of difficulty should be adjusted appropriate to the ability of the pupils, i.e. even the weakest ones could find something that they can do whilst the higher ability group could find some challenging questions to make a distinction score more worthwhile. If too many pupils get zero in an examination or if a lot of them get top marks, it tells you nothing about their abilities relative to one another.

7. Do not set tricky questions

It is a mistake to think that a test/examination is made harder by including some tricky questions in which pupils have to spot a particular word or phrase which appears to be asking about something quite different. Pupils who do spot the trick may feel that they have been very clever (and even the teacher feels the same at having set it sometimes) but whatever is being tested, it is not attainment in the subject. In addition, the questions should be written in simple language and be concisely worded. Questions requiring tedious or complicated calculation should also be avoided.

8. Give clear instructions as how to answer

Instructions should be simple and complete. They should include items such as whether to answer on the examination/ test paper or on an answer sheet, whether working is required, the marks allotted to the questions and the time duration allowed for answering, etc.

9. <u>Avoid setting 'yes or no', 'true or false' questions</u>

Pupils get half of the chance of getting a correct answer by guessing. Pupils should give one or more reasons to such answers.

10. <u>Plan the layout of the assessment carefully</u>

Attention should be given to the layout of the paper so that it is easy to read. The test/examination paper should be typed if possible, and duplicated so that each pupil gets a copy. Adequate space must be allowed for the answers to be written in. Large and clear diagrams should be drawn and they should be fully labelled. Test items should be grouped according to item format, and ordered according to difficulty within each format.

11. <u>Prepare model answers and marking scheme</u>

Model answers and marking scheme should be prepared before the assessment is administered. In fact, right after the questions were written, the answers should be worked out because they may bring to light ambiguity, unrealistic expectations or other deficiencies. Alternative methods for solutions should also be worked out for this will help to gauge the difficulty of the assessment. Decide how the paper is to be marked and what each question is worth. A balanced weighting of mark allocation should be figured out appropriate to the type, length, skill involved and level of difficulty of the questions.

12. <u>Checking of the drafts</u>

After setting the paper, leave it for a few days and review it critically. It is not surprising that a number of shortcomings can be spotted like ambiguities, poor wording, unclear instructions repetitive questions, etc.

13. <u>Second opinion is helpful</u>

Ask a colleague to read through the paper for he/she may spot weaknesses that you have missed.

14. <u>Checking of stencils and duplications</u>

Stencils should be checked carefully before duplicates are made.

Proof-reading of the paper after duplication is also required to spot the misprints and errors.

15. Helping pupils prepare themselves for taking tests

Pupils should be taught how to allot time to the various questions and how to organize and present solutions logically. They should be motivated to do their best on tests and should be encouraged to review the test if time allows and change answers where necessary.

16. After the assessment, study the results carefully

Test results should be discussed with the class as a whole and/or with individual pupils. Very often more information can be obtained from an assessment than just the total marks. An analysis of the results will show ways in which the test items can be revised and the instructional strategy can be improved.

17. Check List for the planning stage in preparing classroom tests

(1) What is the purpose of the test? Why am I giving it?

(2) What skills, knowledge, attitudes, etc. do I want to measure?

(3) Have I clearly defined my instructional objectives in terms of student behaviour?

(4) Have I prepared a table of specifications?

(5) What kind of test (item format) do I want to use? Why?

(6) How long should the test be?

(7) How difficult should the test be?

(8) How will I arrange the various item formats?

(9) How will I arrange the items within each item format?

(10) What do I need to do to prepare students for taking the test?

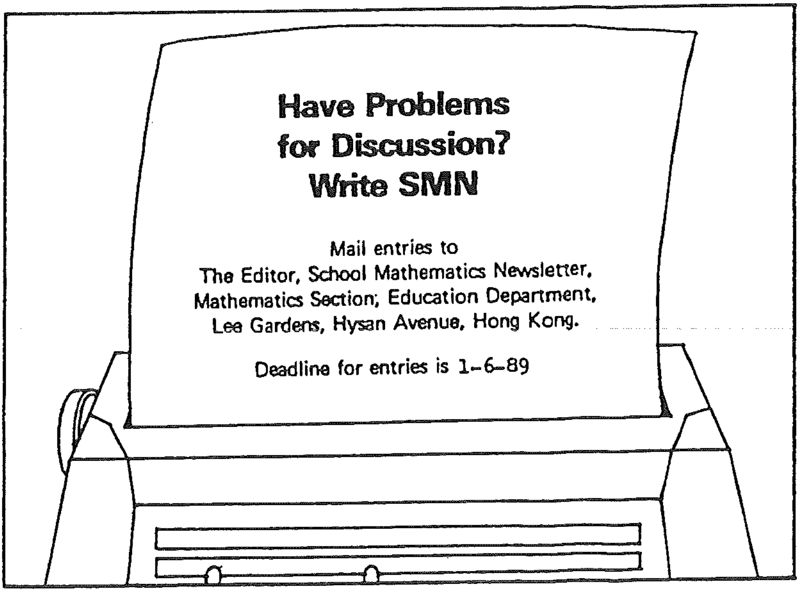
(11) How are the pupils to record their answers to the multiple-choice items? On separate answer sheets? On the test paper?

(12) How is the multiple-choice portion to be socred? Hand or machine?

(13) How is the conventional portion to be scored? With the

help of a model answer and marking scheme?

(14) How are the test results to be reported?



II. Multiple-choice Items

1. Introduction

A multiple-choice item consists of a problem and a list of suggested solutions. The problem may be stated in the form of a direct question or an incomplete statement and is called the stem of the item. The list of suggested solutions may include words, numbers, symbols or phrases and are called options. The pupil is typically requested to read the stem and the list of options and to select the one correct, or best, option. The correct option in each item is called the key while the remaining options are called distractors.

The instructions for a multiple-choice paper should be clearly stated. An example is given below:

This examination consists of 40 items on 7 pages. Each item carries 2 marks. Try to answer every item, but do not spend too much time on any one item. For each item, darken the box corresponding to the option you choose on the answer sheet. Darken only one box per item.

2. Advantages of multiple-choice items

(1) The multiple-choice type tests provide an extensive sampling of course content, due to the large number of questions that can be included.

(2) Multiple-choice items can measure pupils' ability to recall knowledge as well as their ability to apply knowledge.

(3) The nature of the incorrect options selected by pupils provides diagnostic information.

(4) Scoring can be quickly, easily, consistently and more reliably done.

3. Limitations of multiple-choice items

(1) Preparation is difficult and time-consuming.

(2) Multiple-choice items are subject to guessing.

(3) Multiple-choice items are inappropriate for measuring the ability to organize and present solutions, also inappropriate for measuring some problem solving skills.

4. <u>Some suggestions for writing the multiple-choice items</u>

(1) The item stem should be meaningful by itself and should present a definite problem. It should be unambiguous in wording and concept.

(2) Each item should have a single objective.

(3) The item should consists of simple data to avoid tedious manipulation.

(4) The item stem should be in simple wording and should contain words or phrases which would otherwise have to be repeated in each option.

(5) The item should be stated in positive terms.

(6) The item should contain only one correct or clearly best answer.

(7) The item should not contain overlapping options.

(8) The stem should be free of clues to the correct answer. Also, overlapping items should be avoided because the information presented in one item may provide a valuable clue for answering another item.

(9) All distracters should be plausible. Each distracter should attract some pupils with certain misconceptions, otherwise it makes no contribution to the functioning of the item and should be eliminated or revised.

(10) Options such as "none of the above" or "all of the above" should be used sparingly.

(11) The correct answer should appear in each of the option positions approximately an equal number of times, but in random order. A simple way to obtain a random placement of the correct answer is to place all verbal options in alphabetical order and all numerical answers in numerical order.

(12) Items used to measure understanding should contain some novelty. The problem situations must be new to the pupils but not too far removed from the illustrative examples used in class.

(13) Use four or five options.

5. Check list for writing multiple-choice items

(1) Has the item been clearly and concisely presented? Is the main problem in the stem?

(2) Does the item have a single objective?

(3) Are the data sufficient and simple?

(4) Has the item been cast so that there is no repetition of key words or phrasee for each option?

(5) Have all irrelevant clues been avoided?

(6) Have negative statements been avoided? If used, has the negative been underlined or written in capital letters?

(7) Is there only one correct (or best) answer?

(8) Are all distracters plausible?

(9) Have overlapping options been avoided?

(10) Has "all of the above" been avoided?

(11) Has "none of the above" been used only when appropriate?

(12) Are the correot answers randomly assigned throughout the test with approximately equal frequency?

III. Setting a conventional paper

When the purpose of the test/exam. has been defined and the test plan developed, the next stage is question writing. There are several forms which questions can take. Questions can range in form from the open-response requiring extended written answers, such as long questions, to objectively-scored questions such as fill-in-the-blank items and short-answer items. A middle ground between the two forms is represented by the highly structured or 'programmed' form of open-response questions.

Common types of questions

IIIa. Fill-in-the-blank items/Questions requiring answers only

This form of questions has a space(s) on which the student writes the answer to the question.

Principles in writing the items

(1) The question should require answer(s) consisting of a unique word, phrase or number.

(2) For computational problems, the degree of precision or unit expected in the answer should be specified.

Advantages

(1) It measures knowledge of facts efficiently and may also tap high levels of reasoning such as required in inference and organisation of material.

(2) In junior forms, a well set fill-in-the blank question can remedy the students' weak presentation of solution.

Limitation

(1) Since no intermediate step is required, a student may get the answer by guessing or the result will be biased by the students' careless working.

(2) It is less suitable for measuring complex learning outcomes.

IIIb. Short-answer iteme/short questions

It is an item in which the answer can be got with few intermediate steps.

Principles in writing the items

(1) The question should be simple and direct.

(2) It is used for testing one or two concepts.

Advantages

(1) It is suitable for measuring a wide variety of relatively simple learning outcomes and measuring problem-solving ability.

(2) The pupil must supply the steps together with the answers. This reduces the possibility of guessing the answers.

Limitation

It is less suitable for measuring complex learning outcomes.

IIIc. Long questions

In writing long questions, structured questions are preferred. In a structured question, there is normally a stem in which the necessary information and the purpose, as a whole, of the set of questions that follow are given. The situation is broken down into simple components to provide students with a set of questions. Each of the set of questions is related to the information in the stem but not necessarily to each other. In fact, it is better if the answer to a question in a structured set is not dependent on a correct answer to a previous question in the set.

Limitation

(1) A test containing long questions provides limited content sampling. Some pupils do better on some questions while others do better on others. Thus, a pupil's score will depend to some extent on the particular questions asked. The more questions, the less likely a pupil's score will suffer because of inadequate sampling of content and the greater the likelihood that the test will be reliable.

(2) Another serious limitation is their low scoring reliability. However, variations in scoring can be minimized by careful construction of the questions and by setting up specified scoring procedures.

(3) Scoring answers to long questions is very time consuming and laborious.

Principles in writing long questions

(1) Present the material in the stem as concisely as possible. The material should be presented in the most appropriate form. If diagrams, graphs, tables, etc. are necessary to clarify the information, they should be provided. The set of questions should be sufficiently specific and detailed so that the students know precisely what are required and what direction their answers should take.

(2) The problem should be well structured with the related questions written in order of increasing complexity and/or difficulty. Beginning with simpler and/or easier questions

has a motivating effect on the students. Also, this helps to avoid students spending a disproportionate amount of time on the more complex/difficult questions encountered early in the problem and being forced to omit later questions which they could easily have answered.

(3) The complexity of the question and the length of the expected solution should be reasonable. The question should be framed to measure the intended objective and be presented, wherever feasible, in a manner unfamiliar to the students to test if they can apply the concepts/skills learned to a new situation.

Advantages

(1) Though constructing well structured long questions requires considerable time and effort, it is relatively easier to prepare a test containing long questions than to prepare a comparable multiple-choice test.

(2) Long questions are especially appropriate for measuring the pupil's ability to synthesize his ideas and express them logically and coherently in written form. In addition, long questions tend to have a desirable influence on pupils' study habits.

IIId. Check list for writing short-answer questions

1. Are the instructional objectives clearly defined?
2. Did you prepare a test blueprint? Did you follow it?
3. Did you formulate well-defined, clear and unambiguous questions?
4. Did you write the questions in simple language?
5. Did you state all necessary conditions/data?
6. Did you include necessary illustrative diagrams?
7. Did you avoid giving clues to the correct answer? For example, grammatical clues, length of correct response clues?
8. Did you test for the important concepts/skills rather than the trivial?
9. Did you adapt the test's difficulty & length to your students?
10. Did you indicate the degree of precision required? Whether or not the unit of measurement is to be included in the answer?

11. Was each question set on a different teaching point?
12. Did you prepare a scoring key?
13. Did you review your questions?
14. Is this format most effioient for testing the instructional objectives?

IIIe. <u>Check list for writing long questions</u>

1. Are the instructional objectives clearly defined?
2. Was a test blueprint prepared? Was it followed?
3. Is the question restricted to measuring objectives that would not be assessed more efficiently by other item formats?
4. Does each question relate to some instructional objective?
5. Does the question establish a framework to guide the student to the expected answer?
6. Is each question composed of a set of short-answer questions?
7. Are the questions realistic in terms of
   a. difficulty?
   b. time allowed the student to respond?
   c. complexity of the task?
8. Were the important concepts/skills rather than the trivial tested?
9. Were the questions written in simple language?
10. Were all necessary conditions/data provided in each question?
11. Were necessary illustrative diagrams provided?
12. Are the questions novel? Do they challenge the student?
13. Are all students expected to answer the same questions?
14. Has a model answer with marking scheme been prepared for each question?
15. Did you review your questions? Yourself? Another teacher?

EVALUATING CLASSROOM TESTS/EXAMINATIONS

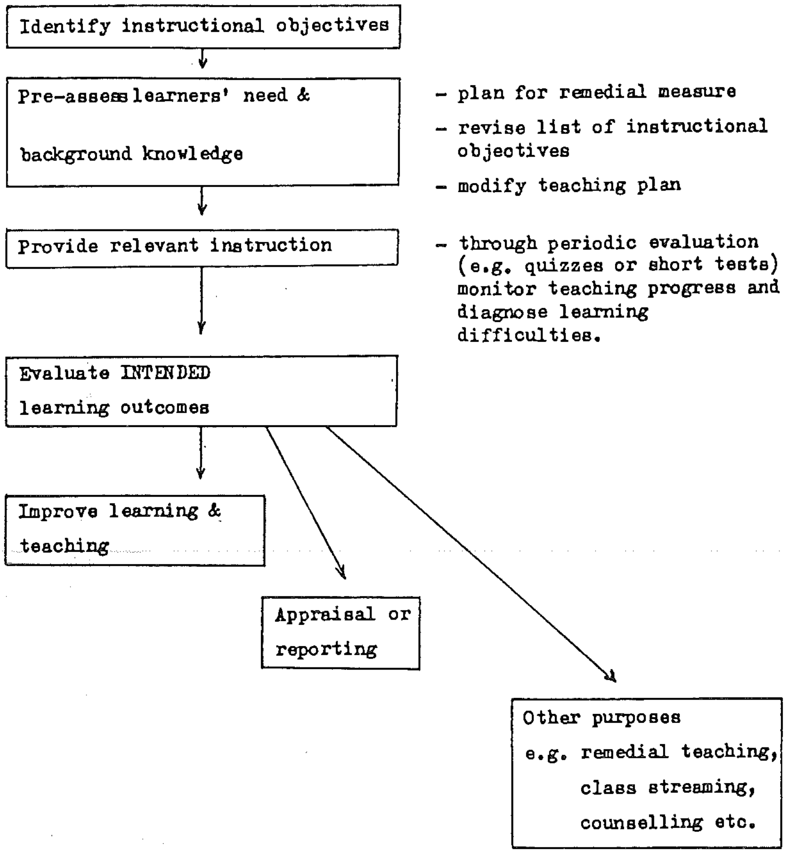
The ultimate purpose of testing, as with all classroom procedures, is to improve pupil learning. Thus, any classroom test we construct should be evaluated in terms of the extent to which it contributes, directly or indirectly, towards this end.

I. A SIMPLIFIED LIST OF LEARNING OUTCOMES

1. Knowledge/understanding
   - terminology
   - speoific facts
   - concepts & principles
   - methods & procedures
2. Application
   - factual information
   - concepts & principles
   - methods & procedures
   - problem-solving skills
3. General skill
   - computational skills
   - performance skills
4. Thinking skill
   - critical
     - distinguish between facts & opinions
     - identify assumptions underlying conclusions
     - identify limitation of given data
   - logical
     - draw valid conclusions
     - generalize
5. Attitude
   - social attitude
   - soientific attitude
6. Interest
   - personal interest
   - educational/vocational interest

II.

A Simplified Instructional Model



III. General considerations in evaluating tests/examinations

The purpose of evaluating tests/examinations is to find out whether they function as intended. This could be done by analyzing the pupils' responses and the following aspects should be considered :

- Did the questions adequately measure the effects of the instruction?
- Were the questions of appropriate difficulty?
- Did the questions adequately discriminate between high and low achievers?
- Were the questions free of defects?
- In case of multiple-choice items, were each of the distracters effective?

IV. Values of evaluating the effectiveness of the test/examination questions?

- Helps in selecting or revising questions for future use and leads to increased skill in test construction.

  As we analyze pupils' responses to items, we become increasingly cognizant of technical defects and the factors causing them. During revision of the items, we obtain experience in rewording statements so that they are unambiguous, rewriting distracters so that they are of a more appropriate level of difficulty.
- Provides a basis for efficient class discussion of the test results.

  Easy items that were answered correctly by all pupils can be omitted from the discussion, and the concepts in those questions causing pupils the greatest difficulty can receive special emphasis. Also, misinformation and misunderstandings reflected in the choice of particular distracters can be corrected.
- Provides a basis for remedial work.

  Analyzing pupils' responses brings to light general areas of weakness which require more extended attention.
- Provides a basis for the general improvement of classroom instruction.

  Materials which are consistently too difficult for the pupils might suggest curriculum revisions or shifts in teaching emphasis. Similarly, common errors among pupils might direct attention to the need for more effective teaching procedures.

V. Suggested evaluating procedures :

Data from item analysis and other statistical procedures will aid the teacher in making judgements in the evaluation process. However, these procedures will not be discussed in this seminar owing to limited time. What we are going to suggest here are some simple and quite practical evaluating procedures.

(a) MC Items Response Chart

The structure of this chart is self-explanatory. Pupils may enter the scores themselves. This chart offers a two-fold perspective on evaluating paper as well as the ability of the pupils. The performance of individual pupil as well as the scoring of an individual MC item can be studied very easily from the chart. Pupils' competence or weakness in respective content area are displayed 'diagrammatically'. Irregularities observed can be further studied by the suggested method V(b).

Question no. / Pupils (Name or No.)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	. . .
1	✓		✓																	
2	✓	✓	✓	✓																
3	✓	✓																		
4	✓	✓	✓	✓																
5		✓	✓																	
6	✓	✓																		
7	✓		✓																	
8		✓	✓																	
9		✓	✓																	
10			✓																	
11	✓	✓	✓	✓																
12			✓																	
13		✓	✓																	
14	✓	✓	✓	✓																
15	✓		✓																	
16		✓	✓																	

V. (b) Examining the Effectiveness of the Item : (for MCQs)

e.g.1	A	B	C*	D	E	omits
	Options					
Upper group # frequencies	0	0	20	0	0	0
Lower group frequencies	4	2	8	3	3	0

e.g.2	A	B*	C	D	E	omits
	Options					
Upper group frequencies	1	19	0	0	0	0
Lower group frequencies	5	9	6	0	0	0

* The key of the item.

\# The upper group usually consists of the best 27% while the lower group the poorest 27% of the papers. However, in case of small classes, it is better to have the top and bottom halves for more reliable judgement.

Irregularties as spotted in V(a) or items which the teacher finds interested in may be examined more closely by this method. By spreading out the scoring of respective options, the effectiveness of the distractors can be studied.

(c) Impression Report for Conventional Questions

A convenient method for markers to jot down his fresh impression on the paper briefly on the question paper itself. Pupils' common mistakes are depicted for teachers' information.

Extracts from Seminar Handouts on Activities for Mathematics in Secondary Schools

Mathematics Section

Advisory Inspectorate E. D.

(a) Fibonacci Sequence and the Golden Section Ratio

(b) Activities on Tessellation

(c) Tower of Hanoi

(d) Frog

(e) Pentominoes

(f) Flexagons

(g) Möbius Strip

(h) List of books

(a) Fibonacci sequence and the golden section ratio

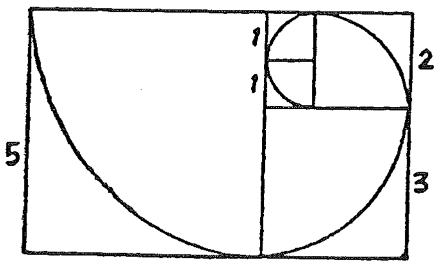
The sequence : 1, 1, 2, 3, 5, 8, 13, ... where the next number is formed by adding together the previous two number is commonly known as the *Fibonacci sequence.

Although the Fibonacci sequence is not of any great importance in pure mathematics, the fact that it has been found to occur both in nature and in art is paradoxical enough to warrant investigation.

The number sequence occurs in nature in many surprising ways. If you look at the scales on a pine cone, you will see that they appear to spiral around the cone. Count the number of such spirals and you will always find it is equal to one of the numbers in the Fibonacci sequence.

Similarly the seeds in a sunflower head also lie on spirals and the number of spirals will again be a Fibonacci number. This spiral is known in mathematics as a 'logarithmic spiral'. Remarkably enough, it is just the kind of spiral frequently found in the arrangements of seeds in many flowers, in the shells of snails and other animals.

The rectangle construction in 'spiralling squares' gives a very practical way of drawing a logarithmic spiral by a quadrant of a circle in each new square which is added.



Now, consider the ratio of any term to the succeeding term of a Fibonacci sequence. Use your calculator to express the ratios in decimal form.

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*The sequence is named after Fibonacci, an Italian mathematician of the 13th Century. In fact, a Fibonacci sequence is generated by taking any two numbers as a starting point and using the rule of always adding the last two numbers to obtain the next number.

For example starting with 2 and 9 :

Fibonacci sequence :

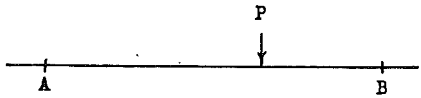
2 9 11 20 31 51 82 133 ...

successive ratio :

$\frac{9}{2}$ $\frac{11}{9}$ $\frac{20}{11}$ $\frac{31}{20}$ $\frac{51}{31}$ $\frac{82}{51}$ $\frac{133}{82}$

No matter what numbers you start with you should have found that the ratios appear to be always getting closer and closer to a number which starts with 1.61803.....

This number was studied in a couple of thousand years by the Greeks in a geometrical context. They wanted to divide a line segment AB at a point P so that the ratio AP : PB equalled AB : AP. If we let AP : PB be x and since AB = AP + PB, then

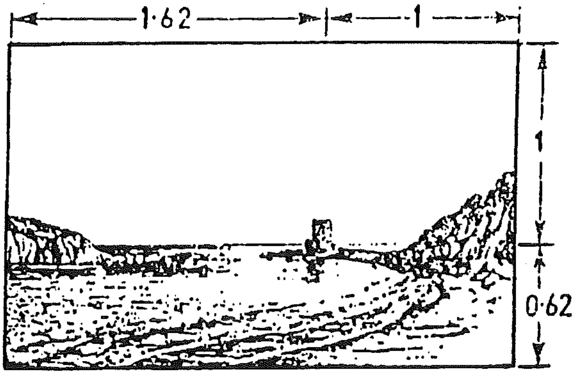


$$x = \frac{AP + PB}{AP} = 1 + \frac{PB}{AP} = 1 + \frac{1}{x}$$

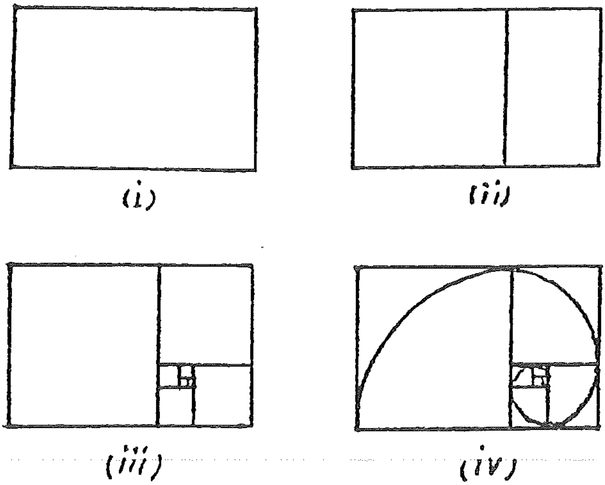
giving $x^2 - x - 1 = 0$, from which x equals to a positive value 1.6180339.....

This ratio is called the golden section ratio and its precise value is $\frac{1}{2}(1 + \sqrt{5})$.

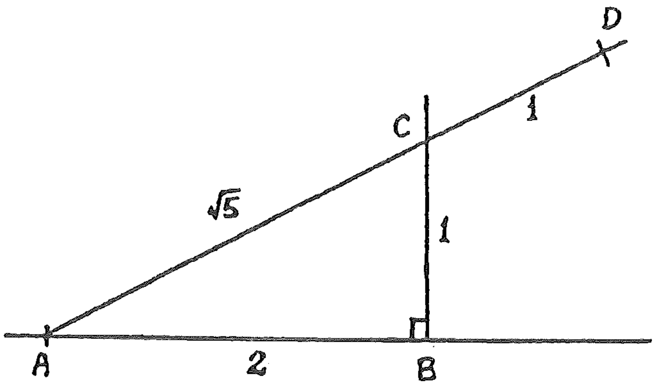
Pychologists have done experiments which suggest that people find the most pleasing rectangular shape to be that with its sides in the golden section ratio. Artists too have been fascinated by this ratio as architects and used it in the design of their pictures or buildings.



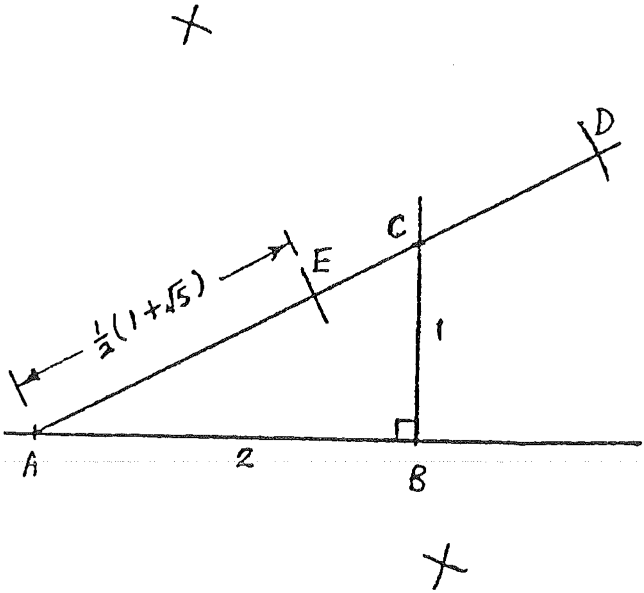
Interestingly if you start with a rectangle whose sides are in the golden section ratio then cutting a square off it leaves a smaller rectangle whose sides are also in the golden section ratio. The figure below shows the result of the continued division of each successive rectangle into a square and a rectangle. Amazing enough, the curve inscribed in the successive squares is again a logarithmic spiral!

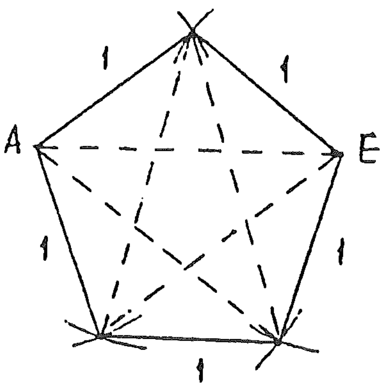


Interestingly too the golden section ratio is the ratio of a diagonal to the side of a regular pentagon and this fact makes it possible to contruct a regular pentagon using only a pencil, ruler and compasses only. The key is to construct a length equals to $\frac{1}{2}(1 + \sqrt{5})$. This can be done as follows :



Construct a right-angle and use your compasses as a pair of dividers to mark off AB equals to 2 units and BC equals to 1 unit. Join A to C and extend. By Pythagoras' theorem AC = $\sqrt{5}$ units. Use the compasses to mark off D, 1 unit from C, then AD is $1 + \sqrt{5}$. Half the length of AD will give the length of $\frac{1}{2}(1 + \sqrt{5})$. The rest should be straightforward.





(b) Activities on Tessellations

(A1) Equipment : tracing papers, thin cards, scissors.

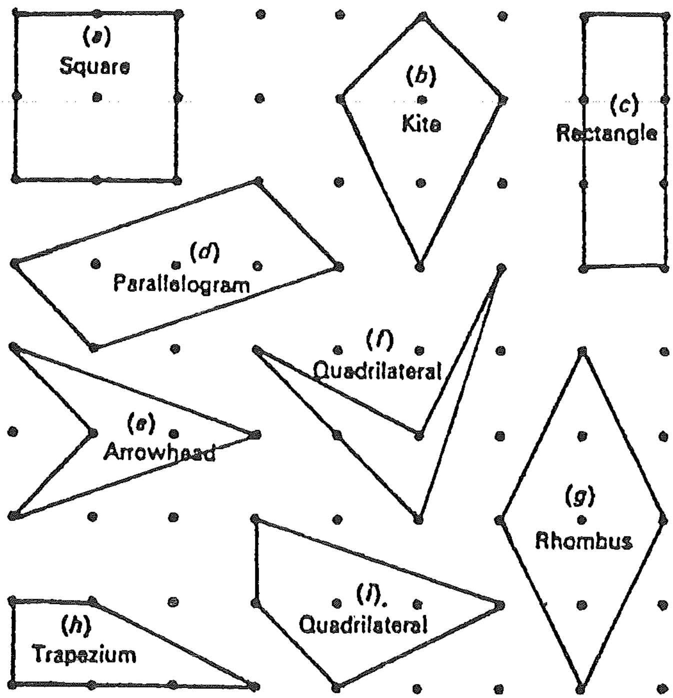
Draw an equilateral triangle. Use tracing paper to help cutting out more identical equilateral triangles. Use the triangles to form a tiling pattern. Repeat the whole process by drawing any triangle.

Points to think about :

(a) What size (in degrees) is each angle of an equilateral triangle?
(b) How many triangles fit around each vertex of the pattern without a gap or an overlap?
(c) How many degrees are there in one whole turn?
(d) What is the connection between the answers to (a), (b) & (c)?
(e) Can any triangle tessellate?

(A2) Equipment : pinboards or spotty papers

Try to form tiling patterns on pinboards or spotty papers from each of the following quadrilaterals.



Points to think about :

(a) What size (in degrees) is each angle of a square?

(b) How many squares fit around each vertex of the pattern without a gap or an overlap?

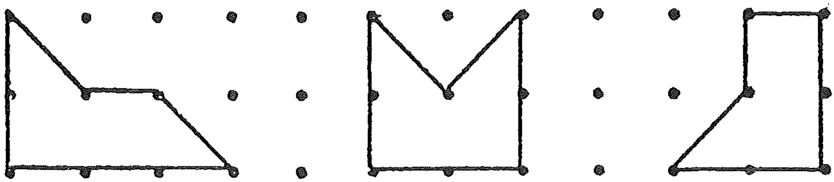
(c) How many degrees are there in one whole turn?

(d) What is the connection between answers to (a), (b) & (c)?

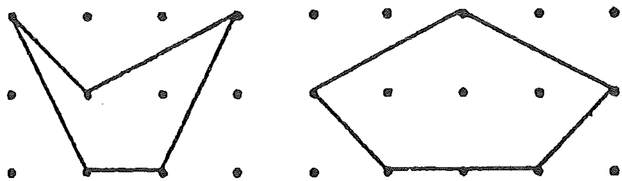
(e) Can any quadrilateral tessellate?

(A3) Equipment : pinboards or spotty papers, tracing papers, thin cards scissors.

Cut out some identical regular pentagons. Try to see whether the pentagons can tessellate to form a pattern. Repeat the activity with other pentagons on pinboards or spotty papers.



(a) (b) (c)



(d) (e)

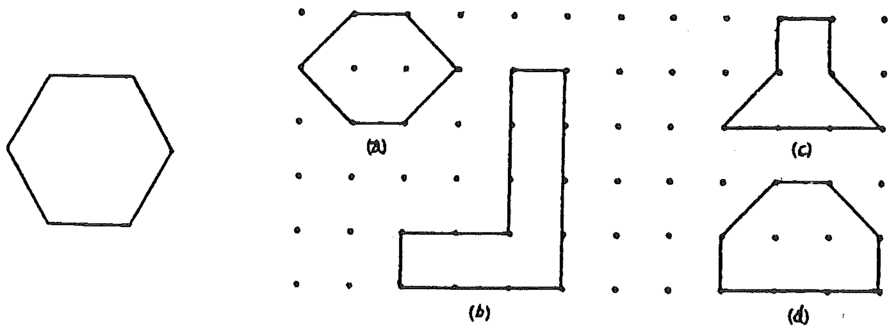
What size (in degrees) is each angle of a regular pentagon, and is it a factor of 360°?

Can regular pentagons tessellate?

(A4) Equipment : pinboards or spotty papers, tracing papers, thin cards, scissors.

Cut out some identical regular hexagons. Try to see whether the hexagons can tessellate to form a pattern.

Repeat the activity with other hexagons on pinboards or spotty papers.



What size (in degrees) is each angle of a regular hexagon, and is it a factor of 360°?

Can regular hexagons tessellate?

(A5) Equipment : tracing papers, thin cards, scissors.

Cut out some identical equilateral triangles, squares and regular hexagons. Think about :

(a) What are the factors of 360°?

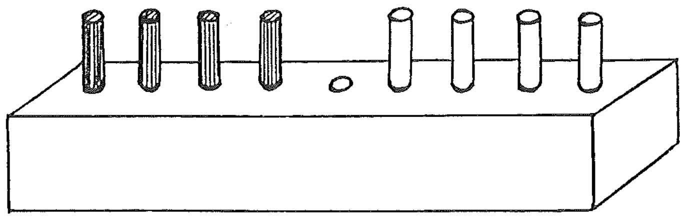
(b) Which of the factors is the number of degrees in an angle of a regular polygon?

(c) List out some combinations of the numbers obtained in (b) so that the sum in each combination is 360.

Try to form tessellations of combinations of polygons according to the results obtained in (c).

(d) FROG

'Frog' is a game relevant to the teaching of mathematical induction. Nevertheless, the game may be treated solely as pastime. Rules and winning strategies are supplied to facilitate quick mastery of the game.



Target : To interchange the positions of two sets of pegs.

Rules : 1. All pegs may move only towards the opposite side and not backwards.

2. A peg may move one place into an empty hole, or it may jump over a peg of the other set into an empty hole.

It has been found out that the minimum number of moves is $n(n+2)$, where n is the number of pegs in a set.

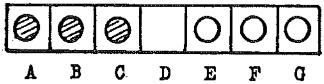
n	3	4	5	6
No. of moves	15	24	35	48

During the game, pupils may be led to discover the formula and hence a better learning atmosphere can be created for the teaching of mathematical induction.

(Winning strategies on separate sheets)

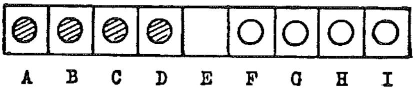
'FROG – SUGGESTED WINNING STRATEGIES

1. n = 3



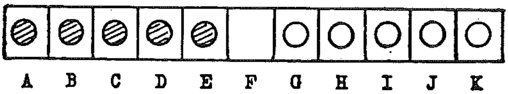
C → D	E → C	F → E	D → F	B → D	A → B	C → A
E → C	G → E	F → G	D → F	B → D	C → B	E → C
D → E						

2. n = 4



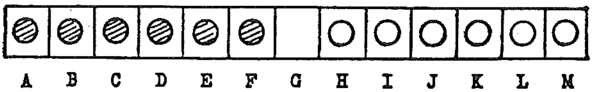
D → E	F → D	G → F	E → G	C → E	B → C	D → B
F → D	H → F	I → H	G → I	E → G	C → E	A → C
B → A	D → B	F → D	H → F	G → H	E → G	C → E
D → C	F → D	E → F				

3. n = 5



E → F	G → E	H → G	F → H	D → F	C → D	E → C
G → E	I → G	J → I	H → J	F → H	D → F	B → D
A → B	C → A	E → C	G → E	I → G	K → I	J → K
H → J	F → H	D → F	B → D	C → B	E → C	G → E
I → G	H → I	F → H	D → F	E → D	G → E	F → G

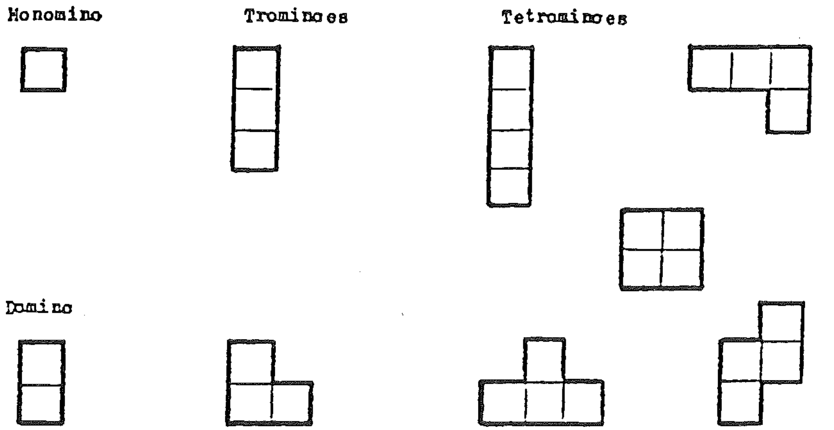
4. n = 6



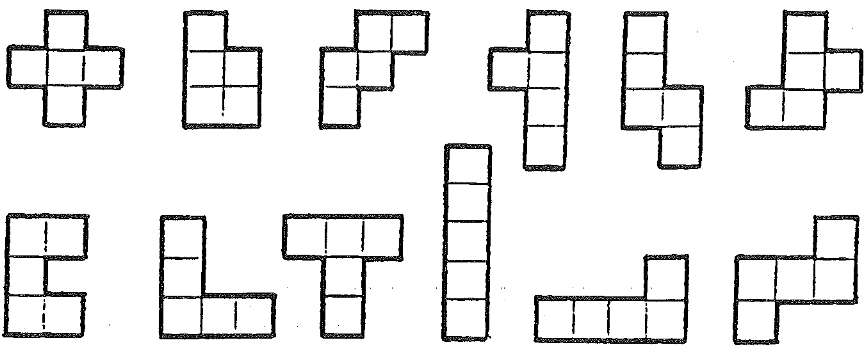
F → G	H → F	I → H	G → I	E → G	D → E	F → D
H → F	J → H	K → J	I → K	G → I	E → G	C → E
B → C	D → B	F → D	H → F	J → H	L → J	M → L
K → M	I → K	G → I	E → G	C → E	A → C	B → A
D → B	F → D	H → F	J → H	L → J	K → L	I → K
G → I	E → G	C → E	D → C	F → D	H → F	J → H
I → J	G → I	E → G	F → E	H → F	G → H	

(e) The Pentominoes

Polyominoes are shapes built up of squares connected edge-to-edge. The first five members of the polyomino family, namely the monomino, domino, trominoes, tetrominoes and pentominoes are illustrated below :



Pentominoes



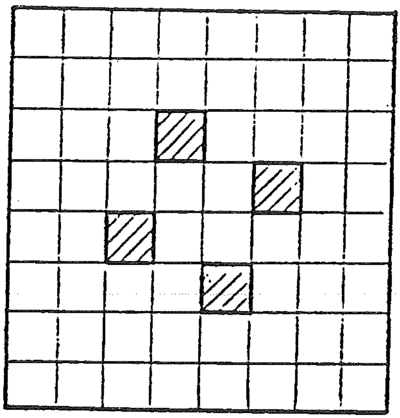
You may notice that there is only one type of domino, two trominoes, five tetrominoes and twelve pentominoes. Asymmetrical pieces, which have a different shape when 'turned over', are considered as identical types.

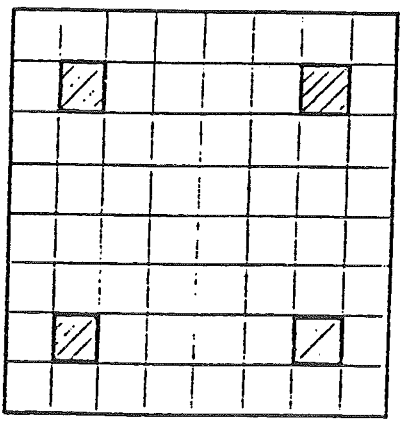
Cut for yourself a set of pentominoes from a thick card and see if you can find ways of fitting them together to form a 10 x 6, 12 x 5, 15 x 4 or 20 x 3 rectangle. There are thousands of solutions altogether but you can be congratulated if you find one for each shape.

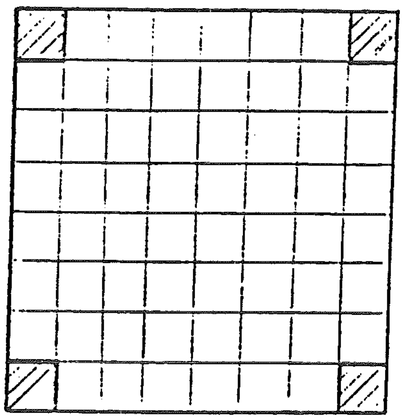
For more challenging ways of playing with the pentominoes, consider the followings :

(1) Will the 12 pentominoes, together with a 2 x 2 square tetromino, form an 8 x 8 chessboard? HOW?

(2) If the square tetromino is replaced by four monominoes, can you form an 8 x 8 chessboard with the monominoes at the positions indicated by the shaded regions in each of the following diagrams?







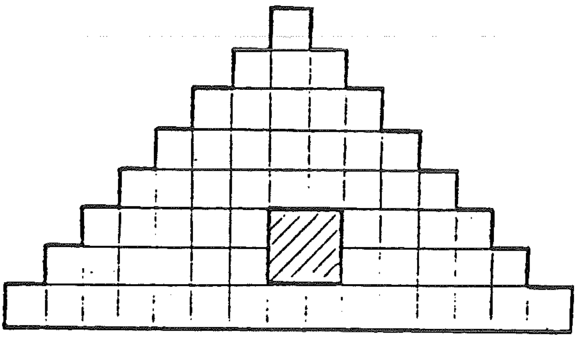
(3) 'The Triplication Problem' - Select one pentomino, then use 9 of the remaining ones to form an enlarged shape of the chosen piece. The enlarged shape will be three times higher and wider than the chosen one.

(4) 'The Double-double Problem' - Form any desired shape with 2 pentominoes. Duplicate it with two other pieces. Then use the remaining 8 pieces to form the same shape but twice as large.

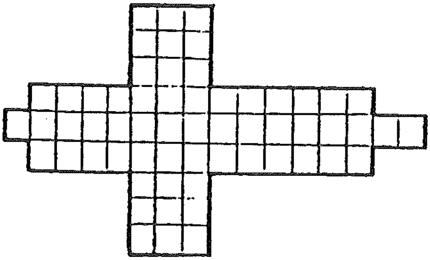
(5) Divide the 12 pentominoes into three groups of 4 each. Find a 20-square shape that each of the three groups will cover.

(6) Divide the 12 pentominoes into three groups of 4 each. Subdivide each group into two pairs of shapes. For each group find a 10-square shape that each of the two pairs will cover.

(7) Can you form the 64-square pyraimd shown below with the 12 pentominoes and a 2 x 2 square tetromino?

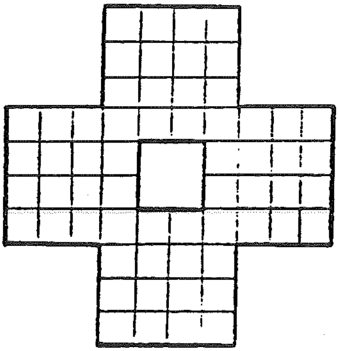


(8) Can you form the 'Cross' shown below with the 12 pentominoes?

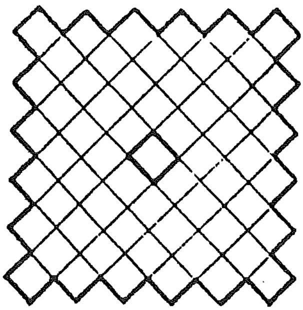


There are patterns which are still unsolved. They are neither constructed nor proved impossible. Below are two of them for you to challenge your pupils' mentality.

(1)

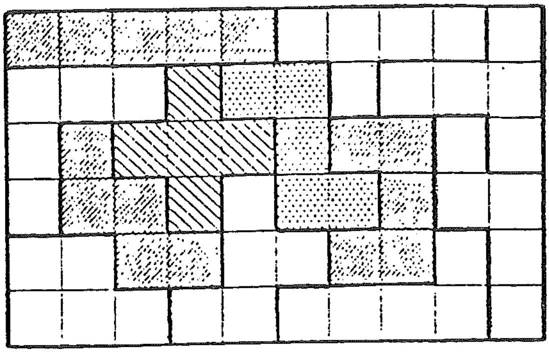


(2)

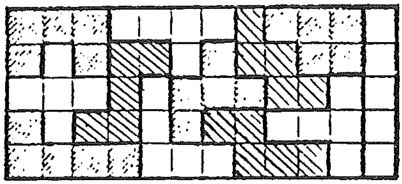


Solutions

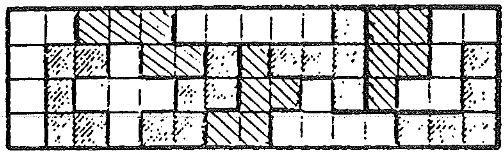
10x6



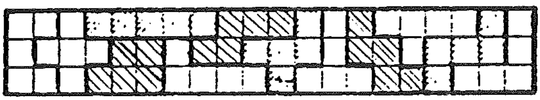
12x5



15x4

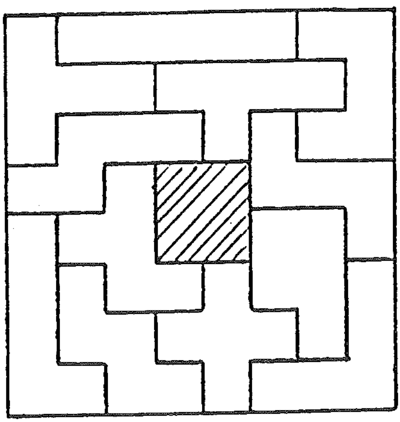


20x3

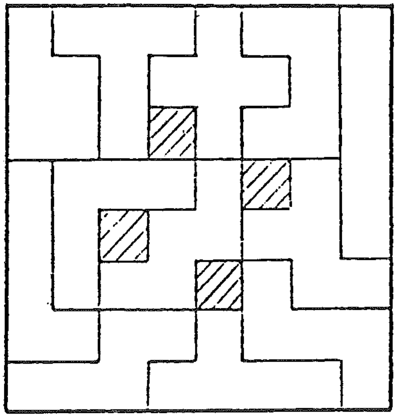


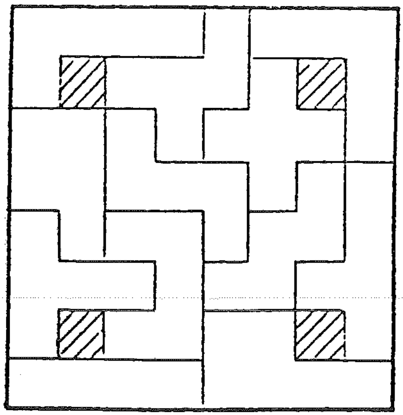
Problem (1)

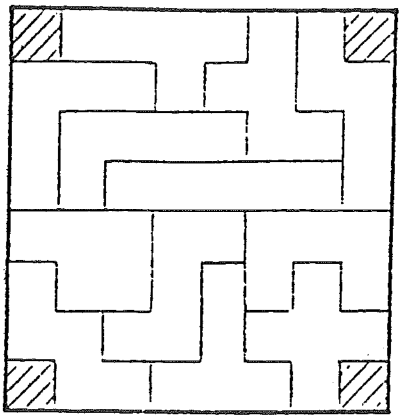
8x8 chessboard



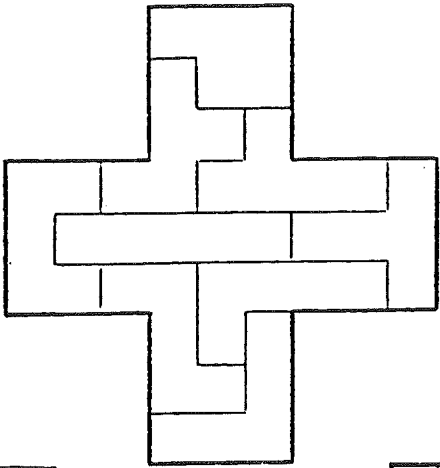
Problem (2)



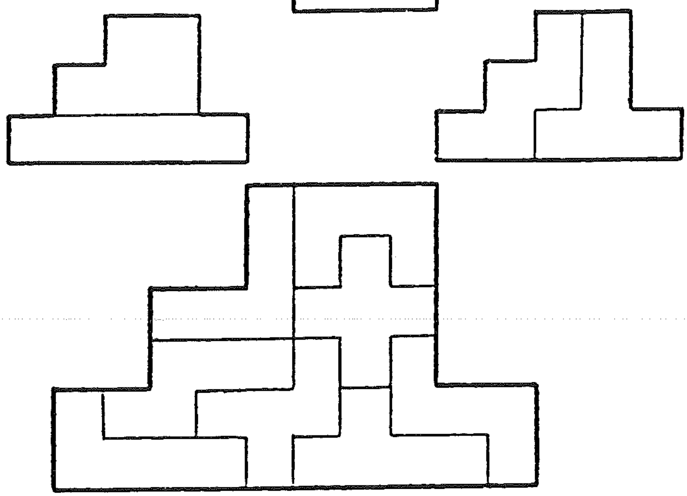




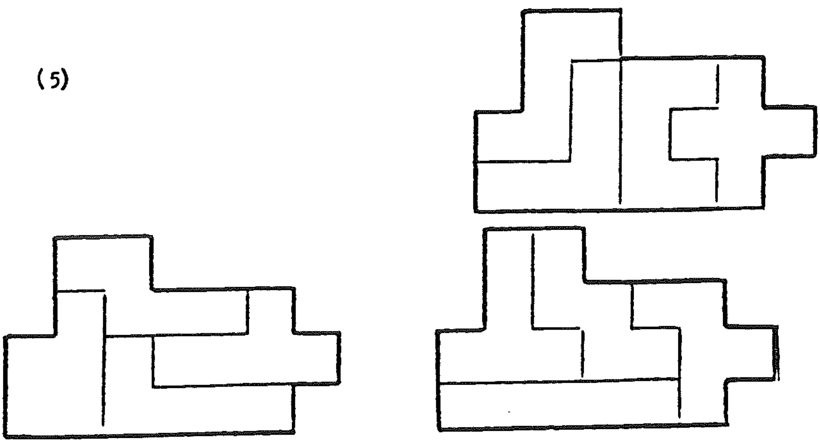
(3)



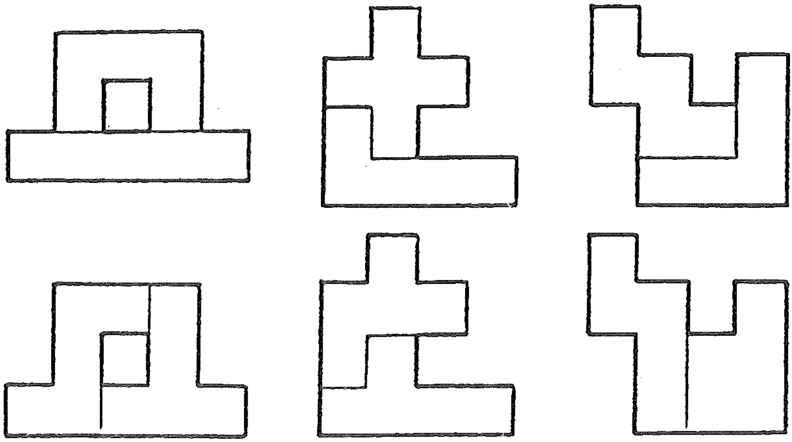
(4)



(5)



(6)



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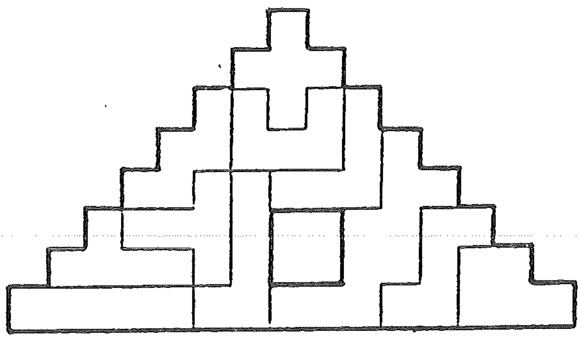

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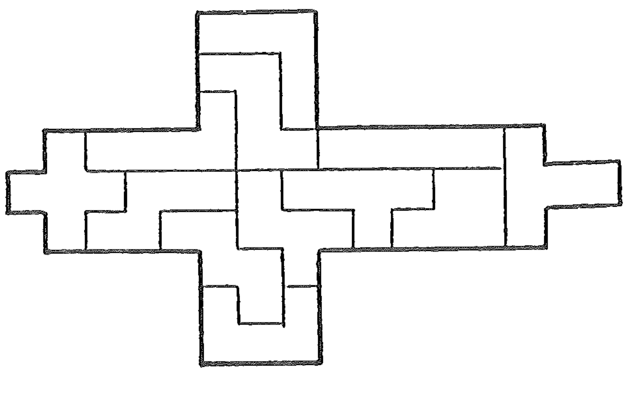
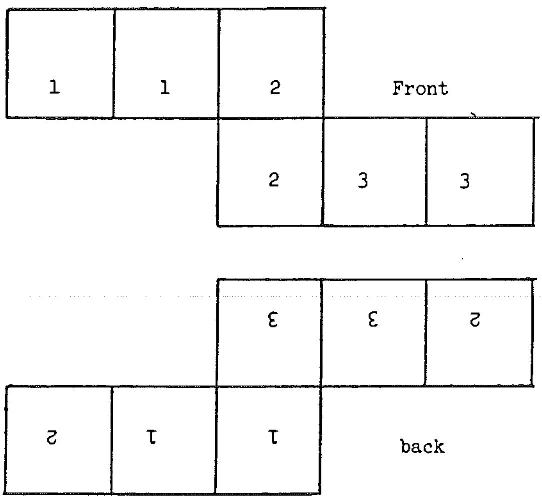


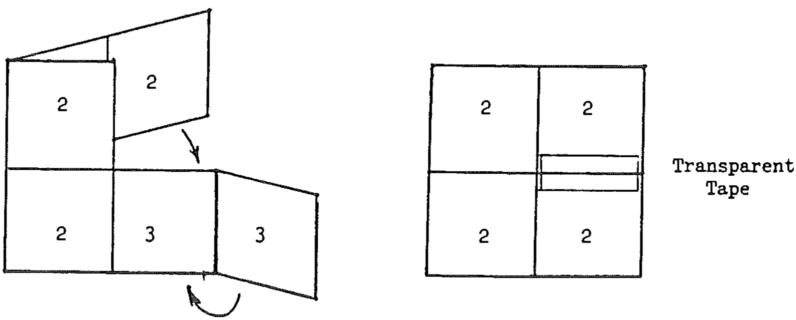
(f) The Magic of flexagons

Flexagons are paper polygons, folded from straight or crooked strips of paper, which have the fascinating property of changing their faces when they are flexed.

1. Tri-tetraflexagon (three-faced four-sided flexagon)

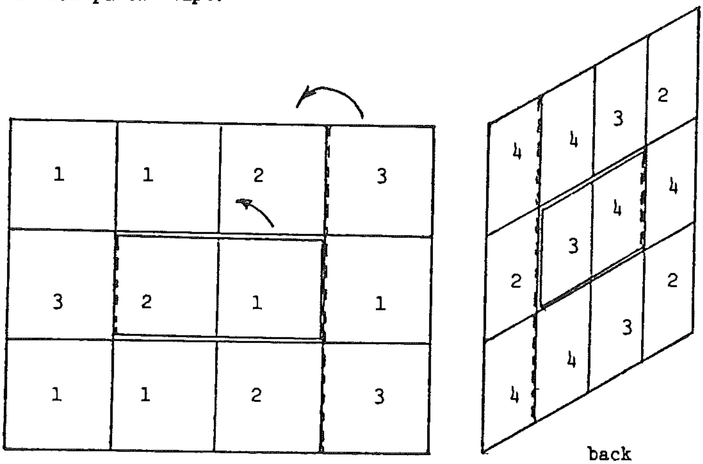
   It is folded from a strip of paper as shown in the figures below. Colour/number the small squares on each side of the strips as indicated. Fold both ends along the dotted lines as shown and join the two edges with a piece of transparent tape.





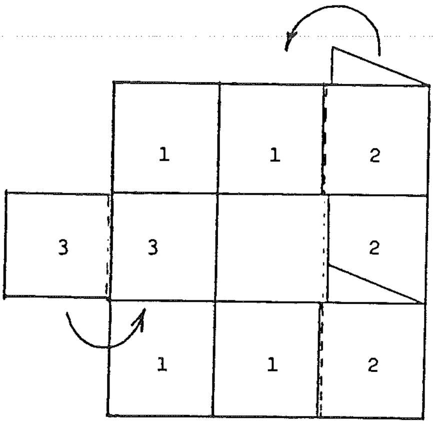
2. Tetra-tetraflexagon (four-faced four-sided-flexagon)

As shown in figures (1), (2) and (3), fold along the dotted lines in the directions indicated by the arrows and join the end edges with a transparent tape.

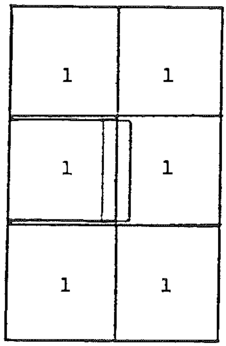


back

(1) Front

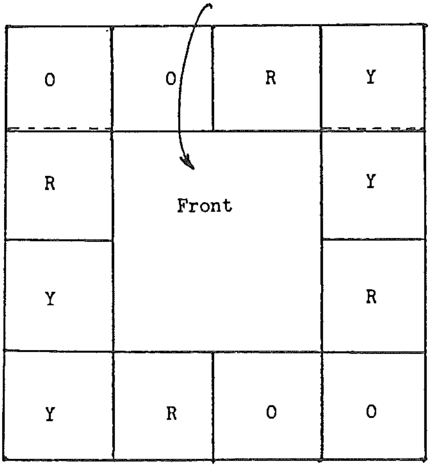


(2)

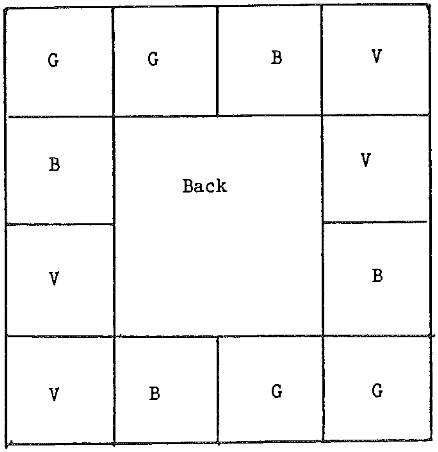


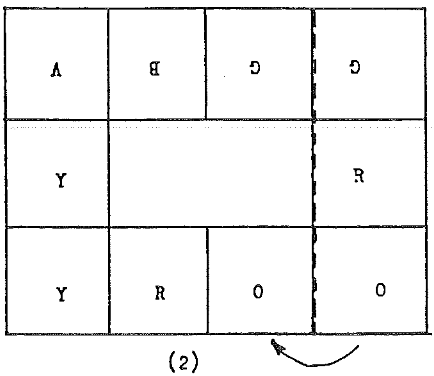
(3)

3. Hexa-tetraflexagon (Six-faced four-sided flexagon)

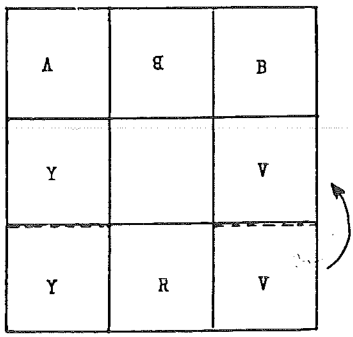


(1)





(2)



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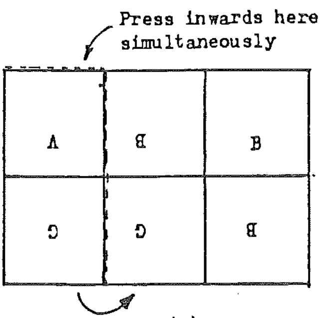
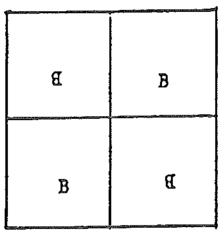


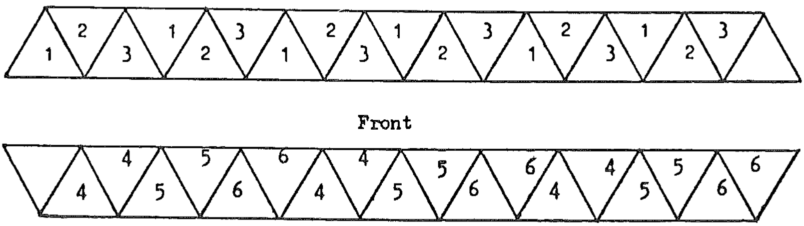
(4)



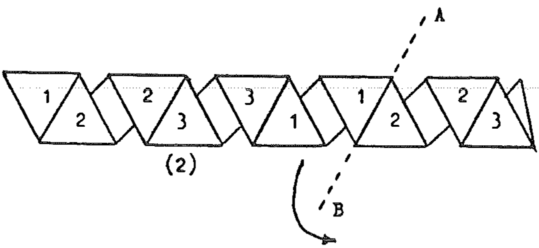
(5)

4. Hexa-hexaflexagon (Six-faced six-sided flexagon)

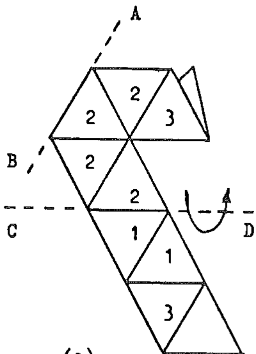
To make a hexa-hexaflexagon, we start with a strip of paper which is divided into 19 equilateral triangles, 18 of them numbered and one blank as in (1). The Strip is folded so that adjacent identical numbers on the back face come together i.e. 4 on 4, 5 on 5 and so on, to obtain (2). Then fold the resulting strip along AB to obtain (3). And then along CD to obtain (4). Finally paste the blank triangles together.



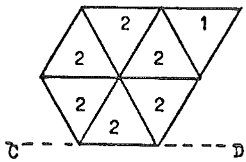
(1) back



(2)



(3)



(4)

(g) Möbius Strip

Take a long rectangular piece of paper ABDC, give it a half-twist and join the ends so that C falls on B, and D on A as shown in Fig. 1 to obtain a Möbius Strip.

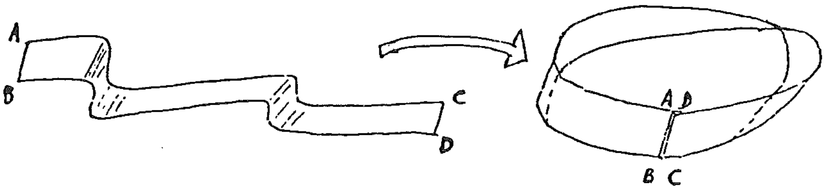


Fig. 1

The Möbius Strip is a one-sided surface. If you paint one side of it, you would be painting both sides. Draw a straight line down the centre of the strip, extending it until you return to the point at which you start. Now separate the ends of the strip and you will find that both sides are covered by the straight line even though in drawing it you did not cross any edges. Any two points on the Möbius Strip may be connected by merely starting at one point and tracing a path to the other without lifting the pencil or carrying it over any boundary.

Cut a Möbius strip in half with a pair of scissors along a line drawn down the centre to obtain a ring as shown in Fig. 2

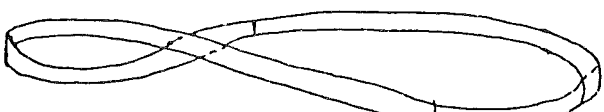


Fig. 2

Cut a Möbius strip with a pair of scissors along a straight line about $\frac{1}{3}$ of the width of the strip from the edge. Can you predict the result?

If you cut along a line a third of the way between one edge and the other, and cut until you return to the starting point, the Möbius strip opens into a large one linked with a smaller one.

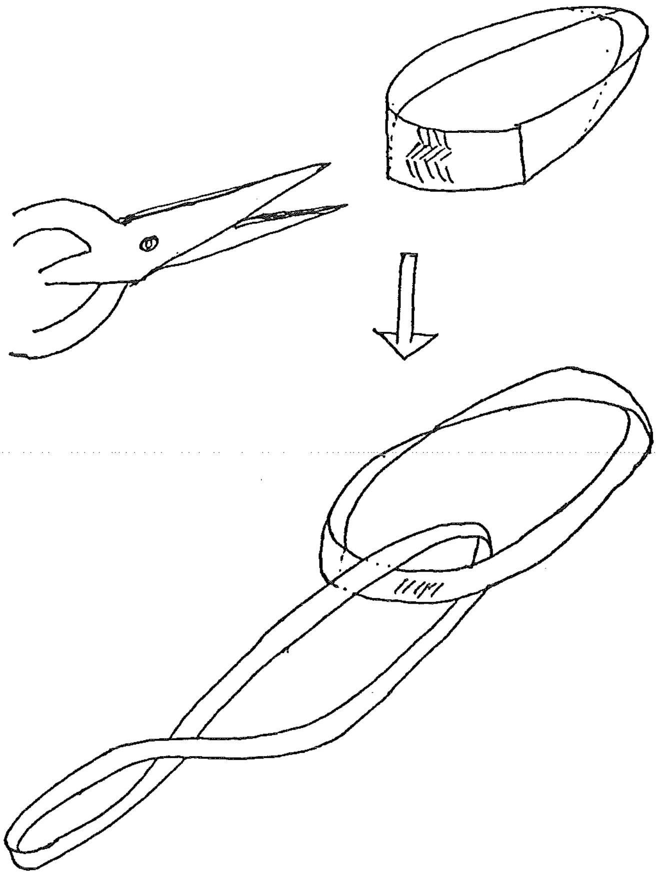


Fig. 3. "Trisecting" the given Möbius strip

If the strip ABDC is twisted through 360° before joining, a "Möbius strip of the second order" results. It has two sides and has two boundary curves, but they are linked together. Figure 4 shows a Möbius strip of the second order.

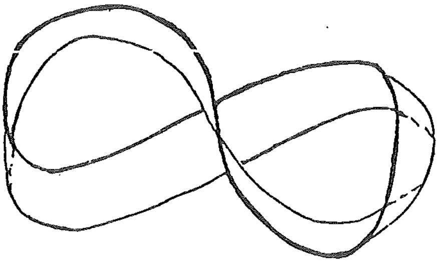


Fig. 4

Generally, if there are n half-twists before joining, we obtain the Möbius strip of the nth order. If n is odd, the surface is one-sided and possesses a single boundary curve which is knotted for $n \geqslant 3$. Figure 5 shows a Möbuis strip for $n = 3$.

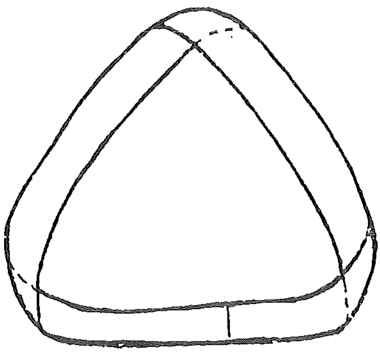


Fig. 5

(A) Check the following by cutting Möbius strips of various orders down their centre-lines.

(i) If n is even, two strips similar to the original result, linked together in the same way as the boundary curve.

(ii) If n is odd, only one strip results, and is knotted for $n \geqslant 3$. It has 2n + 2 half-twists.

(B) If the strips are trisected, the centre strip will resemble the original, but the outer strip will be single (if n is odd e.g. See Fig. 3 for n = 1) or a pair (if n is even, e.g. See Fig. 6 for n = 2), like the result of bisection, and they will be linked to the centre ring.

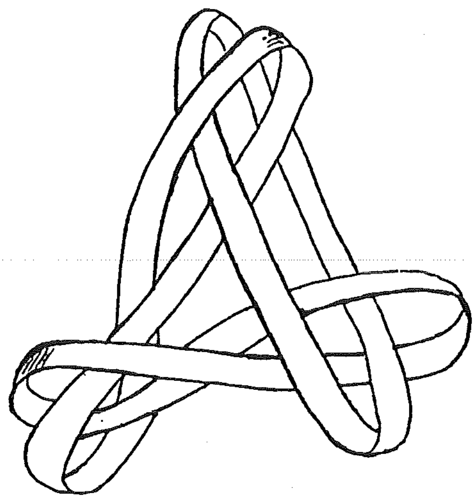


Fig. 6

Mobius strips which are mirror images of the above figures also exist. They have similar properties.

(h) List of books

The following books related to mathematics activities in schools are available at the Mathematics Teaching Centre;

1. Geometry for Enjoyment
   Granada Publishing Ltd.
   D. S. Fielker

2. The Impossible in Mathematics
   National Council of Teachers of Mathematics
   Irving Adler

3. Mathematics from Outdoors
   Chatto & Windus (Educational) Ltd.
   E. T. Norris

4. Mathematics Through Paper Folding
   The National Council of Teachers of Mathematics, Inc.
   Alton T. Olson

5. Paper Folding For the Mathematics Class
   National Council of Teachers of Mathematics
   Donovan A. Johnson

6. Geometric Exercises in Paper Folding
   Dover Publications, Inc., New York
   T. Sundara Row

7. Mathematical Challenges II
   National Council of Teachers of Mathematics
   Thomas J. Hill

8. Puzzles & Graphs
   National Council of Teachers of Mathematics, Inc.
   John N. Fujii

9. Movement and Pattern
Oxford University Press
D. Paling, C. S. Banwell & K. D. Saunders

10. Making Models
Oxford University Press
D. Paling, C. S. Banwell & K. D. Saunders

11. Cubes
Cambridge University Press
David S. Fielker

12. Tessellations
Cambridge University Press
Josephine Mold

13. Solid Models
Cambridge University Press
Josephine Mold

14. Exploring The Pinboard
The Educational Supply Association Ltd.
K. Lewis & D. P. Ambrose

15. A History Of π (Pi)
The Golem Press
Petr Beckmann

16. A Concise History of Mathematics
Dover Publications, Inc. New York
Dirk J. Struik

17. Games & Puzzles for Elementary and Middle School Mathematics
National Council of Teachers of Mathematics
Seaton E. Smith, Jr. & Carl A. Backman

18. Mathematical Puzzles and Diversions
Cox & Wyman Ltd.
Martin Gardner

19. Mathematical Carnival
Alfred A. Knopf, Inc.
Martin Gardner

20. Mathematical Diversions
Dover Publications, Inc. New York
J. A. H. Hunter & Joseph S. Madachy

21. The Master Book of Mathematical Recreations
Dover Publications, Inc. New York
Fred. Schuh

22. Mathematical Snapshots
Oxford University Press
H. Steinhaus

23. How To Develop Problem Solving Using a Calculator
National Council of Teachers of Mathematics
Janet Morris

24. Mathematical Experience
Chatto & Windus for The Schools Council
P. J. Floyd & Others

25. Topics for Mathematics Clubs
National Council of Teachers of Mathematics
LeRoy C. Dalton & Henry D. Snyder

26. Activities in Metrio Measurement
Michigan Council of Teachers of Mathematics
Albert P. Shulte & Others

27. Activities from the Mathematics Teacher
National Council of Teachers of Mathematics
Evan M. Maletsky & Christian R. Hirsch

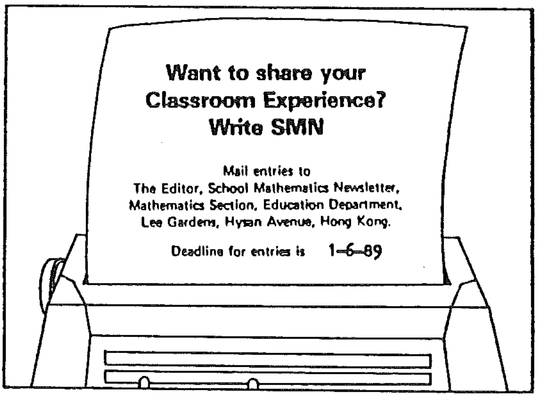
28. Mathematical Activities
Cambridge University Press
Brian Bolt

29. Activities for Junior High School and Middle School Mathematics
National Council of Teachers of Mathematics
Kenneth E. Easterday, Loren L. Henry & F. Morgan Simpson

30. Mathematical Models
Oxford University Press
H. Martyn Cundy & A. P. Rollett

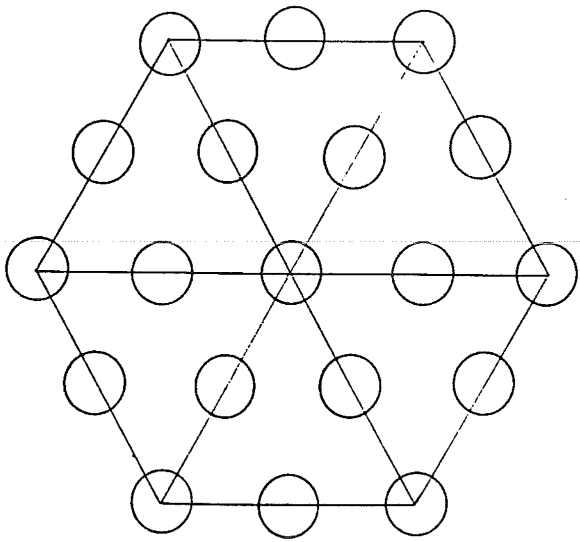
31. Polyhedron Models
Cambridge University Press
Magnus J. Wenninger

32. Mathematical puzzles
Bell & Hyman Limited
Stephen Ainley



PASTIMES

1. Interesting Patterns



Try to fill in the above figure with numbers from 1 to 19 in the nineteen circles so that the sum of the three numbers on any side of any equilateral triangle as depicted in the figure is 22. Can you re-arrange the nineteen numbers so that the sum is 23.

2. Interesting array

Guess the number that follows 3613.

3, 5, 13, 85, 3613, ...

3. 'Similar' Partition of Rectangles

A1 : 594 x 840 mm

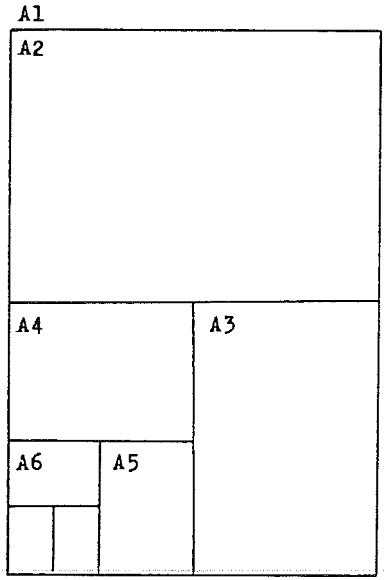
A2 : 420 x 594 mm

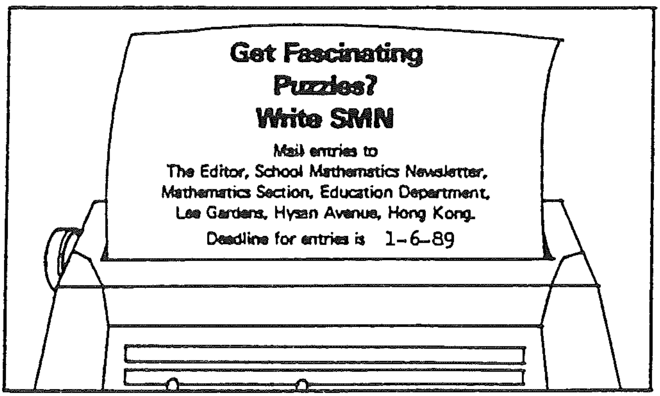
A3 : 297 x 420 mm

A4 : 210 x 297 mm

A5 : 148 x 210 mm

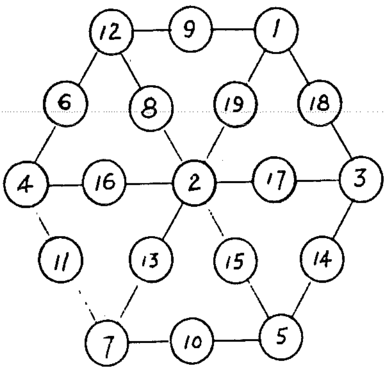
The A-series paper can be treated as a family of similar rectangles. Can you think of some rectangles which can be divided into a finite number of smaller rectangles each of the same size and similar to the original one?



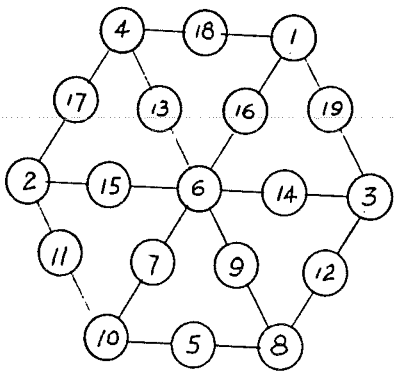


ANSWERS TO PASTIMES

1.



Sum = 22



Sum = 23

2. The term that follows is 6526885

$$\frac{3^2 + 1}{2} = 5 \quad , \quad \frac{5^2 + 1}{2} = 13 \quad , \quad \frac{13^2 + 1}{2} = 85$$

$$\frac{85^2 + 1}{2} = 3613 \quad , \quad \frac{3613^2 + 1}{2} = 6526885$$

i.e. $$T(n) = \frac{T(n - 1)^2 + 1}{2}$$

Or

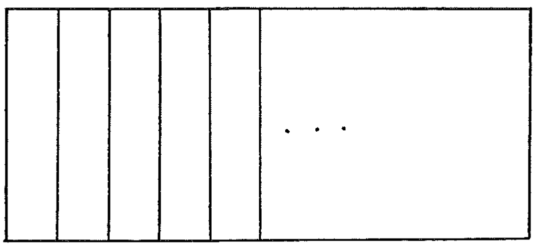
$3 = 1 + 2$	;	$1^2 + 2^2 = 5$
$5 = 2 + 3$	;	$2^2 + 3^2 = 13$
$13 = 6 + 7$	;	$6^2 + 7^2 = 85$
$85 = 1806 + 1807$	;	$1806^2 + 1807^2 = 6526885$

i.e. $$T(n) = \left[\frac{T(n-1)}{2}\right]^2 + \left\{\left[\frac{T(n-1)}{2}\right] + 1\right\}^2$$

where $[x]$ denotes the least integer function.

Readers may be interested to know the equality of these two apparently different expressions can be proved.

3. length : breadth = $\sqrt{n}$: 1



n rectangles

DO YOU KNOW ?

The International Mathematical Olympiad

The International Mathematical Olympiad (Hong Kong) Organizing Committee of the Hong Kong Mathematical Society has prepared a team of six senior secondary students to represent Hong Kong to participate for the first time in the International Mathematical Olympiad, i.e. the 29th IMO held in July 1988 in Canberra, Australia. The IMO has been in existence since 1959. This year, there were 49 countries and territories sending teams to the Olympiad. The members of the Hong Kong team were selected from 300 prominent students nominated by their principals and intensively trained for months to solve difficult mathematical problems in a variety of areas. The actual contest was held in Canberra College of Advanced Education on 15 & 16 July 1988. As a result, the Hong Kong Team won two bronze medals and a honourable mention. Congratulation on their success!

The next IMO will be held in Federal Republic of Germany in July 1989. The initial selection contest has been held in July 1988. Students with distinguished performance in the training sessions will be selected as members of the Hong Kong team to partioipate the 1989 IMO.

Hong Kong Mathematics Olympiad

The Heat Event of the Fifth Hong Kong Mathematics Olympiad was held on 12 December 1987. One hundred and seventy-five secondary schools participated in the competition. Forty schools with the highest scores were selected to take part in the Final Event held on 13 February 1988 and the results were encouraging. The Champion was Ying Wa College. The 1st runner-up was Ying Wa Girls' School and the 2nd runner-up was Rosaryhill School.

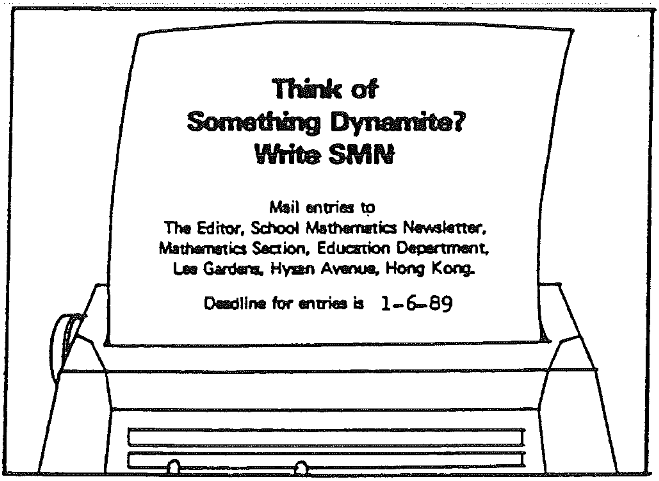
Glossary of Terms Commonly Used in Secondary School Mathematics

This glossary provides Chinese translations of those English terms commonly used in the teaching of Mathematics in secondary schools and is intended to facilitate the wider use of Chinese as the medium of instruction. The glossary issued by the Curriculum Development Committee, Education Department, Hong Kong, in 1988, would be available soon.

/ .....

Audio-Visual Resources Catalogue 7

This catalogue contains listings of audio-visual materials classified in accordance with the mathematics curricula offered in most primary and secondary schools. It has already been issued to schools. All audio-visual materials in the catalogue are available for loan to schools upon application to the Audio-Visual Resources Library, Visual Education Section, Advisory Inspectorate, Education Department, Room 228, Lee Gardens, 2nd Floor, Hysan Avenue, Causeway Bay, Hong Kong. (Tel. 5-8392362).



FROM THE EDITOR

I wish to express my sincere thanks to those who have contributed articles and to those who have helped in the preparation of this issue of SMN.

Grateful acknowledgement is made to the Editor of the Graduate magazine and Professor Ed Barbeau, University of Toronto for their permission to reproduce the article "Problem-solving as an Art Form."

Readers are cordially invited to send in articles, puzzles, games, cartoons, etc. for the next issue. Contributions need not be typed. Write : The Editor, School Mathematics Newsletter, Mathematics Section, Education Department, Room 528, Lee Gardens, Hysan Avenue, Hong Kong.

For information or verbal comments and suggestions, please contaot the editor at 5-614364.

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