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School Mathematics Newsletter（SMN）

Foreword

The School Mathematics Newsletter（SMN）is for mathematics teachers. SMN aims at serving as a channel of communication for mathematics education in Hong Kong. This issue includes articles written by academics and teachers. The first three articles are contributed by academics about their insightful views on Mathematics education. Other articles involve different areas, including suggestions of effective strategies in implementing school-based STEM education; learning and teaching of mathematics on specific topics; strategies to promote reading to learn in mathematics; story about history of mathematics and gifted education in mathematics, etc. I hope all the readers can get some fascinating insights in mathematics education.

SMN provides an open forum for mathematics teachers and professionals to express their views learning and teaching in mathematics. We welcome contributions in the form of articles on all aspects of mathematics education. Please send all correspondence to:

The Editor, School Mathematics Newsletter,
Mathematics Education Section
Curriculum Development Institute

Room 403, Kowloon Government Offices
405 Nathan Road
Yau Ma Tei, Kowloon
email: schmathsnewsletter@gmail.com

We extend our thanks to all who have contributed to this issue.

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1. 我們對數學教育的看法[1]

張僑平[2]、梁玉麟[3]、陳葉祥[4]、黃家鳴[5]、黃毅英[6]、鄧國俊[7]

前言

任何時期都會有人提出新的教育政策、理念和構想。這些可能來自教育行政當局、教育學者、國際趨勢甚至各界人士。作為關心教育的人，我們除了逐一回應這些新舉措外，也必須了解自身的取向和立場，否則容易變成無奈執行又或只能在這些新猷的既定框架內討論。

本文的作者是幾個志趣相投的數學教育工作者，在一段時期，我們透過電郵、手機社交應用程式等討論這幾年在香港教育界，特別是數學教育界推出的一些新意念。本文綜合了當前的這些新意念以及我們的想法，於這裏和大家分享。

數學的本質

1. 要探討數學教育，先須了解數學的本質。雖然學科之間不是壁壘分明，但隨著它悠久的歷史發展，數學已形成了一門「知識領域」（discipline），且成為人類文化的一個重要組成部分。所謂知識領域，是它擁有與其他知

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[1]作者按筆畫序。
[2]香港教育大學。
[3]香港浸會大學。
[4]香港中文大學。
[5]格拉斯哥凱爾文學院。
[6]香港教育大學。
[7]香港浸會大學。

識領域不同的詮釋、解讀不同現象的方式以及特有的處理和解決問題的取向與方法。

2. 一般來說，數學（包括概念、法則、技巧和方法等）的發展和形成是人們在觀察現實世界時提取了一些關鍵因素然後歸納出通則和規律（pattern），而從這些通則和規律中提煉出一些<u>數學對象</u>（mathematical object）。例如人類活動中涉及「分物」，逐漸提鍊出數字這數學對象，並探討除法及其法則。在抽象過程中，可能會超越實際情境的考慮，但卻可以發展和形成獨特的數學方法去詮釋、解讀現實世界。數學的發展亦會進一步抽象到更高深、較遠離現實世界的數學（例如由物件數量到數字，然後到代數及數系等）。這就是在《為甚麼要學習數學？》[8]中所說的，數學「源於生產實踐」、「從『歸納』到『演繹』」的過程。例如我們把自由落體看待成二次方程，我們假設自由落體是一個點（即「點質量」：point mass）的運動軌跡，中間除了涉及<u>抽象化</u>，也涉及<u>理想化</u>。

3. 所謂用數學方法處理問題，不局限於用現有、既定的數學解讀現象，也可以包括在出現相對複雜的現象時衍生出新的數學分支。

[8]蕭文強（1978）。《為甚麼要學習數學》。香港：學生時代出版社。（第二版）香港：香港新一代文化協會（1992）。（增訂本）台北：九章出版社（1995）。

4. 上面（第 2 點）所說的數學對象超越了實際情境的考慮，亦有程度之別。以上面自由落體為例，當我們把風速、浮力等考慮在內，形成更細緻的數學建模時，這便又考慮了實際情況。無論如何，由於數學重視可推廣性，故此從各種現象抽象起來、逐漸擺脫具體個別實際情境是必要的。

大眾數學

5. 承接第 1 點，數學既自成一門知識領域，它就如藝術等一樣，不是每個人都需要學，也不一定人人都能學會。故此也未必人人都可以進入「數學內圍」（esoteric mathematics）[9]。縱使如此，我們還是認為數學的某些元素，尤其是數學思考和問題解決方法，對普羅大眾（並非從事數學或相關行業的人）是有所裨益的。認識到數學的這種價值，從事數學研究或者數學教育的人就需要有更開放的心態，讓不同的人有機會接觸甚至受益於不同程度、不同方面的數學。倘若把數學局限在一小撮有「數學天賦」的人身上，這就和「人人都有機會接觸不同知識領域和發揮潛能」的社會趨勢背道而馳。數學既然作為人類文化的一部分，一般人能夠學會從「數學的角度」或方式看待周圍的事物自然是有價值的。這便是「大眾數學」的基礎。

6. 雖然說普羅大眾可能只需要數學思考和問題解決的方法（所謂「過程」：process）等元素，甚至他們在成年人

[9] 大概為高等數學。

生活中根本用不著技術層面的數學內容[10]（或「結果」：product）。不過，雖然學任何課題都是希望藉之培養數學思考和問題解決的方法，但這些方法又不能空洞地培養，仍需透過學習代數運算技能和幾何證明的技巧等等，來學習其中的數學思考、推理和問題解決。

7. 那麼，我們是否希望所有人總是用「數學角度」去解讀現象和解決問題呢？以解決問題為例，我們其實可以有不同方法（包括付錢委託別人解決、祈禱、接受輔導……），數學未必就是最佳或唯一的解決方法，問題解決能力的培養也並非數學才有。正如我們不需要事事都用「藝術角度」去考慮，就算受過基本數學訓練的人也不必凡事都得用「數學角度」去看待事物。不過，有受過基本數學訓練的人就有能力（在有需要時）用數學去解讀和處理問題。正如我們可以用藝術的角度看數學圖形，也可以用數學的角度解釋藝術。數學的角度給我們提供了多一種有力的思維方式和解決問題的方法。

8. 綜合上兩點來說，「大眾數學」既不是硬要所有人進入高等數學又或為了普及而淺化數學這門知識領域，而是更積極地讓不同人（包括學生）感受到數學可以提供一套思考方式和解決問題的方法。我們相信兩者並不割裂。數學思考方式和解決問題的方法能為有志進入高等數學的學生提供基礎，也能協助學生面對未來世界的挑戰。

[10] 如解二次方程、三角學、尺規作圖等。

數學學習

9. 那麼數學是怎樣學會的呢？當然每個學習者有自己的學習和建構知識的方式（建構主義的基本想法），但既然數學基本上是從現實（不局限日常生活，也包括科學及技術）問題中逐步抽取出關鍵因素，然後用數學技巧去處理，故此「學」數學基本上便是學懂這個**過程**[11]（無論你會否叫它做數學化過程[12]）。籠統而言，「教[13]」數學便是引領學生走這個過程。

10. 這個過程的其中一端便是生活數學或現實情境數學。它們可以是引發數學化過程的一個好的開端，但若停留在此，便很難、甚至不能進入數學的內圍。

11. 不過，數學學習既不是只顧教授數學的技術內容，提供「作為結果的數學」（mathematics-as-an-end-product[14]）；也不是停留在用數學去組織（formulate）各種問題（例

[11] 黃毅英（2007）。數學化過程與數學理解。《數學教育》25 期，2-18。

[12] 這個過程在被廣泛引用的 Freudenthal, H.(1991). *Revisiting mathematics education (China lectures)*. Dordrecht, the Netherlands: Kluwer Academic Publishers 中有不少表述，其中也提到數學化與設基化、形式化、圖式化（schematising）是近義詞。作者一直用分詞（participle: “ing”「化」）表明這是一個過程。

[13] 教的過程十分複雜，於此不贅。這裏所指的只是「成年人世界」所訂定的一種「假設性學習軌跡」（hypothetical learning trajectory）罷了。

[14] Siu, F. K., & Siu, M. K. (1979). History of mathematics and its relation to mathematical education. *International Journal of Mathematical Education in Science and Technology, 10*(4), 561-567.

如我們從文字題中設立了方程後，還得用數學方法去解方程），又或只是把數學應用到其他知識領域（如科學及技術）的各種問題上。

12. 用現時已知的數學（即上面所說的「作為結果的數學」）去解釋一些學生正在學習的數學概念或運算法則（例如用 $(a_n 10^n + \cdots a_2 10^2 + a_1 10^1 + a_0 10^0) \times (b_m 10^m + \ldots b_2 10^2 + b_1 10^1 + b_0 10^0)$ 去解釋乘法）未必對學生理解數學有很大幫助。反而，我們應引導學生探討這概念或法則如何逐步發展出來（「製作過程中的數學」：mathematics-in-its-making[15]）。

13. 然而，這個數學化或抽象化過程（縱使針對個別課題）所需的時間可以很長，可能由初小一直到高中（甚至更後）才能完成。因為在歷史上，一個數學概念（比如函數，又或負數的運算以至數系的擴展）的建立往往也是經過漫長的過程。故此，要慎防把數學過早地形式化、系統化、符號化[16]。

14. 這個學習過程以至上述所說的時間長短，是如何佈置的呢？雖然歷史基本上不可能重演，「學習」也不完全是引領學生重新把歷史的路走一遍，但認識數學概念及技巧在歷史上的演化過程會對上述第 9 點提到的引領學生（即「教」數學）會有很大的啟示。

[15] 同註 14。

[16] 見梁鑑添關於新數學的文章：《抖擻》編輯委員會（1981）。《香港數學教學論叢》。香港：抖擻。

數學理解

15. 上面提到借鑑數學的歷史發展，但與此同時，猶如第 9 點所指出，每個學習者有自己的學習和建構知識的方式。特別地，（多重）表像（representation）當然有助於學生理解和建構知識，但最重要的是學生能建立內在表像和自行建構，而非我們規定他們建構哪些表像來。

16. 眾所周知，我們不應讓學生不求甚解，只以成功解題為指標，要著重理解。但甚麼是理解呢？近代研究均發現（數學)理解和學生內部知性網絡的連繫的程度(degree of connectedness）、連繫的結構強度有密切關係，概念性理解和操作性理解密不可分，「先理解後運算」的說法也未必全面[17]。

17. 讓學生自行建構會否出現不同人得出南轅北轍的概念呢？從現象圖式學（phenomenography）的觀點來看，縱使對於同一概念，不同人所建構出來的知性網絡會有所不同。但沒必要過份恐慌，亦不必刻意壓抑那些「在課程上不正確」（即不符合課程規範和要求）的連繫，因為很有可能它們會自然在「社化」的過程被「萎縮」掉。例如有些人會認為數字有顏色（甚至氣味、喜惡等），如「2」字是綠色[18]之類，這是正常不過的。但隨著對數字「2」的不斷運用，「顏色」一直沒有產生任何作用的

[17] 同註 11。

[18] 原因可以多樣，例如小時候「2」字的積木是綠色。

話[19]，漸漸這關於「顏色」的連繫自然會「消失」掉（正確來說只是隱藏了）。故此，我們無須刻意禁止學生的想法，否則，這與鼓勵學生主動建構的理念背道而馳。如果我們必須為學生佈置「學習軌跡」時，亦要建基於學生各自的認知結構，而非只考慮數學的知識結構。

18. 在概念性理解和操作性理解之間，建構過程的規範與自由之間，並不存在一道不可踰越的鴻溝[20]。例如學習分數加法，我們引領學生初步理解加法的原理後，仍需讓他們熟悉操作方式，如擴分、通分等，而在操作不同情境的分數加法（即運用變式）中加深對分數加法的理解（反正縱使具備純熟數學訓練的人在進行分數加法時也只會用擴分、通分等的法則去操作，但到複雜情境時，他們卻能動用當中的概念）。

19. 社會建構主義學者 Anna Sfard[21]亦有類似主張。她提出人類社會中的「常規（例如運算程序）與創造」[22]（routines

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[19] 當然若有某些數字記憶法用得上又是另話——如顏色因數：colour factor。

[20] 同註 11。

[21] Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* New York, N.Y., U.S.A.: Cambridge University Press.

[22] 例如初中學生學習有向數的加減運算常規時，一方面要認識和熟稔小學的的加減運算常規，但亦要肯跳出已有框框，建立新的常規，並將以往所學重新組織理解，創造出新的有向數操作性和概念性理解。

and creativity）和「探索與習俗（成規）」[23]（explorations and rituals）是並存而非互相排斥對立。所以，學生在學習數學過程中遇到的「潛規則」或非日常生活常見的「嚴謹數學語言」[24]，某程度上是不可避免的儀式習俗，學生要接受這些儀式習俗的洗禮，才可以進入更深層的數學學習與思考。

數學與共通能力

20. 在第 1 點，我們指出，數學是人類文化的（一種）結晶[25]，隨著悠久的歷史發展它已形成了一門知識領域。如第 2 點所說，數學（猶如其他領域）已逐漸**自然**地形成了它的獨特思維及處事方式，例如數學問題解決能力、科學創意思維、人文的價值觀等，它並不是人為的界定或「割裂」（compartmentalise）出來的。因此，不能說學科（包括數學）學習是把知識「割裂」，甚至窒礙能力培養。學科與<u>跨學科</u>兩者對學生的能力培養是可以相輔相承的。

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[23] 例如教師教學生用粵語或普通語背誦乘數表，這近似儀式習俗，基本上每間學校每個課室沒有大分別（用英文來背誦則有頗大分別），但教師在引導學生探索並建構乘法的操作性和概念性理解時，教師和學生皆有很多空間去選取不同的教學活動和建構方式。

[24] 小學例子可見：陳葉祥（2014）。$4\frac{3}{2}$是不是帶分數?《數學教育》36 期，37-9。
中學例子見：張家麟、黃毅英 (2010)：《從「微積分簡介」看數學觀與數學教學觀》。香港：教育局課程發展處數學教育組。

[25] 同註 8。

21. 不過，數學，猶如其他學科一樣，在歷史發展上已形成其有別於其他學科的思維模式，有不可取代性。雖然學科、跨學科能力相輔相承，以問題解決為例，一般問題解決能力和數學問題解決能力有密切關係，但只有前者算不上是內圍數學的學習，至於是否每人須進入數學的內圍，前已有討論（見第 5 點）。

22. 所以我們不反對各種課程統整[26]（包括 S.T.E.M.[27]）的課程措施及跨學科共通能力的培養。不過，正如所有的課程及教學設計均要小心考慮其目的、實施和成效。於數學而言，如果一些統整課程聲稱有數學元素，作為課程實施者和接受學習的人自然就要考究這種統整是否能帶出有意義的數學學習。縱使數學科需要配合宏觀的教育政策，數學教育工作者亦應反思政策是否合理及有權參與其製訂以至檢視其施行與成效。

數學內外

23. 第 13 點所說的在數學化過程中學生的準備情況（readiness）、第 17 點所說的課程與教學的佈置、第 18 點所說的「建構過程」，以及第 19 點所說的「儀式習俗」，不只需要老師的專業判斷，亦要參考各種教育理論（如教育心理學、課程論，甚至哲學、社會學等）所提供的學理基礎和建議。

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[26] 課程統整的其中一個目的在於打破學科「樊籬」。

[27] Science, Technology, Engineering, Mathematics：指科學、技術、工程、數學。這其實蘊含四個板塊的課程統整。

24. 數學既然是人類文化的一部份，它的教學應同時對「育人」有所顧及。這包括一些品德培養、作社會批判等。這些雖然不屬於數學或數學學習的範圍，但教師在教學過程中自然會和學生共享這些價值。

結語

儘管上面我們提出了對數學教育的一些立場和想法，我們認為大家同意與否並不重要，況且在未來陸續還會有新的教育舉措出現，難道我們又要找一些人回應和撰寫新的想法？我們深信，假如每一位教育工作者（包括數學教育工作者），無論是前線人員還是官方或學者等「後勤人員」，能不斷通過實踐探研、反思(即成為「學養教師」scholar-teacher 或「具反思的實踐者」reflective practitioner），不難得出合理的結論，從而真正提升教學。這樣既不會隨波逐流，亦不會被人牽著鼻子走。假若能有這樣一代的「學養教師」，我們一定會引領學生學得更好。

2. 數學教育與價值觀的培養[1]

鄧國俊

香港浸會大學

前言

上一期(第二十一期)的學校數學通訊有三篇文章以 STEM 教育為主題。羅浩源教授的〈STEM 教育：以數學作起點來推動 STEM 教育的挑戰〉，引領讀者思考數學在 STEM 教育的定位和角色，並探討如何以數學作為起點來配合 STEM 教育這個的挑戰；關子雋、簡嘉禧老師的〈校本 STEM 教育經驗分享〉和林嘉康校長的〈李炳學校的「STEM 教育」〉，則分享他們在數學科推行 STEM 教育的校本經驗，使讀者對前線實踐多一分了解。這三篇文章除了使筆者思考科學、科技及工程的高速發展對數學教育的沖擊而獲益良多外，亦使筆者反思近年社會、文化及政治的急劇演變是否亦為數學教育帶來挑戰？

美國英國近年趨勢：我們不單只教數學

2016 年美國總統大選，特朗普險勝對手希拉莉，使國家社會更為撕裂。同年 12 月 1 日，美國全國數學教師協會 (National Council Teachers of Mathematics) 會長 Matt Larson 發表一篇題為〈我們不單只教數學〉[2]的署名文章，當中提

[1] 感謝黃毅英教授於文章撰寫過程中向筆者提供不少寶貴意見。

[2] https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Matt-Larson/We-Teach-More-Than-Mathematics/

到：「最近在我國發生的事件，其所帶來的挑戰對許多教育工作者來說是前所未見的。作為數學教育工作者者，我們不能倖免於當前的政治氣候和情緒化的環境。……雖然我們教的科目是數學，但我們要記住我們是將數學教給學生。我們的學生在課堂上的成功取決於我們作為教師的能力，無論他們的背景如何，我們都有責任為每一位學生營造一個具安全感和包容性的課堂環境。……作為教育工作者，我們需要同時維護我們對民主原則和我們在民主進程中的所承諾擔當的角色，並確保不同觀點在交流分享時，能得到建設性和包容性的對待，同時確保學生福祉不會受到任何人的威脅。」

位處歐洲的英國，情況亦有相似之處。為防止脆弱的個人（尤其是年青學生），被激進思潮影響，甚至被捲入恐怖主義漩渦，英國政府早於 2012 年便推出傳揚英國價值觀的教育活動。其中一個方向就是要求學校及各學科教師（包括數學科）在課堂教授和探討英國價值觀：民主、法治、個人自由，以及尊重和包容各種不同信念信仰（Carroll, Howard, & Knight, 2018, p.2）。

價值觀與數學教育

價值觀的培養是學校教育中提升學生情感素質的重要一環。但由於不少人（包括數學教師）把數學看作具結構性的純知識體系，又或者是一套無容置疑而又具實用價值的客觀真理和運作規則，從而認定數學是價值無涉、價值中立，甚至是價值超然的學科，故與培養學生價值觀扯不上任何關係。雖然這觀點在上世紀末已受到挑戰，包括來自

哲學／認識論的分析（例如：Bishop 1988a; Bishop 1988b; Ernest 1991; Restivo 1992），以及從教學角度所作的思考（例如：Bishop et al. 1999; Wilson 1986; Winter 2001）。換言之，仍有一定數量的數學教育工作者認為，數學不是價值無涉的學科，數學教育不應只集中培訓學生學習那些所謂價值中立的數學知識和技能。

Bishop (2001) 將數學教育中價值觀簡單分為兩大類：一般教育價值觀和數學價值觀。一般教育價值觀源於社會對學校培養學生價值觀的期望，以及培育學童成長發展和社會化的要求。它們可以是紀律、心智、社群、文化、美學、經濟、道德、政治或宗教等相關的價值觀。在官方課程文件中，有些國家會在其數學教育的宗旨和目標中，開宗明義地說明其中一些相關教育價值。以新加坡為例，其自 1996 年啟動國民教育[3]以來，無論在國家或學校層面，都有不少具體實施方案[4]。2000 年的數學課程綱要，建議將當前與國家相關的社會事件作為設計數學應用題的情境，並鼓勵學生思考討論這些問題的答案及其含意。常見的情境包括：新加坡的歷史和地理，個人開支和儲蓄，日常生活的水電消耗，以及資源的減耗—再利用—回收。新加坡國立教育學院黃冠麒教授於 2003 年，更進一步提出「國民教育×數學教育」框架，以激發本地及外國學者討論如何將國民教育及相關價值培養注入數學課堂（Wong, 2015, p.114-5）。

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[3] 新加坡的國民教育，旨在增進國民凝聚力、提高國家生存發展能力、使學生對個人及國家未來充滿信心。它亦強調培養學生對新加坡的歸屬感和情感根源。

[4] https://ne.moe.edu.sg/ne/slot/u223/ne/index.html

至於數學價值觀，Bishop (2001)提出的例子包括：理性主義(rationalism)、客觀主義(objectivism)、控制(control)、進步(progress)、開放(openness)和神秘(mystery)，這些價值通常與西方數學相關[5]。有關數學價值觀及態度的培養，新加坡2012 年的官方數學課程文件提出五個重要的情感素質：1. 數學可信可靠且有用；2. 學習數學可以是有趣歡樂的；3. 欣賞數學的美感和力量；4. 有信心使用數學；5. 堅持不懈地解決問題（Wong, 2015, p.197）。

近二十年香港情況

1997 年香港回歸後，2000 年推行大規模教育改革，中小學的課程架構由「學習領域」、「共通能力」和「價值觀和態度」三個部分組成。政府當局有關培養學生數學價值觀和態度的建議，包括：1. 發展學習數學的興趣；2. 展示對參與數學活動的熱忱；3. 具有靈敏的觸覺，能體會數學在日常生活中的重要性；4. 展示在日常生活中能應用數學，以澄清自己的論證及挑戰別人論據的信心；5. 能與其他人分享意見及經驗，以及合作完成數學課業／活動和解決難題；6. 充分了解並履行個人在群組工作中的責任；7. 在群組工作時，應持有開放的態度；8. 而在討論數學問題時，亦願意聆聽及尊重他人的意見，對他人的貢獻予以重視及懂得欣賞；9. 能獨立思考，從而解決數學問題；10. 具有鍥而不捨的鑽研精神，努力嘗試解決令人困惑的數學難題；10. 欣賞數學的精確性、美感和在文化方面的貢獻，以及其在人類事務上所發揮的巨大作用（課程發展議會，2002，p.24）。

[5] Bishop (1988a, 1988b, 2001)對這些西方數學價值觀不是全盤認同，而是作出批判反思。

與此同時，當年的教育改革亦提出四個關鍵項目，即德育及公民教育、從閱讀中學習、專題研習和運用資訊科技進行互動學習。數學科有關德育及公民教育的價值觀培養，有關當局建議可以下列的方式引入：1. 通過解決問題，讓學生培養出對不同 題解（數學問題未必只有一個題解）的正確態度。雖然有些題解較其他方法有效，但 很多時候其實只是觀點與角度而已；2. 在數學課堂中引入生活的例子，有助增強學生關注數學與現實的關係；3. 通過組織一些與數學有關的專題研習或課外活動，讓學生有機會充分發展其探究思維、接受責任、學會與人合作，以及培養其領袖才能和社交 技巧（課程發展議會，2002，p.36）。

2008 年，有關當局出版《新修訂德育及公民教育課程架構》[6]，強調知識的掌握和價值觀的培育是互相緊扣，故建議學校將培育學生價值觀和態度的工作，與各個學習領域及科目互相結合，彼此相輔相成，為學生提供一個整全的學習經歷。課堂教學實踐方面，則建議學校採用「生活事件」作為學習情境，而「生活事件」包括六個範疇：「 個人成長及健康生活」、「家庭生活」、「學校生活」、「社交生活」、「社會及國家生活」及「工作生活」，每個範疇有不同的學習事件，以配合學生在不同學習階段的需要。以數學科的「社會及國家生活」範疇為例，該文件所列舉的相關學習情境包括：透過討論課題「容量」，讓學生學會珍惜食水和善用能源，以實踐環保的生活習慣（小一至小三）；透過討論課

[6] https://www.edb.gov.hk/tc/curriculum-development/4-key-tasks/moral-civic/revised-MCE-framework2008.html

題「速率」，讓學生尊重法治，明白遵守交通規則的重要性（小四至小六）;欣賞日常生活中具有對稱性及經變換的平面幾何圖形（例如區徽）和認識及欣賞有關畢氏定理的不同證明（包括中國古代勾股定理所用的方法）（中一至中三）；認識抽取調查樣本的不同技巧及製作問卷的基本原則，以及評估從新聞媒介、研究報告等不同來源所獲得的統計調查報告（中四至中六）。

香港的相關實證研究

關於香港數學課堂中的價值觀研究，較早期的有 Leung (1992)的北京、香港和倫敦的數學課程比較研究，當中有關價值觀的結論包括：1. 北京教師傾向於對數學這門學科有一個較堅固穩定的看法，而倫敦教師則有具啟發性的和變化的看法，香港教師看法則在兩者之間；2. 北京教師強調努力，而倫敦教師強調能力，香港教師再次在兩者之間；3. 北京教師強調課堂上的冷靜和嚴肅，香港教師則強調效率，而倫敦教師則強調享受和靈活性。Lau (1996)以定量研究方法，探討香港初中數學教師的數學與教育價值觀。她的研究結果顯示，雖然教師對透過數學教學培養價值觀表現出積極的態度，但在個人、知識和文化這三類價值觀中，實際上只有知識價值被重視強調，而個人和文化價值則被輕視忽略。

近期探討價值觀在數學教育中的角色的研究，有陳葉祥、黃毅英（2014；2015）於 2011 年開展的宗教信仰、數學信

念與數學教學的關係的研究[7]。是項研究共分三個階段：第一階段以問卷方式分析教師的宗教與數學教學信念的關係，這階段結果證明持不同宗教的數學老師的數學及數學信念傾向有分別；第二階段以深度訪談分析教師的宗教與數學教學信念的關係，他們共訪問了十五位香港數學老師，包括五位佛教徒、五位基督徒及五位無宗教信仰者，訪問的焦點是了解他們對數學的本質及有效的數學教與學的看法，並且他們認為宗教與數學教學信念的關聯。結果顯示，雖然有宗教信仰的數學老師未必會將其整套信念投射到數學教學之中，但是他們會從其宗教信念取出一些價值觀、世界觀，或道德標準，以轉化成數學教學信念及教育理念，而無宗教信仰的數學老師亦會有這種價值觀及信念的轉化；第三階段探討源於宗教的世界觀的數學教學信念在課堂上的實踐，在這階段的研究分析六位數學老師如何嘗試把他們自己的宗教信念實踐於數學教學中。結果顯示，這六位老師的整合模式有頗大分別，包括：1. 融合模式；2. 過渡模式；3. 宗教滲入數學；4. 數學課附加宗教；5. 隱藏模式；6. 宗教作為一種教學的態度[8]。

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7 研究對象除香港數學教師外，還包括中國大陸和台灣地區的數學教師 (Leu, Chan, & Wong, 2015)。

8 研究者強調，該研究焦點並不在於那種整合模式較佳。因為教師的宗教價值觀在數學教學上的實踐，受著不同的因素影響，而其採用的整合模式亦然。

給數學加添趣味價值，抑或還數學一片「靜」土？[9]

行文至此，相信有部份讀者已生「一波未平（STEM），一波又起（價值觀）？！」之嘆。回顧上世紀九十年代的目標為本課程提出的現實情境教學，曾引起「數學教育之生活化與數學化」[10]之爭，黃家鳴 (1997, 1998, 2001) 曾深入及細緻地探討這課題，現將其當年的主要論點簡述如下：「情境化—數學化」的矛盾由來已久，原因是數學家頗為強調數學之嚴謹性與抽象性，而前線老師及數學教育工作者雖然認同數學的嚴密性和邏輯性，但他們更關心普及教育下所面對的課堂實況。一方面，並不是所有學生將來的職業均與數學有關（更遑論要成為數學家）；另一方面，普及教育亦帶來個別差異和學習動機兩大問題。因此，他們提出種種方法，如：實驗、遊戲、利用情境化和生活化事例……等，以提高學生興趣，並照顧不同能力的學生，以培養數學思維。然而，亦有數學教育工作者指出，如果課程編排和教學設計，缺少了對數學學理以至有關教學細節的關注和討論，數學的特質便會被淡化而至課堂教學形同「兒戲」，當年數學科目標為本課程改革，正是為此而引

9 筆者曾以數學教育專業知識，參與撰文評論一宗社會事件，以表明數學與社會關懷及價值觀思考的緊密連繫。該文標題為〈給數字加添趣味，抑或應還數字一片靜土？〉（許為天、鄧國俊，2009），評論一幢實際只有三十三層的西半山豪宅為「好意頭」而將頂層複式名為八十八樓的「數字風波」事件。

10 中文大學課程與教學學系，曾於 1999 年 2 月 19 日舉辦一個「數學教育之生活化與數學化」研討會，研究其中的矛盾與整合。

起爭論[11]。

無論是於數學科推動 STEM，又或是透過數學教學培養學生價值觀，在在需要引入生活化情境，故皆可看成是上世紀九十年代「情境化—數學化」之爭的延續。但由於過去二十多年科學、科技及工程的高速發展，與及社會、文化及政治的急劇演變，故兩者對數學教育所衍生的另一波挑戰的深度和幅度，相信是前所未見的。作為關心教育的人，必須了解自身的取向和立場[12]，以逐一回應這些挑戰，否則只能在官方文件政策的既定框架內作討論，或無奈地在課堂教學中執行。

結語：數學教育的 STEM 與 PATH？

我在大學修讀土木工程，亦曾於香港地鐵（港鐵前身）地盤實習，對 STEM 的理念與實踐有點具體認識，亦明白 STEM 對社會建設及經濟發展的重要性；畢業後從事前線數學教育工作十年，之後進入大學從事數學課程研究及師資培訓工作，對教書育人亦有一定的體驗感受，亦明白良好教育對文明社會價值觀培養的重要性。正是這些學歷與經歷，驅使筆者不時思考這個問題：我們是否偏重 STEM 的推動而忽略價值觀的培養？

[11] 筆者則曾嘗試從理念及實踐角度，思考如何回應「情境化—數學化」這挑戰（鄧國俊，2014; 2015）。

[12] 今期《學校數學通訊》〈我們對數學教育的看法〉一文，就是分享筆者與幾位志趣相投的數學教育工作者的一些想法和立場。

美國政治科學學系[13]教授 Andrew Hacker，於 2016 年出書批評美國近年過於偏重 STEM，並批評部份高中和大學一年級的數學課程與現實割離，建議增加社會文化思考及價值批判的內容。書中更記述其坐言起行的一個教學實驗：2013 年他向其學院的數學學系毛遂自薦，親自為學院的一年級學生設計並教授一門必修數學課程，內容主要為定量推理（Quantitative Reasoning），其目的是使學生能夠靈活地運用數學方法於不同社會議題的研究探討，尤其是統計的運用和分析（Hacker, 2016）。該書出版後，引來數學家的猛烈批評，亦引起社會各界的關注和討論[14]。

筆者雖然不盡同意其對數學本質及數學教育的看法，但認同教育工作是充滿價值判斷和人生意義的思考[15]，數學教育亦不能例外[16]。如果數學教師未能反覆地從理論與實踐

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13 Department of Political Science at Queen College in New York.

14 有興趣讀者可登 http://devlinsangle.blogspot.com/2016/03/the-math-myth-that-permeates-math-myth.html 及 https://themathmyth.net/ 了解詳情。

15 Postman(1996)曾提出警告，教育不能單為物質建設及經濟發展服務，否則教育便完蛋(the end of education)。他以積極態度及正面角度出發，為教育的終極關懷(the end of education)提出以下選項：保持生物多樣性及地球可持續發展；發揮人性光輝美好一面；來自不同種族文化背景的公民能和諧共融地相處；為未來世代編織美好的將來。

去反思數學教育的終極關懷，很容易誤入數學科目本質對理性和邏輯思考的盲目崇拜，以致思考形式變得冷漠、僵化、算計及非人性化，和教學信念流於工具理性化，而教師本入則變成只會關注學生考試成績的教書匠。

由於篇幅所限，筆者不打算在此探討「香港近年有關數學數學教育的研究討論是否偏重 STEM 的推動而忽略價值觀的培養？」這問題。只希望在本文完結前，引用 Andrew Hacker 為要抗衡 STEM 霸權而提出另類取徑所寫的一段文字，以為讀者提供一個思考起點（Hacker, 2016, Kindle Locations 223-229）：

> *我想在完結前提出另一個教育口號：PATH。我個人的構想是 Philosophy, Art, Theology, History*[17]*（讀者自己或可嘗試構想成：Poetry, Anthropology, Theater, Humanities*[18]*？）……我們正處於關鍵時刻，因為我們現正落後於競爭對手於 PATH*[19]*的渴求與追尋。如果我們的國家要保持其道德和文化形象地位，我們每年必須承擔或開創 100 萬或更多個與 PATH 領域相關的職*

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[16] Ernest (1991)以哲學及社會學理論為基礎，嘗試把關心數學教育的社會團體分為：工業訓練者；科技實用主實者；古典人文主義者；進步主義教育工者；大眾教育工作者。此外，為分析這五類社會團體的數學教育理念與實踐，他將各團體所抱持的相關觀點細分成十二項：政治意識形態；數學觀點；道德價值；社會理論；兒童理論；能力理論；數學教育目的；學習理論；數學教學理論；教學資源採用；數學評估理論；社會分流理論。

[17] 哲學、藝術、神學、歷史。

[18] 詩歌、人類學、戲劇、人文學科。

[19] 而非 STEM。

位。否則，或許我們仍會繼續是世界上最富裕的大國，但我國的文明肯定會踏上衰落之道！

作者電郵：kctang@hkbu.edu.hk

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3. 動手操作學數學：在 STEM 思維之下

譚克平、謝舒琪

國立臺灣師範大學

很多學生害怕數學，認為數學只在乎繁複的計算，經常玩符號的遊戲，或者是要寫一道又一道艱澀的證明題，久而久之，這樣的學習經驗，逐漸導致學生對於所學習的數學內容感到枯燥乏味，不但覺得數學十分抽象，而且產生數學與日常生活幾乎毫無關係等刻板印象。學生對數學這樣負面的態度，還常會一直持續到出社會之後，例如日本有一位文學家曾經指出，他在學校所學的幾何知識在生活中一點用處也沒有，唯一可以派上用場的，是當他走路時還是會運用三角形兩邊長的和大於第三邊的性質。

那該如何改變如此不理想的情況？我們的建議是加入一些非傳統的數學學習內容，這些內容最好能夠讓學生動手操作，增加在數學課堂中體會到參與數學活動的樂趣，並且在操作過程中能夠學習到數學推理，如果該素材能夠符合STEM的理念，提供跨學科學習的可能性，即不單只可以從數學的角度進行探究，而且還可以從工程學中設計與分析的眼光來進行研究，或者是所習得的知識是可以應用於科學的研究之上。若能找到這樣子的學習素材，將有機會可以讓學生瞭解到學習數學不是只能夠坐著聽講，而是可以實際參與在其中，不但可以發揮個人的創意，而且所學習的數學知識是有具體功能的，既可以應用於生活上，而且在現代科學中也有應用的價值。問題是有這樣子的素材嗎？

學習結理論

若要配合上述考量的話，我們認為結理論就是一個適合的素材，它原則上是符合STEM課程整合的理念，尤其是栽培學生設計與分析的能力方面。從古至今，具體的結與人們的生活密不可分(Ashley, 1953)，結理論在近代的數學與科學的研究中，是一個非常蓬勃的主題，而且在數學教育界也有學者嘗試開發一些活動，向學生介紹基礎的結理論(Turner, & Griend, 1996)，因此，我們推薦不妨在中學階段教導基本的結理論，因為在初中階段學生擁有的數學知識比國小階段多，在初中階段學習結理論可以吸收到較廣泛的知識。

可能有讀者會懷疑，中學階段學結理論是否會太難？我們建議用下述的方式來做思考，即使中學生的生活經驗有限，但結是一個他們生活中常見的素材，例如大部分學生都有打鞋帶的經驗，如果我們能夠從打出生活結的活動出發，逐漸引進數學結的基本概念，對學生來說應該是很容易接受的，尤其是當他們發現每天打出來的結居然可以從數學的眼光來看待，對於提升學習數學的動機或多或少會有幫助。作為一個數學主題，結具有親切感與新鮮感，它著重數學推理與空間能力，而且不牽涉到繁瑣的計算，學生比較容易對其產生親切感。學習數學結的相關概念，藉由動手操作的方式可以幫助學生增加學習課綱所涵蓋的數學概念，這是因為透過學習結理論可以用另外一個方式來介紹投影、方位、函數、轉換(transformation)、等價、不變量等重要數學概念，從而增進學習遷移(transfer)的成效，並且增加學生數學推理、表徵轉換、空間轉換等核心能力。因此

我們認為學習結理論應該不致於會增加學習負擔，在中學階段學習不但不算是過早學習，反而可以鞏固一般傳統課程中所涵蓋的數學概念。是故，我們認為以結為素材，在適當且合理的課程開發條件之下，對七、八年級學生而言應是有啟發性的數學學習教材。

再者，在日本已經有學者將基本的結理論開發成為適合小學、初中與高中學生學習的課程，例如可參考Kawauchi & Yanagimoto (2012) 的著作，他們的研究顯示，只要配合正式課綱的數學內容來編排結理論的學習內容，學生在對的時機點以及具備相關的數學先備知識，即能夠掌握結理論相關的內容。然而，在華人數學教育界中，這類型的研究比較缺乏，十分可惜，因此我們進行了一個研究計畫，嘗試開發一套適合初中學生學習的結理論課程，並透過打出生活結來引入相關數學結的概念。而在開發相關教材與活動的過程中，我們提出一個有趣的問題，我們心中存疑是否存在一套打結的基本步驟或動作，可以方便學習者有系統性地學習打出各種不同的結，本文的目的是要介紹這個有趣的問題以及我們初步的心得。

生活結

古人結繩記事，現代社會雖然有文字、有電腦，已經不用結繩那麼麻煩了，可是結在日常生活中仍是隨處可見。不少人在小時候的生活經驗中，即常常有打結的機會，第一個可能學會打的結就是單結，接著為了要綁鞋帶、綁禮物而學會打蝴蝶結。一般人幾乎都是在習以為常的情況下學會這種打結技巧，這些動作甚至可以不假思索、很純熟地

完成。在臺灣，初中還會安排童軍課，很多學生也因此學會許多不同打繩結的方法，例如八字結、平結等等。這些結各有不同的結構，可以用在許多地方，做很多不同的功能與用途，而在學習打結的過程中，可能會透過不同的方法，例如背口訣、看錄影帶、看圖打結等等。如果只是要打一、兩個結，上述的手法即已足夠，但問題是，繩結有超過一千多種(Ashley, 1993)，如果每個結的打法都需要一一學習，不但煩瑣，而且除非經常使用，否則並不容易記得。因此我們想自問，是否存在一套基本的打結步驟或動作，讓初學者只需學習這些動作，即可以透過該等動作輕鬆且靈活地打出大部份的結？

有鑑於前述的問題，本文以下將介紹我們的研究團隊所整理出來的六個打結基本動作，學習者藉由學習這些基本動作後，可以加以組合變成不同的繩結，這六個動作很有潛力成為一套有系統而且容易學習的打結方法。

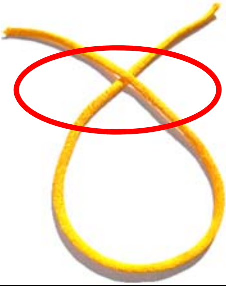
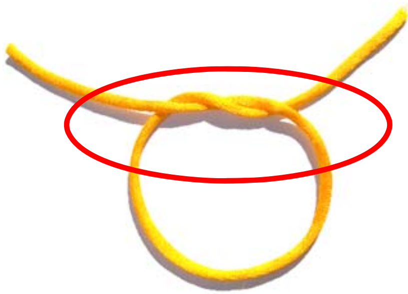
動作的說明

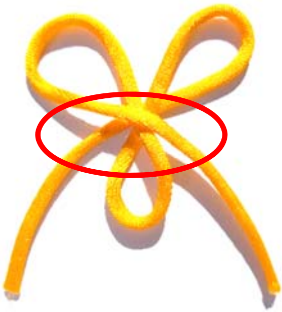
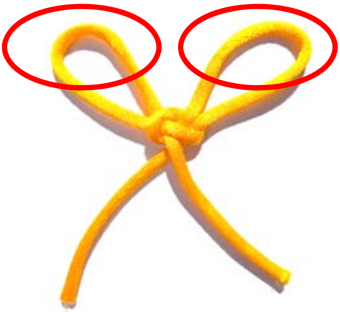
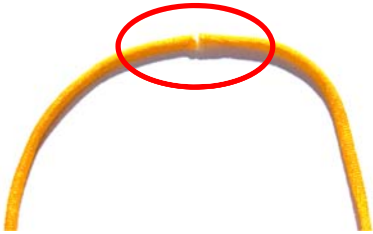
- ✧ 動作一：將繩的兩端交叉形成一個繩圈。
- ✧ 動作二：穿越繩圈，而此動作又分為從上往下穿以及從下往上穿兩種。
- ✧ 動作三：將繩的兩端拉緊，該動作通常是打出結的形狀後，再將繩子的兩端拉緊。
- ✧ 動作四：先將繩子的左右各打一個動作一，再將兩個繩圈交叉重疊，此動作通常用在打出蝴蝶結時會使用到。

✧ 動作五：通常在動作四後，如果需要成為一個結的時候，再做一個動作二，接著拉緊兩個繩圈。

✧ 動作六：當一個結打好之後，如果要將它變成一個封閉的結，只需要將兩個端點相連。

表一、打結的基本動作

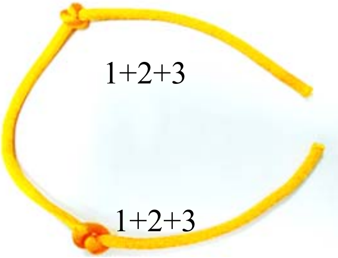
動作	圖示	說明
1		線交叉形成一個繩圈
2		前提：先完成動作 1 動作：將壓在上面的端點從下往上穿，並穿越繩圈。(或將壓在下面的端點從上往下穿，並穿越繩圈。)

動作	圖示	說明
3		前提：先完成動作 1 與動作 2 後 動作：拉緊兩端點
4		前提：在繩的左邊完成動作 1，在繩的右邊也完成動作 1 動作：將兩繩圈的交叉點重疊
5		前提：先完成動作 4 再完成動作 2 動作：拉緊兩繩圈
6		動作：兩個端點相連

在初步整理出六個基本動作後，研究團隊嘗試使用這些基本動作分析生活中常見的結，生活中常見的結很多都可以簡化成為反覆使用這些基本動作來完成。反之，也可以運用這六個基本動作打出不少常見的生活結，甚至創造出新的結。我們將一些打結的步驟整理成一個表格，如以下表二所示。

表二、運用六個基本動作分析生活中常見的結

生活結	圖例	動作
單結		1+2+3
平結		1+2+3+1+2+3

雙單結		(1+2+3)+(1+2+3)
八字結		1+1+2+3
生活結	圖例	動作
蝴蝶結		1+1+4+2+5
三葉結		1+2+3+6

研究團隊認為，與其教很多特殊的結與打法作為學習零星的例子，倒不如教導基本動作的打法更為精簡有效，一方面在教與學的過程中方便溝通如何打出某個結，另一方面則可以讓學生運用這些基本動作加以排列組合進而打出新的結，換句話說，這六個動作可以視為結的產生因子，以方便分析。

此研究為初步介紹拿繩子打結，但打結有些時候會牽涉到左右與上下的關係，我們正在嘗試做更深入的處理，加入其他的基本動作，並留待在合適的場合再做詳細介紹，不在此進行相關報導。

討論

很多家長在教導小朋友為自己綁鞋帶時，需要做很多次的示範，缺乏一套適合的語言去說明打結的動作。此外，有些教導童軍課的老師在教導打結時也是以示範為主，通常需要學生跟著老師的步驟來打，同樣也缺乏溝通的語言，而學生一恍神可能就跟不上老師的步驟，建議可以考慮使用我們的基本動作與語言來介紹與教導學生，在教導六個基本動作後，當要教導學生打出某一個結的時候，可以跟學生說待會打結需要哪幾個基本動作以及先後次序為何，然後再帶著學生去打這些結，因為這些基本動作是有系統性的，容易瞭解也容易記得，應該是一個很好的教導工具，學習打結不用單靠模仿教師的打結步驟，而且可以利用這些結去創造一個新的結。

由於要打出一個生活中常見的結需要進行分析，要創造出

一個屬於自己的結也需要進行設計，當這個課程將這些打結的基本動作過渡到結圖的學習後，我們還可以引導學生自己分析打出結的性質，因此學習打結基本上是符合STEM課程的基本精神。

本文最後想按照研究團隊的經驗，提出關於教導學生打結所用材料的建議，為了要開始此課程，我們嘗試使用不同的繩子，發現並不是任何繩子都適合，我們整理出幾個比較特別的材料，如緞帶、麻繩以及鬆緊帶等，研究團隊發現在打結過程中如果使用了有寬度的材料(例如緞帶)，在打結動作上會造成操作上的困難；麻繩因為不具有任何彈性，再加上有脫線的問題，在視覺判斷結的形狀較不容易；而鬆緊帶是使用了圓鬆緊帶，因為較不易脫線且富含彈性，較容易判斷結的形狀，且只要繩段夠長，操作上非常容易。

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4. 由一個非一般的解題方法到《九章算術》的工程問題

李柏良

多年前的經歷

(小明是一個中一學生)

老師：若單獨使用甲水管為一水箱注水，最快需 3 小時才可裝滿；若單獨使用乙水管為同一水箱注水，最快需 4 小時才可裝滿；若同時開啟甲乙兩條水管為這水箱注水，最少需多少時間才可注滿這水箱？

(過不了多久，小明筆也沒有動一動，便立刻回答)

小明：$1\frac{5}{7}$ 小時

(老師有少許驚詫，小明如何算得那麼快)

老師：非常好！小明的答案是對的！我現在給你再出一題更複雜的問題！現在又添多一條水管，若單獨使用丙水管注水，最快需 2 小時便可注滿水箱；若同時開啟甲乙丙三條水管注水，最少要多少時間才可注滿一水箱呢？

(又過不了多久，小明動筆寫下了三兩個數字，便又再回答)

小明：$\frac{12}{13}$ 小時

(這一次，老師更為詫異，看看小明寫的，只有 8、6、12 三個數字)

老師： 好！小明你可以告訴我你的計算方法嗎？

小明： 第一題，假設我同時開啟甲乙水管 12 小時，甲水管能注滿 4 箱水，乙水管能注滿 3 箱水，共有 7 箱水，所以要注滿 1 箱水，只需 $\frac{12}{7}$ 小時，即 $1\frac{5}{7}$ 小時；第二題，假設我同時開啟甲乙丙水管 24 小時，甲水管能注滿 8 箱水，乙水管能注滿 6 箱水，而丙水管能注滿 12 箱水，三條水管合共可注滿 26 箱水，所以用三條水管來注滿 1 箱水，只需 $\frac{24}{26}$ 小時，即 $\frac{12}{13}$ 小時。

以上的案例，是我多年前的親身經歷，學生的方法更直接、簡潔。

較為傳統的解題方法是先計算每條水管一小時能裝多小箱水，然後再計算一同開啟兩條、三條水管一個小時的存水量，從而算出注滿一箱水所需時間。

話說小明的解題方法較為不傳統，查實類似的算術方法，是我國古代常用的方法。

『鳧雁相逢』

劉徽(魏，約公元 263 年)、李淳風(唐，公元 602-670 年)注釋的《九章算術》卷第六均輸篇的第二十題，一般稱之為『鳧雁相逢』，有以下的描述：

今有鳧起南海，七日至北海；雁起北海，九日至南海。今鳧雁俱起。問何日相逢？

按現今的說法是：

今有鳧由南海飛到北海，需時七日；雁由北海飛到南海，需時九日。若鳧和雁同時起飛，分別從南海和北海相向飛行，二鳥於何日相遇？

『鳧雁相逢』與剛才小明的注水問題相當類似。《九章算術》在提出問題後，會提供答案(『答』)和一般的解題方法(『術』)：

答曰：三日十六分日之十五。
術曰：并日數為法，日數相乘為實，實如法得一日。

『術』可以如下理解：

『并日數為法』：將日數相加，鳧的七日加雁的九日得 16，以 16 為除數(『法』)；
『日數相乘為實』：將日數相乘，得 63，作為被除數(『實』)；
『實如法得一』：以除數為一個單位，求實所佔的分比；簡單來說，就是求商，得 $3\frac{15}{16}$ 日。

《九章算術》為《算經十書》其中的一本，《算經十書》包括：《周髀算經》、《九章算術》、《海島算經》、《孫子算經》、《張邱建算經》、《五曹算經》、《五經算術》、《緝古算經》、《數術記遺》、《夏侯陽算經》，是唐初的最高學府『國子監』算學館指定的十部課本，而《九章算術》是我國現存最早的古算書之一。其作者、成書年代不詳，專家估計不是出自一人之手，而是由西周(公元前 1044 年–771 年)到漢初(公元前 202 年–220 年)1000 年間，經多人的整理、編纂和修訂成書。《九章算術》全書共有九卷，故稱『九章』，全書的組織，是以『問、答、術』的形式程示，全書共有 246 題和 202 術。

惟每一題的『術』，主要是講述算法，是以並不容易掌握和理解『術』的背後意義。有見及此，約於公元 263 年，劉徽為《九章算術》作注。及後，李淳風再為《九章算術》注釋。

在『鳧雁相逢』這一個示例，劉徽、李淳風提供了以下的注釋：

按：此術置鳧七日一至，雁九日一至。齊其至，同其日，定六十三日鳧九至，雁七至。今鳧、雁俱起而問相逢者，是為共至。并齊以除同，即得相逢日。故并日數為法者，并齊之意；日數相乘為實者，猶以同為實也。

一曰，鳧飛日行七分至之一，雁飛日行九分至之一，齊而同之，鳧飛定日行六十三分至之九，雁飛定日行六十三分至之七。是為南北海相去六十三分，鳧日行九分，雁日行七分也。并鳧、雁一日所行，以除南北相去，而得相逢日也。

劉徽、李淳風共提出了兩個解法。第一個解是假設鳧雁共飛了六十三日(同其日)，鳧飛了九至，雁飛了七至，以六十三日，鳧雁共完成十六至(并齊)，63 除以 16(并齊以除同)，得完成一至的時間為三又十六分之十五日。

而第二個解法與現今教科書的解法差不多：因為鳧七日一至，所以鳧每日飛 $\frac{1}{7}$ 至(鳧飛日行七分至之一)，而雁則每日飛 $\frac{1}{9}$ 至；即鳧每日飛 $\frac{9}{63}$ 至，雁飛每日飛 $\frac{7}{63}$ 至，并鳧雁一日所行，9 分加 7 分得 16 分，以 $\frac{16}{63}$ 除 1 至，得 $3\frac{15}{16}$ 日。

在我初始接觸古算時，非常不習慣，就以《鳧雁相逢》為例，它的『問、答、術』不易理解。閱讀古籍有以下的一些難點。其一，古籍是沒有標點符號的，所以必要參考前輩、學者提供的評注、點校；其二，古籍用的古字與今日的用字，同字不同寫；如『麁』即現今『粗幼』的『粗』，是以要經常查找字典；其三，古代疇人對某些技術非常熟識，而我們則不然，如劉徽、李淳風在注釋中的『鳧飛日行七分至之一，雁飛日行九分至之一，齊而同之』一段內，

輕描淡寫的說到『齊而同之』；查實，『齊而同之』是有關『齊同術』，即現今的異分母分數通分的加減法；千多二千年前，分數的加減並不是一件易事。《九章算術》在卷第一方田章用到『合分術』，劉徽、李淳風便在注釋中就『合分術』，進一步解說如何『合分』，如何進行異分母分數的加法：

> 臣淳風等謹按：合分知，數非一端，分無定準，諸分子雜互，群母參差，『麄』細既殊，理難從一。故齊其眾分，同其群母，令可相并，故曰合分。
>
> 術曰：母互乘子，并以為實。母相乘為法。母互乘子；約而言之者，其分『麄』；繁而言之者，其分細。雖則『麄』細有殊，然其實一也。眾分錯雜，非細不會。乘而散之，所以通之。通之則可并也。凡母互乘子謂之齊，群母相乘謂之同。同者，相與通同共一母也；齊者，子與母齊，勢不可失本數也。方以類聚，物以群分。數同類者無遠；數異類者無近。遠而通體知，雖異位而相從也；近而殊形知，雖同列而相違也。

按以上說法，處理異分母分數的加減，基本上是使用擴分和約分而矣。以 $\frac{a}{A}+\frac{b}{B}$ 為例，先『母互乘子，并以為實』，以 $aB+bA$ 為分子(『實』)，『母相乘為法』，以 AB 為分

母(『法』)，便可得 $\dfrac{aB+bA}{AB}$。若以 $\dfrac{1}{6}+\dfrac{1}{9}$ 為例，可得 $\dfrac{1\times9+1\times6}{6\times9}=\dfrac{15}{54}$。

因為由此分數加法得到的和，未必是最簡分數，是以《九章算術》亦討論到如何用『約分術』將分數約簡：

> 可半者半之；不可半者，副置分母、子之數，以少減多，更相減損，求其等也。以等數約之。

當然，若分子分母都是偶數時，可將分子分母同時折半來約簡，如 $\dfrac{4}{6}$，分子、分母同時折半得 $\dfrac{2}{3}$，惟分子分母不全是偶數時，約分得要看看二者的最大公因數了，古時，稱最大公因數為『等數』，如以上的分數 $\dfrac{15}{54}$，古人是如何求 15 和 54 的『等數』呢？

往後便需要『副置分母、子之數，以少減多，更相減損，求其等也』了。置 54、15，以少減多，得 39、15；再以少減多，得 24、15；再得 9、15；再得 9、6；再得 3、6；最後得 3、3；兩數相等了，等數就是 3，即為 15 和 54 的最大公因數。以上的過程，減的次數是多了些，可用帶餘除法來解決，無論如何，進行的就是『輾轉相除法』；查實，

乘法可視為連加，而減法可視為連減，是故古算有『除者減也』的說法。『以等數約之』後，$\frac{15}{54}=\frac{5}{18}$

『五渠注池』

我的老毛病是愛將問題加鹽加醋，你解決了我兩條水管的問題嗎，我再加多你一條，三條水管又看你如何解！

但《九章算術》更誇張、更厲害，跨的更大步，它不問三條，它問『5條水管又如何』!一般稱卷第六均輸篇的第廿六題為『五渠注池』，問題如下：

今有池，五渠注之。其一渠開之，少半日一滿；次，一日一滿；次，二日半一滿；次，三日一滿；次，五日一滿。今皆決之，問幾何日滿池？

答曰：七十四分日之十五。

術曰：各置渠一日滿池之數，并，以為法。以一日為實。實如法得一日。

其一術：各置日數及滿數。今日互相乘滿，并，以為法。日數相乘為實。實如法得一日。

其題意為：

今有水池，由 5 條水渠注之。若只由第 1 條渠注之，三分一日可滿(當時稱三分一為少半，在劉徽的注解中，有『此其一渠少半日滿池者，是一日三滿池也』)；若只由另外一條渠注之，一日可滿一池；若只由另外一條渠注之，二日半可滿；若只由另外一條渠注之，三日可滿；若只由另外一條渠注之，五日可滿。今由 5 條水渠注之，問幾何日滿？

接下來的『術』指出『各置渠一日滿池之數，并，以為法。以一日為實。實如法得一日』，計算每一天每一條渠注池的數，加起來作為分母(『法』)，再以一日為分子(『實』)，求出的商(『實如法得一』)，便是日數。由此，若五渠注之，一日可得到『四滿十五分滿之十四』，再由倒數便可得注滿一池水的日數。

劉徽、李淳風的注釋又說：

『此猶矯矢之術也。先令同於一日，日同則滿齊。自鳧雁至此，其為同齊有二術焉，可隨率宜也』

『矯矢之術』是指卷第六均輸篇第二十三題的方法，第二十三題亦是一道工程問題，因為要計算 $\frac{3}{1}+\frac{1}{1}+\frac{1}{2\frac{1}{2}}+\frac{1}{3}+\frac{1}{5}$，所以亦屬『同齊』問題，注釋指出『同齊』的兩個方法，在『鳧雁』已有解說。

要計算$\frac{3}{1}+\frac{1}{1}+\frac{1}{2\frac{1}{2}}+\frac{1}{3}+\frac{1}{5}$，注釋提供了一個頗具特色的算法：『各置日數及滿數。令日互相乘滿，并，以為法。日數相乘為實。實如法得一日』。為進一步講解，注釋說：『列置日數於右行，及滿數於左行。以日互乘滿者，齊其滿；日數相乘者，同其日。滿齊而日同，故并齊以除同，即得也』

第一步，是要布局；為準備計算$\frac{3}{1}+\frac{1}{1}+\frac{2}{5}+\frac{1}{3}+\frac{1}{5}$『列置日數於右行,及滿數於左行』如右圖。

滿數	日數
3	1
1	1
2	5
1	3
1	5

第二步『日數相乘者，同其日』，計算通分後的分母。日數相乘得 $1\times1\times5\times3\times5=75$

第三步『以日互乘滿者，齊其滿』，通分後，將分子相加。

滿數	日數
③	1
1	①
2	⑤
1	③
1	⑤

$3\times1\times5\times3\times5=225$

$\frac{3}{1}=\frac{225}{75}$

滿數	日數
3	①
①	1
2	⑤
1	③
1	⑤

$1\times1\times5\times3\times5=75$

$\frac{1}{1}=\frac{75}{75}$

滿數	日數
3	①
1	①
②	5
1	③
1	⑤

$2\times1\times1\times3\times5=30$

$\frac{2}{5}=\frac{30}{75}$

滿數	日數
3	①
1	①
2	⑤
①	3
1	⑤

$1\times1\times1\times5\times5=25$

$\frac{1}{3}=\frac{25}{75}$

滿數	日數
3	①
1	①
2	⑤
1	③
①	5

$1\times1\times1\times5\times3=15$

$\frac{1}{5}=\frac{15}{75}$

再『并齊以除同』，$75\div(225+75+30+25+15)=\frac{75}{370}=\frac{15}{74}$，(其中 75 和 370 的等數 5 可由輾轉相除法求得)，『即得也』。

剛才『五渠注池』的算法，看似是有些笨拙及欠少許說服力，因為 $\frac{3}{1}+\frac{1}{1}+\frac{2}{5}+\frac{1}{3}+\frac{1}{5}=3+1+\frac{1}{3}+\frac{3}{5}$，真正需要進行分數加減的只是 $\frac{1}{3}+\frac{3}{5}$ 而矣。

『五渠注池』是有改善的空間，但亦不失為展示用到以上列表方法的一個簡例。

由小明的兩條水管問題到此，是時候做總結了。回顧小明的三條水管問題：

> 今有池，3 渠注之。其一渠開之，3 時一滿；次，4 時一滿；次，2 時一滿。今皆決之，問幾何時滿池？

『列置日數於右行，及滿數於左行。以日互乘滿者，齊其滿；日數相乘者，同其日。滿齊而日同，故并齊以除同，即得也』

滿數	時數
①	3
1	④
1	②

$1 \times 4 \times 2 = 8$

滿數	時數
1	③
①	4
1	②

$3 \times 1 \times 2 = 6$

滿數	時數
1	③
1	④
①	2

$3 \times 4 \times 1 = 12$

『日數相乘者，同其日』，$3 \times 4 \times 2 = 24$ ；『并齊以除同』，

$24 \div (8+6+12) = \frac{12}{13}$ 。

就今天的說法，《九章算術》的算法太過『注入式』、太過『機械化』、以強記背誦的方法來計算，似乎並不可取。惟重組古人的思考過程、知識的發展、算法的發明和發展，亦非常重要；是故撰寫本文是為與大家分享。因力有不逮，當中表達不清晰，不順暢或錯誤處，謹在這先致歉，盛蒙賜教。最後，僅以東漢王充(公元 27-97)《論衡　謝短篇》的話語作結。

『夫知古不知今，謂之陸沉。夫知今不知古，謂之盲瞽。』只知道古代而不知道現代，是迂腐；只知道現代而不知道古代，是無知。

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5. Heron's Formula Revisited

CHAN Sai-hung , LEE Kwok-chu

According to Encyclopaedia Britannica, Heron of Alexandria (flourished c. AD 62, Alexandria, Egypt), also called Hero, is a Greek mathematician. The Heron's formula can be found in his Book 1 of *Metrica*. According to the current curriculum, Heron's formula is introduced in senior secondary mathematics curriculum. Generally, students will meet the formula when they learn trigonometry. However, the proof given in the text is not the one adopted by Heron. Heron employed various propositions from Euclid's Elements to establish his formula. In this article, we would like to reproduce his original proof and introduce other proofs, all of which are different from the existing local textbooks provided. Besides, we would like to discuss how we used these proofs in learning and teaching.

Before looking into Heron's proof, we start with some preparation.

The following is an inscribed circle with radius r and in-centre I. That is, (i) ID, IE, and IF are perpendicular to AC, AB and BC respectively, (ii) AI, BI and CI are the angle bisectors of $\angle BAC$, $\angle ABC$, and $\angle ACB$ respectively.

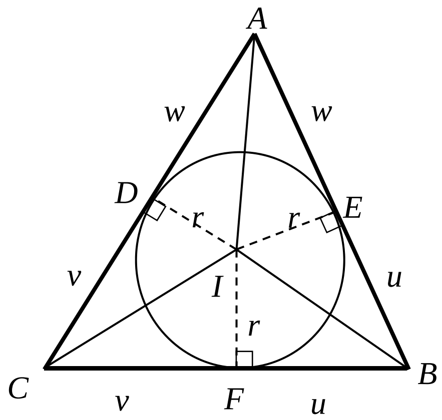


Figure 1

In figure 1, $AE = AD = w$, $BE = BF = u$, $CD = CF = v$, $BC = a$, $AC = b$, $AB = c$, $IF = ID = IE = r$ and semi-perimeter = $s = \frac{1}{2}(a+b+c)$

It can be proved that the area of the triangle = rs. The proof is not difficult and can be used as an exercise for students. The area of $\Delta ABC = \frac{1}{2}(IE \times AB + IF \times BC + ID \times AC) = \frac{1}{2}(rc + ra + rb)$

$= \frac{1}{2}r(a + b + c) = rs$

In fact, "the area of $\Delta ABC = rs$" can be viewed as follows: (see figure 2)

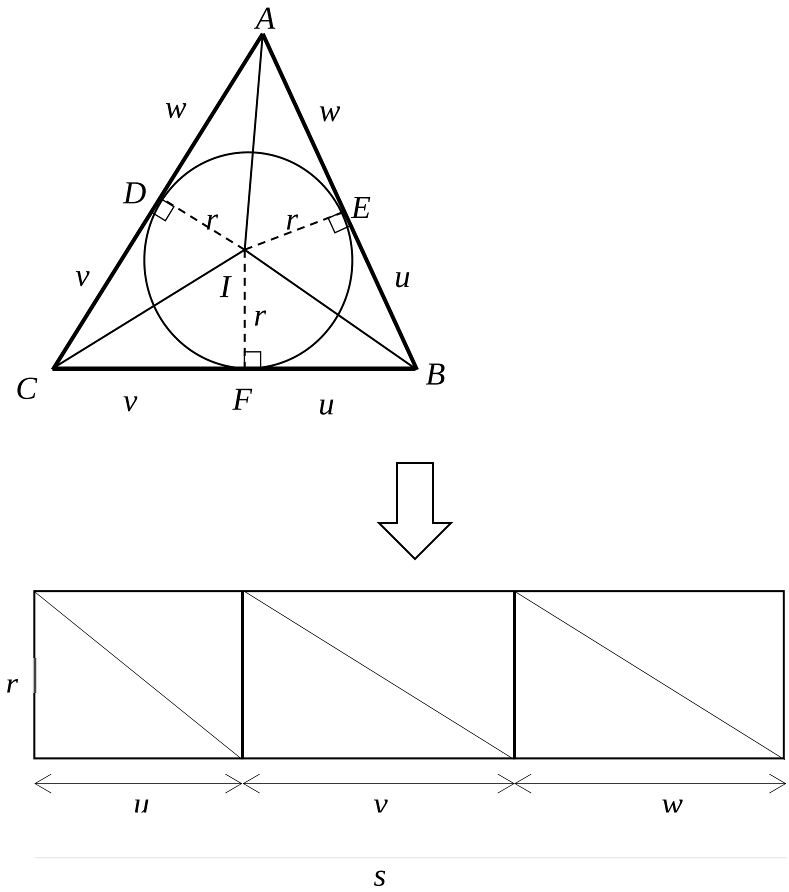


Figure 2

Now, let’s look at how Heron proved his formula.

Further, we denote $\angle DAI = \angle EAI = \alpha$, $\angle EBI = \angle FBI = \beta$, and $\angle FCI = \angle DCI = \gamma$. Then we construct two lines. One passes through I and is perpendicular to CI and another line passes through B and is perpendicular to CB. The former one intersects with BC at R. The two lines meet at Q. That is, both $\angle CIQ$ and $\angle CBQ$ are 90°. Hence, I, C, Q, B are concyclic. See figure 3.

Note that $s = AE + BF + CF$.

Therefore, $AE = s - (BF + FC) = s - BC = s - a$,

$CF = s - (AE + BF) = s - (AE + EB) = s - AB = s - c$, and

$BF = s - (AE + CF) = s - (AD + DC) = s - AC = s - b$.

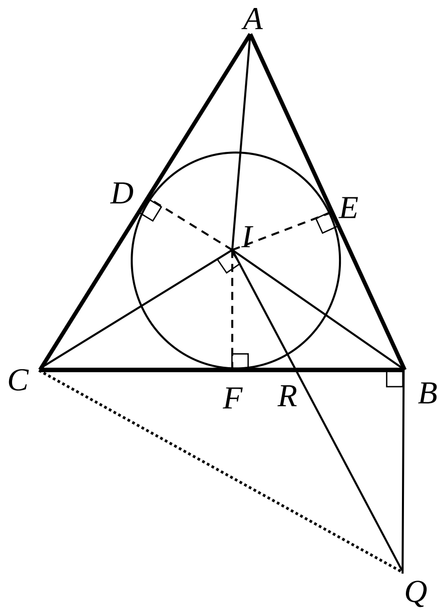


Figure 3

Next, we are going to show that $\Delta AIE \sim \Delta CQB$ and derive $\dfrac{a}{s-a} = \dfrac{BR}{FR}$. (See figure 3)

In ΔCIB, $\angle CIB = 180° - (\beta + \gamma) = 180° - \dfrac{1}{2}(180° - 2\alpha)$

$= 90°+\alpha$. Hence, $\angle QIB = \alpha$.

As I, C, Q, B are concyclic, $\angle CQB = 180° - \angle CIB = 180° - (90° + \alpha) = 90° - \alpha$. Hence, $\angle BCQ = \alpha$.

Thus, in ΔAIE and ΔCQB, we have $\angle EAI = \angle BCQ = \alpha$, $\angle AIE = \angle CQB = 90° - \alpha$ and
$\angle AEI = \angle CBQ = 90°$.

Hence, $\Delta AIE \sim \Delta CQB$.

Thus, we have $\dfrac{BC}{AE} = \dfrac{BQ}{EI}$

$$\frac{a}{s-a} = \frac{BQ}{r}$$

Also noted that $\Delta IFR \sim \Delta QBR$, we have $\dfrac{BQ}{IF} = \dfrac{BR}{FR}$

$$\frac{BQ}{r} = \frac{BR}{FR}$$

Therefore, we obtain $\dfrac{a}{s-a} = \dfrac{BC}{AE} = \dfrac{BR}{FR}$. …………….(*)

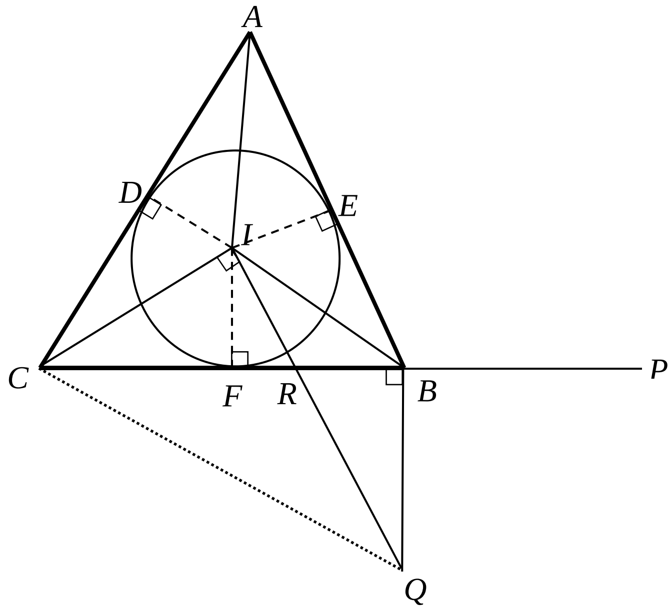


Figure 4

Then we produce CB to P such that $AE = BP$ and show that $\frac{CP}{BP} = \frac{BF}{FR}$. (See figure 4)

$\frac{CP}{BP} = \frac{CB + BP}{BP} = \frac{CB}{BP} + 1 = \frac{BC}{AE} + 1$ (by construction, $AE = BP$)

$$= \frac{BR}{FR} + 1 \quad \text{(by (*))}$$

$$= \frac{BR + FR}{FR} = \frac{BF}{FR}$$

That is, we have $\frac{s}{s-a} = \frac{s-b}{FR}$ ……………….(**)

In ΔCIR, as $\Delta CFI \sim \Delta IFR$, we have $\frac{IF}{CF} = \frac{FR}{IF}$. Hence, $IF^2 = FR \times CF$. That is, $r^2 = (s-c)FR$. ……..(***)

The final step is to make use of (**) and (***) to derive the Heron's formula.

From (**) and (***), $\frac{s}{s-a} = \frac{s-b}{FR}$

$$\frac{s}{s-a} = \frac{(s-b)(s-c)}{r^2}$$

$$sr^2 = (s-a)(s-b)(s-c)$$

$$s^2r^2 = s(s-a)(s-b)(s-c)$$

$$sr = \sqrt{s(s-a)(s-b)(s-c)}$$

Area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$. QED

It is noted that various propositions in Euclid's Elements are used in the Heron's proof. For instance, the proposition 22 in the Book III of the Elements (the sum of the opposite angles of quadrilaterals in circles equals two right angles) is used. Some propositions in the Book V of the Elements are also used.

Euclid's Elements is introduced in junior form mathematics curriculum. However, it is found that students would forget what Euclid's Elements is about when they study senior form mathematics. The plausible reason is that we seldom mention "Euclid's Elements" afterward. Instead, students were used to finding angles or lengths in the rectilinear figures or in the circles after learning numerous geometric theorems. Teachers may use this opportunity to remind them geometric theorems they learnt are from Euclid's Elements.

Though the proof given by Heron looks a bit complicated compared with the proofs given in the textbooks, we think that the original proof provides a good exercise for students to revise what they learnt. It can be a self-directed learning assignment. Teachers may split the original proof into different parts and let their students complete the whole proof.

As a matter of fact, there are other less complicated geometric proofs. Interested readers may find the proofs in the reference 1 and 2. Teachers may ask their students to read them and explain how they obtain the formula. For those who are interested in history of mathematics may find HPM Newsletters resourceful.(See reference 3) The newsletter does not only mention the original proof of the Heron formula but also how the several ancient Chinese mathematicians proved this formula or its equivalent expression. The ancient Chinese mathematicians mentioned includes Qin Jiushao(秦九韶) who was born in the Southern Sung Dynasty, Li Shanlan(李善蘭) who was born in the Qing Dynasty and Mei Wending(梅文鼎) who was also born in the late Ming Dynasty and was grown up in Qing Dynasty.

Learning and Teaching

Studying the proof is one of ways to enhance our solving problems techniques. It can be challenging even for smart students or motivated students. In this part, we would like to share some ideas how we make use of the original proof to design an exercise for a self-directed learning material. Teachers may set guided questions to complete the Heron's formula. Here are some suggestions: At the beginning of the proof, teachers may ask their students to prove that the area of a triangle equals the product of radius of inscribed circle and half of the perimeter of the triangle. That is, area of the triangle is rs. Then students may be asked the following questions:

(i) Explain why I,C,Q and B are concyclic.

(ii) Show that $\Delta AIE \sim \Delta CQB$ and $\Delta IFR \sim \Delta QBR$. Hence, prove that $\frac{a}{s-a}=\frac{BR}{FR}$.

(iii) Prove that $\frac{s}{s-a}=\frac{s-b}{FR}$.

(iv) Show that $\Delta CFI \sim \Delta IFR$. Hence, prove that $r^2=(s-c)FR$.

(v) Using the results of (ii),(iii) and (iv), prove that area of

$$\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}\ .$$

(See Appendix I)

When students complete the worksheet, teachers may facilitate students to adopt a deep learning strategy to study the proof as mathematicians do by asking some questions. Certainly, teachers may ask their students to read the original proof directly and then ask them questions if lesson time is tight. The difficulties of the questions vary depending on the capability of the students. Here are some suggested questions:

(i) In the Heron's proof, he produced *CB* to *P* and made use of it to establish $\frac{s}{s-a} = \frac{s-b}{FR}$. Is there an alternative way to establish the expression without producing *CB* to *P*?

(ii) Provide another way to rewrite the process in finding $\angle QIB$ and $\angle BCQ$.

(iii) If students read the proof directly, teachers may ask if there is possibility that line *IQ* cuts *CB* at *B* and why.

After further discussion, students could understand more about the proof. Their critical thinking skills may be enhanced as well.

Reflection

(i) The proof done by Heron contains a lot of topics in existing mathematics curriculum. It is good for students to do revision when studying the proof.

(ii) As mentioned before, there are other geometric proofs that are less complicated. Teachers may ask their students to study them and present the proofs in class.

(iii) Apart from geometric proof, existing textbooks provide other methods. Teachers may ask their students to present the idea of the proof. If the students have learnt the trigonometry section in extended module 2, teachers may discuss the proof using trigonometric identity to prove the formula. (See Appendix II)

(iv) It is also a good exercise for students to prove that the Qin Jiushao's formula $\frac{1}{2}\sqrt{a^2c^2-(\frac{a^2+c^2-b^2}{2})^2}$ is equivalent to Heron's formula. If needed, teachers may give some guidelines to their students to complete the proof. Moreover, we can ask students which one they prefer and why.

(v) Heron's formula is one of the popular built-in formulae in calculators. It is found that students could not recall it correctly. To help them recite it correctly, we may adopt a small further investigation, teachers may introduce Bramagupta's formula which states that area of a cyclic quadrilateral with sides of length a, b, c, and d is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where the s as the semi-perimeter of the cyclic quadrilateral. Teachers may ask their students to compare Bramagupta's formula and Heron's formula and ask them what we could obtain if d tends to zero. It seems that students learnt something that is not in the curriculum. However, through the discussion, students at least realize that there are four factors (s, $s - a$, $s - b$, and $s - c$) under the radical sign and reduce the possibility that they forget s under the radical sign.

Reference

1. Kung, S.H. (1992). *Another Elementary Proof of Heron's Formula*. Mathematics Magazine V65.5, p.337-338.

2. 柯志明 (2010)。有圖為證:希羅公式(Heron's Formula)。《數學教育》，30 期，91。

3. HPM 通訊第九卷第四期。

Appendix I (A sample of the worksheet on derivation of the Heron's Formula given by Heron.)

Worksheet

This exercise helps you reproduce the proof of Heron's Formula.

First, you draw a triangle ABC and then draw an inscribe circle such that the inscribe circle touches AB, BC and AC at E, F and D respectively. Then label the in-centre as I. Denote $BC = a$, $AC = b$, $AB = c$.

Semi-perimeter $= s = \frac{1}{2}(a+b+c)$ *and the radius of the inscribed circle* $= r$

Question 1

Show that the area of $\Delta ABC = rs$.

Question 2

Find the segments in your drawing equal to the length of $s - a$, $s - b$, and $s - c$.

(Hint: find the lengths of *AE*, *BF*, *CF*, *AB*, *CD*, and *BE*)

Next, we try to add some lines on your drawing. Construct two lines such that one passes through I and is perpendicular to CI and another line passes through B and is perpendicular to CB. The former one cuts BC at R. Label the intersection of the two lines as Q.

Question 3

Explain why I, C, Q and B are concyclic.

Question 4

Show that $\Delta AIE \sim \Delta CQB$ and $\Delta IFR \sim \Delta QBR$. Hence, prove that $\dfrac{a}{s-a} = \dfrac{BR}{FR}$.

Then, we add a line along CB by producing CB to P such that AE=BP.

Question 5

Prove that $\dfrac{s}{s-a} = \dfrac{s-b}{FR}$ (Hint: Start with $\dfrac{CP}{BP} = \dfrac{CB+BP}{BP}$ and note that $CP=s$. You may use the result in question 4 in the derivation)

Question 6

Show that $\Delta CFI \sim \Delta IFR$. Hence, prove that $r^2 = (s-c)FR$……..(***)

Question 7

Using the results of questions 4, 5 and 6, prove that area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$.

Appendix II

Refer to figure 2 and denote $\angle DAI = \angle EAI = \alpha$, $\angle EBI = \angle FBI = \beta$, and $\angle FCI = \angle DCI = \gamma$.

Note that $s > a$, $s > b$, $s > c$.

$$\tan\gamma = \frac{r}{v} = \frac{r}{s-c}$$

$$\tan\beta = \frac{r}{u} = \frac{r}{s-b}$$

$$\tan\alpha = \frac{r}{w} = \frac{r}{s-a}$$

Note that $\alpha + \beta + \gamma = 90°$. Thus, $\alpha + \beta = 90° - \gamma$.

Therefore, $\tan(\alpha + \beta) = \tan(90° - \gamma)$

That is,

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{1}{\tan\gamma}$$

$$\tan\alpha\tan\beta + \tan\alpha\tan\gamma + \tan\beta\tan\gamma = 1$$

$$\frac{r^2}{(s-b)(s-a)} + \frac{r^2}{(s-c)(s-a)} + \frac{r^2}{(s-b)(s-c)} = 1$$

$$r^2(s-c) + r^2(s-b) + r^2(s-a) = (s-a)(s-b)(s-c)$$

$$r^2[3s - (a+b+c)] = (s-a)(s-b)(s-c)$$

$$sr^2 = (s-a)(s-b)(s-c)$$

$$s^2r^2 = s(s-a)(s-b)(s-c)$$

$$sr = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Area of } \Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$$

6.　自主學習之數學科閱讀指導經驗分享

張樂華老師
前油蔴地天主教小學(海泓道)副校長

1. 數學閱讀的特殊性和重要性

閱讀是一個完整的心理認知歷程，數學閱讀能力的培養，除了以語文閱讀能力為基礎，還要理解數學文本。這樣，就需要學習閱讀數學的特殊技能，如理解數學的圖表和圖像、術語和詞彙、符號和公式，以及數學程序知識；而數學閱讀建基於學生已有知識，在過程中，需要學生進行假設、證明、想像和推理的心理認知歷程；若學生習得這些技能，有助他們理解和建構數學知識，建立自學能力。

學科閱讀是引導學生邁向自主學習的途徑，而數學課本內文有其獨特性，由於文本涉及數學語言，圖像語言及符號語言，學生所需要的閱讀技巧有別於語文科，故指導學生閱讀數學課本有其必要性(秦麗花，2007)。數學閱讀教學可說是一種數學語言的教學，數學語言具符號化、邏輯化、嚴謹性及抽象性，特點就是具精確的特質(秦，2007引述Astrid, 1994)。筆者也認同課本為學生學習思考的媒介，是邁向獨立學習的工具，故應用相關的理論和策略於數學教學。基於數學語言的特殊性和精確性，在教學過程中，會運用不同的方法，循序漸進，指導學生反覆閱讀，從而引起學生對數學文本的閱讀興趣，提升他們的自主學習能力，也能促進數學科的學與教成效。

2. 數學閱讀的教學策略

筆者在數學教學過程會指導學生如何進行數學閱讀，所應用的策略是參考 Reader's Handbook、秦麗花教授的文獻及其他相關研究的建議。筆者會根據學生的數學素養基礎和實際的學習情況，靈活調適數學閱讀的指導步伐和方法。秦教授(2007) 歸納一些與數學閱讀相關的文獻，並引用Jean(1995)的數學文本閱讀教學歷程，將教學分三個階段，各階段有其教學重點。下表為筆者以三個階段為本，在各階段的教學重點內加上了自己的教學心得，並將曾使用的策略一併列出，以供讀者參考。

階段	階段一	階段二	階段三
閱讀歷程	閱讀前	閱讀中	閱讀後
教學目標	連結學生已有知識，為閱讀作準備	激發推理思考，促進閱讀成效	穩固學習重點、擴展概念連結
主要教學重點	閱讀版面及標題、專有詞彙、概覽學習活動	剖析單元結構、解讀圖文、鉅觀例題、找出重點、學習專有詞彙和符號、歸納重點與摘要	整理和學習做筆記、找出不明白的地方進行思考或詢問

階段	階段一	階段二	階段三
方法和策略	提問、猜測、聯想、設訂閱讀目標	注意粗體和斜體字、可視化策略、大聲思考、減緩閱讀速度、重複閱讀、提問、判斷、探討難點、討論活動、分析圖表和圖像、合作學習	以概念圖整理內容、寫數學日誌、自擬題目、設計數學遊戲、連結生活的敍事

3. 數學閱讀教學的實踐

為培養學生閱讀數學文本的習慣，學期初首節數學課便引導學生進行課本概覽及以談話和問卷調查方式去了解全班學生對數學學習的感覺。為方便讀者閱讀，筆者將於下文中自稱為教師。

在課本概覽方面，首先，教師以提問引導學生認識課本目錄的編排和各單元學習的內容結構，目的是讓學生理解自己將要學習的內容所屬的範疇(「數」、「圖形與空間」、「度量」、「數據處理」和「代數」)，這有助學生學習歸類與連結。

3.1 閱讀前 -- 回想與預測．提問與思考

3.1.1 讓學生認識課本的結構和各單元內容

透過提問，引導學生概覽課本，下面為提問舉隅：

✧ 看看這本書的目錄，作者把你們要學的內容分為多少個單元?

✧ 單元一是學甚麼? 單元二是學甚麼?單元三是學甚麼?

✧ 小學的數學學習有「數」、「圖形與空間」、「度量」、「數據處理」和「代數」五個範疇，「五位數」，這個單元是屬於哪一個範疇? 為甚麼你會這樣想?

3.1.2 讓學生準備閱讀單元的內容

透過提問引導學生回想，連結學生已有知識，鼓勵學生預測和聯想。下面教師以3上A單元一: 五位數為例作提問舉隅。

✧ 看看第一課:五位數(一) 的學習重點是甚麼?

✧ 猜猜甚麼是「萬位」? 它代表甚麼? (教師引導學生說出個位、十位、百位和千位的意義)

✧ 「數值」是指甚麼?

✧ 你們有用過「數粒」和「算柱」嗎? 它們有甚麼用途?

✧ 想想甚麼是「記數」? 「記數」用來做甚麼? (用日常生活例子引導學生聯想)

✧ 第二課:五位數(二)的學習重點是甚麼?

✧ 五位數(二)和五位數(一) 的學習重點甚麼分別?

3.2 閱讀中— 圖像閱讀與專有詞彙

教師著學生閱讀五位數的首頁，與學生討論圖像和數學語言，引導學生找出其概念和意義。最後，以提問去評估學生對於問題的理解程度，並引導學生了解自己所學。下課前，教師鼓勵學生按著自己的能力嘗試進行延伸閱讀，提醒學生可先進行有策略的整體瀏覽，再找出符合題意的部份訊息，進行焦點閱讀。教師提問和指導程序舉隅如下:

✧ 想一想，圖A是表示甚麼?

✧ 那些「方塊」和「長條」表示甚麼? 有甚麼意義?

✧ 「9個1」、「9個10」、「9個100」、「9個1000」是表示甚麼?

✧ 「原有」的意思是甚麼?

✧ 注意B2和C對話匣內的句子，作者為甚麼要把句子放在匣內?

✧ 為甚麼作者要用紅色去表現「五位數」這三個字?

✧ 看看圖C，哪一條數柱放有珠子? 有多少粒珠子? 它有甚麼意思? 作者想說明甚麼?

✧ 著學生發聲朗讀C部份，然後與學生討論內容，如「千位數字是0，表示0」作者想說明甚麼?

✧ 閱讀課文的第一和第二頁後，你們明白了甚麼? 你們知道甚麼? 你們懂得甚麼?

4. 閱讀後 – 讀寫結合．鞏固和記錄

Dickson (1995)認為學科閱讀指導要結合閱讀與寫作，因為敘事能對所學賦予意義。學生除動手操作或動筆練習數學外，還可以不同的方式，如圖像或概念圖等去表達對數學的想法。教師每次進行單元閱讀後，便鼓勵學生用自己的語言去撰寫數學日誌，對學習內容重新說明、解釋、敘述、記錄摘要或自擬問題等，以加深及鞏固學生對學習內容的理解，並讓教師了解學生的學習狀況，如對數學概念的正誤。透過寫的歷程，加強概念的獲得和學習的成效，或甚建構知識。

在閱讀後，教師有時會設提示，引導學生思考，如「我發現…」、「在這單元中，我學會…」、「我覺得最難的是…」、「我不明白的是…」等；也有時因應學生的能力和發展階段，鼓勵學生用以下三個觀點來作反思，以提升學生的後設認知:

i. 個人對學習的觀點: 如對學習的喜好，原因、發現的陳述；
ii. 比較性的觀點: 比較兩種概念或解難方法；

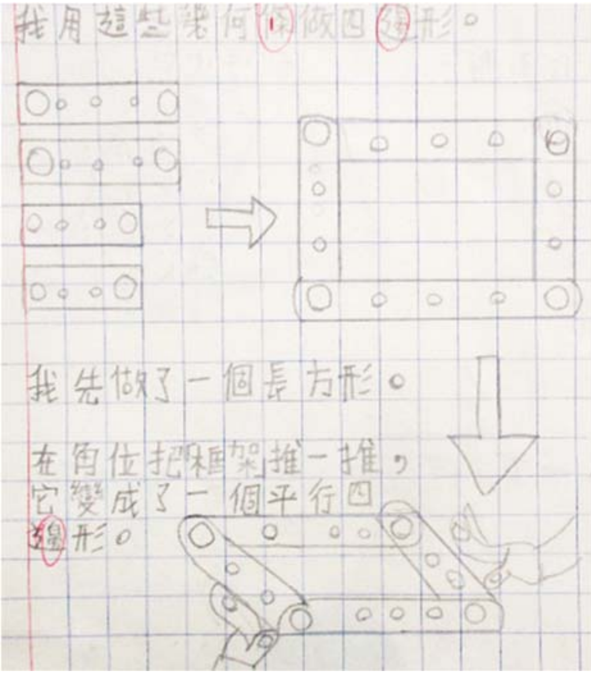
iii. 評鑑性的觀點: 對課本、例題、活動、解難方法等進行評鑑。

下表為數學日誌課業設計的例子:

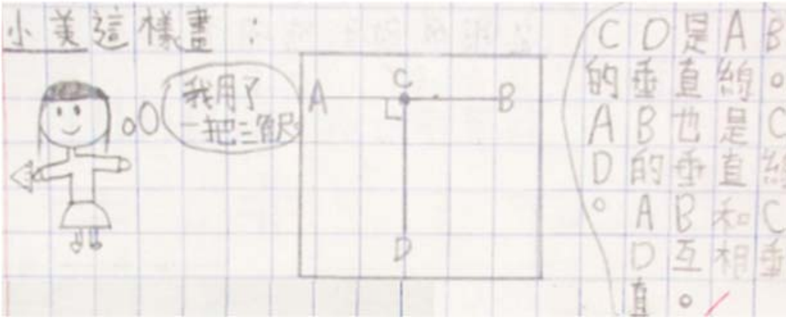
請同學完成下列各項任務：- **重溫** 3 上 B 第 15、16、17 課及**閱讀第 43 頁**，並用文字及繪圖表達下面各項。 畫出**四組**不同的**平行線**，試說說平行線的**特點**。

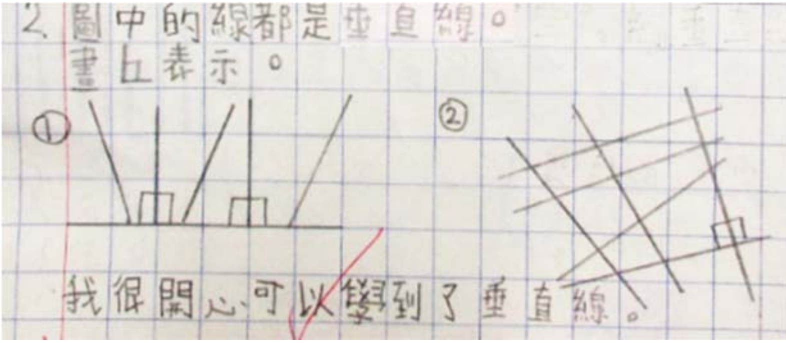
以下是一些學生的數學日誌的例子:

動手製作平行四邊形

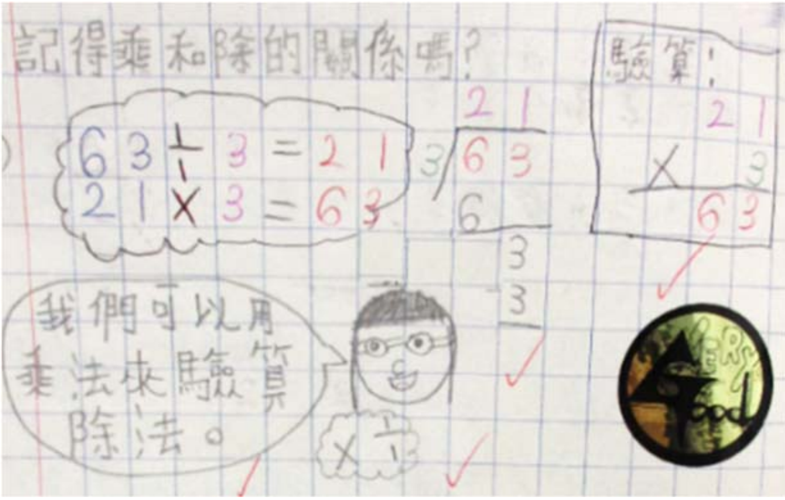


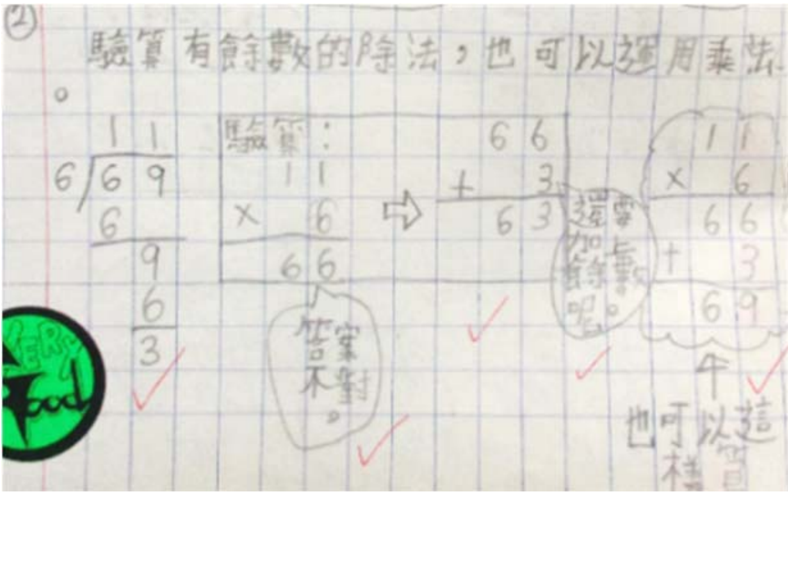
用三角尺畫一對互相垂直的線


小美這樣畫：
我用了一把三角尺
A
C
B
D

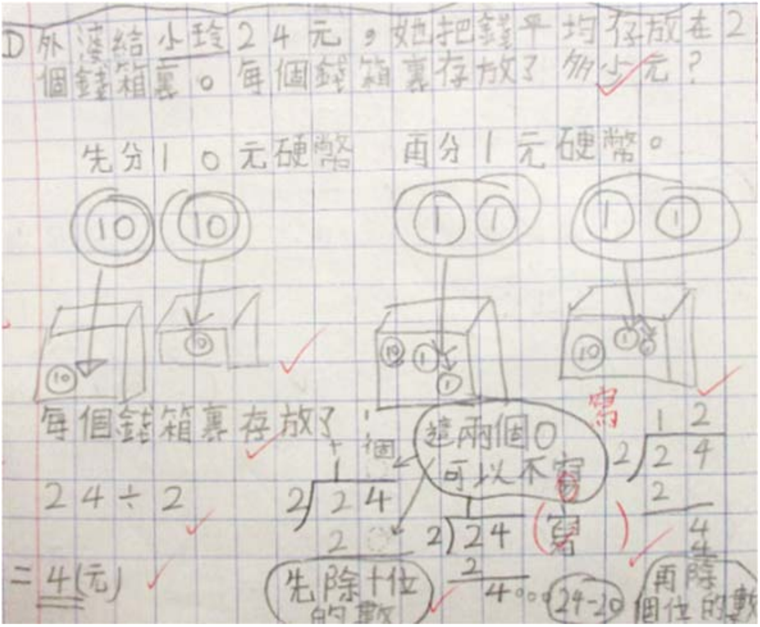

2. 圖中的線都是垂直線。
畫⊾表示。
①
②
我很開心可以學到了垂直線。

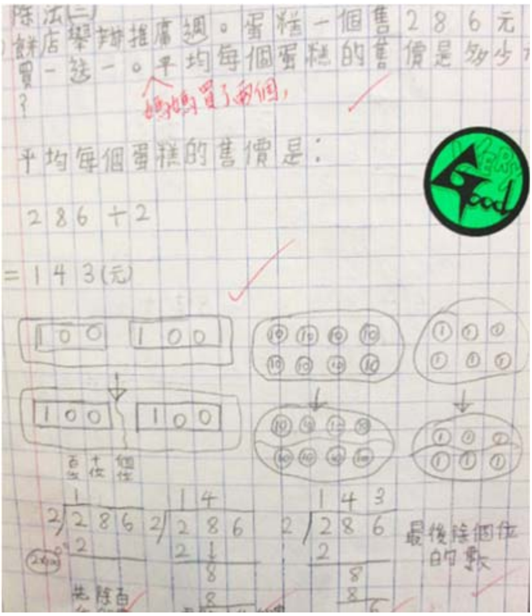
陳述如何以乘法驗算除法





自擬問題展示除法的理解





5. 教學時要注意的事項

讓學生自行進行數學閱讀前，教師要先瀏覽單元內容，以評估學生是否能理解該單元的數學專有詞彙、圖像和符號等；若學生未能理解，教師可先指導學生進行導讀，以免學生理解錯誤，產生不正確的數學概念，造成學生閱讀上的困難，甚至失去對數學閱讀的興趣。若學生沒有足夠的閱讀能力或是明辨性的思考，暫不讓他們自行閱讀或作預習功課。

數學閱讀要注意難點切入，主要是三個技巧的指導，包括解讀數學詞彙的意義、理解符號和圖像與解難問題，教師在教學時要鼓勵學生在遇到不明白的地方，可放慢再讀，讀出聲音及重複閱讀，讓學生聯想文字與符號的意義。

5.1 加強數學語言的指導

數學領域各範疇內容包含不同的概念和專有詞彙，如「圓周」、「對邊相等」；特殊詞彙，如立體圖形的「頂點 vertex」，學生要理解「頂點」這詞彙的互用性，與物理的詞彙「當彈簧抵達彈升之頂點 apex 瞬間」，或生活中的詞彙「達到了壓力的頂點」，三個「頂點」概念不同。

數學教師要指導學生認識專有詞彙，也要教導他們不熟悉的詞彙，如「垂直距離」，或透過上下文的脈絡才可理解的替代性詞彙，如「它的」、「哪些」等。這些專有詞彙包含新的概念，對閱讀能力弱、沒有先備知識及沒有彈性思考的學生較難理解。

教師不能讓學生以望文生義的方式去解釋數學詞彙的意義，而是要加強數學詞彙教學，設計多樣化活動，鼓勵學生發聲思考，幫助學生看出文字與數學之間的關係和連結概念與專有詞彙，建立概念性的知識。故在準備階段前，教師需要判斷學生是否能理解數課本的內容和概念，重溫學生的已有知識，誘發學生閱讀動機。

5.2 加強閱讀符號和圖像的指導

由於數學圖像種類多，不同的圖像代表不同的概念理解，學生在理解圖像表現較弱,所以教師在閱讀前會運用預測、自我提問、澄清、反思等策略，並設計問題以循序漸進引導學生去理解。對於數學概念的學習，會透過探究活動，給予學生明確指導，並在教學時反覆提問，給予適當的文字說明，以加強幾何圖像、表徵圖像和數學符號替代意義的理解，協助學生獲得完整的概念。

5.3 加強後設認知能力

理解應用題是學生較感困難的，許多時學生看不懂題目的意思，儘管有計算能力，也未能解決問題，教師要在教學過程時要設問題，並指導學生反思，例如「你閱讀這個問題後,學到些甚麼？」、「這個符號的意義真的代表9嗎？」、「原有數粒9000粒，多放1000粒後，真的共有數粒10000粒嗎？」、「你讀了甚麼？」、「那是什麼意思？」、「你理解關鍵詞彙嗎? 」、「你能解釋如何使用符號嗎? 」、「你清楚理解例題的意義嗎? 」等等。教師會在評估及考試後讓學生進行自我反思，填寫自我反思表或自設反思問題。透過自我反思教學，不單可以幫助學生建立和提升後

設認知的能力，亦可以幫助學生鞏固數學語言的學習。頁88的附件為筆者於評估或考試後給學生作自我後思用的表格，內容因應課題而設計。

6. 總結

數學閱讀指導是很重要的，數學教科書(課本)是課程內容的一個重要文本，一種思考和學習工具。數學課本不只是用作練習計算，教師可進一層次，指導學生閱讀數學文本，從中擷取數學概念。數學教師該裝備自己成為具有學科知識、學科教學知識和學科教學能力的學習促進者，並指導學生如何應用課本去閱讀，讓他們透過讀與寫去探索和建構知識，讓學生能「學著去讀」(learn to read)。數學閱讀的能力和習慣要慢慢建立，若教師持之以恆，幫助學生透過多次的學習經驗，建立基本數學閱讀的習慣和能力，這些能力能助學生面對未來的學習，強化自學能力，而學與教的成效也將相繼提升。經驗所見，學生由最初的讀不懂，至學期末，學生的預習能力越來越強，學習興趣越來越高，指導閱讀所需時間相對越短，整體成績往上升，可見數學閱讀是具促進教學成效的功能，也是值得推行的。

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附件

2014-2015第四次數學評估反思

姓名：:________________(　　)　　　班別：3______　　　日期：______________

	1 分析自己錯誤的原因，在適合的格子內加上✓號	
	平時沒有認真溫習、認真做功課和認真改正	
	不小心審題或不理解題意	應用題
	運算技巧未熟練，導致速度慢	應用題
	未能以恰當的文字闡述:題解或解釋答案	應用題
	寫錯字/錯別字	應用題
	未能列寫正確的算式	應用題
	計算錯誤	應用題
	不小心處理退位或進位	應用題
	未能背熟乘法表	應用題
	速度慢，未能於時限內完成	應用題
	不小心計算	式題

	2 分析自己錯誤的原因，在適合的格子內加上✓號	
	不明白題意時沒有再重新閱讀題目	
	不理解題目的要求	
	不明白如何計算時間	時間
	未能掌握三位數除以一位數	除法
	未能恰當處理答題或有餘數問題	除法
	未能運用括號列總式計算	應用題
	未能背熟乘數表	乘、除
	未能掌握閱讀及計算容器的刻度	容量
	混淆了容量的單位(升、毫升)	容量
	未能掌握比較容器的容量	容量
	評估時沒有集中精神及思想有點混亂	心理素質

	3 我的反思
	我還可以有進步空間
	我滿意自己的表現
	我會繼續努力求上進
	整體表現頗佳，我仍然會繼續努力
	我的表現未如理想，我要繼續努力

	4 我的反思
	我要專心上數學課
	我要多做練習
	有不明白的地方，我會向別人請教，把概念弄清晰
	上課時要多思考，多動腦筋
	有不明白的地方要多發問

5 為 改進/保持自己的學習成績，我會有以下的計劃(例：我會努力學習、多做練習、上課用心、多思考、多發問..)

____

6 為改進自己的學習成績，我要訂立以下的目標

____

7 我對這次學習的感想 / 我的疑問 / 其他

____

7. 小學數學教師的裝備

羅錦輝老師(中華基督教會基慈小學)

每年五、六月期間都有機會與一些到校面試的數學教師交流，討論一些有關數學概念的問題，以下是其中的一些對談：

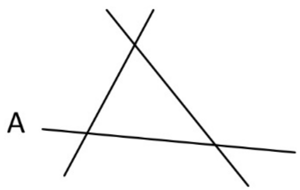
情境一（數學教師面試）：

例子一：

我：請問一個圖形怎樣才是一個閉合圖形？

師：由圖形一點出發，沿邊界一直走，可以返回該點的，就是閉合圖形

我：（在紙上繪畫一個圖形）這個是不是閉合圖形？



師：這個是呀！（毫不疑惑）

我：那如果從A點出發(A是後加的)，怎樣畫可以回到起點呢？

師：……

例子二：

另一位面試教師的對話：

我：分數除法為何要教學生將除數轉為乘以該數的倒數？可以將被除數轉為倒數然後相乘嗎？可以不做倒數直接相

除嗎？

師：我們可以先用摺紙的方法解釋，再用圖說明，最後化為算式表示。

我：十分好，但這些方法怎樣解釋上述的問題？

師：… …

另一個情境。在現職數學教師的圈子中，亦常見一些喋喋不休的討論，例如：

情境二(一些數學教師的討論)

1. 批改乘法的算式時，乘數與被乘數的位置交換了，是否錯呢？有需要改正麼？
2. 中午12時正的英文縮寫應怎樣？12:00a.m./ 12:00p.m. /12:00n.n.？
3. 一個沒有直角、四邊長度相等的四邊形，學生答平行四邊形是否合理，是否給分？
4. 折扣中八五折寫成85折可以嗎？最少在日常生活中，店鋪都是這樣寫的！

從上述例子可以看出，部分教師對數學概念、數學表達方式的應用習慣及教學方法等十分混淆，這個通常與數學教師對自身本科知識的要求及興趣不無關係，可想而知，教師對數學沒有興趣，怎能讓學生對數學提起興趣？加上部分同工認為小學數學教師的工作十分輕省，書容易教，簿容易改，亦不須專科專教。因此，有些原本任教其他科目的教師會提出轉教數學，當作小休。此外，部分數學教師對教學內容認識不深，教學上完成課本的內容，但學生仍未能掌握當中的概念和意義。

另一方面，小學生的學習階段、認知發展需要循序漸進，故教師亦不能直接將其數學概念講解一次而期望學生能掌握，必須通過生活應用及日常經驗了解身邊的事物與數學之間的關係。到底對小學數學教育的嚴謹性、包容性及趣味性應該怎樣拿捏？相信大家都明白，小學數學教育主要的方法都是由學生粗疏但具體的經驗開始，認識和探究數學世界精密而抽象的本質，所以教師對教學內容需要有一定的了解，在相關數學概念的框架下設計一系列教學活動內容，通過實作、討論，鋪排引領學生探究相關知識及概念。

A. 怎樣裝備自己

數學教師並非學者或大學教授，無需、亦無時間過份深入研究相關概念知識，那麼應如何裝備自己，以應付日常我們會過遇到的數學教學問題？

小學數學課本內容，一般都建基於該範疇的核心概念，從日常生活的例子中引入及導引學生進入認識相關的知識和技能，故教師在設計教學流程及活動、學生答案表達正確與否，都靠老師的專業判斷。而要提升專業，有以下三項建議：

1. 多閱讀：

   進修可謂不二法門，數學本科知識的建立，非朝夕之間能以達成，本科訓練須花年日才能打好基礎，好讓教師能得以知其所以然。然而，在課堂活動設計及運用上，也須要融會貫通，才能應用得宜(起碼來面試的也不乏

一些主修數學本科的教師，問著也可能會“答非所問”的)。

所以除了進修外，數學教師也宜多閱讀相關的文章或數學教育相關的期刊，以了解現今教學趨勢、課題的研究及討論等文章，促進本科知識的運用。相關文章數量頗多，目下比比皆是，也無需舉例了，至少教育局數學教育組出版的這本期刊也具閱讀價值呢！

2. 多探究：

無論教授什麼學科，定期做學科探究也有助專業成長，尤其與其他大學院校或教育局合作研究相關課題項目，大學伙伴協作計劃等，能有助教師對該課題有更深入的了解，藉以探究所學課題的概念點、學生難點處理、課題單元的整合等。如我校曾作出探究的課題有小二處理比較應用題的研究、於二年級課程內加入三角形課題是否合適、及至應用資訊科技(如平板電腦)於數學教育等等。多作這些教學研究，積少成多，累積及連結相關的課題，探究新的教學方法，與時並進，提升教學效能。

3. 多交流：

教學方法與日俱增，多與其他教師交流可提升我們的教學專業，而當中觀課是促進交流的平台，不論是同儕觀課、專家到校觀課，教學研究的觀課等專業交流觀課，對了解不同的教學法應用於學生學習上，都會有不同的效用。從觀察及討論中，很多時都能對教學有所啟發。

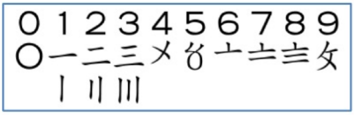
以上雖說是老生常談，亦是本人十數年來學習數學的經驗，學海無邊，期望一同共勉。

B. 再論上述幾個問題

處理以上的問題，可將其歸納為以下三個方向：

1. 約定俗成

使用習慣會隨年月有所轉變。例如以往常用的花碼，我相信年輕的同工也未必看過，可能從書本或紀錄片才有機會找到！似往常用的表示方式，隨時間而流逝。



八五折這個寫法為甚麼不能寫成85折？最明顯的原因是"85"會被理解為"八十五"，如果一折表示10%，即85折將由原價的85%變成850%了！兩者相差甚遠。但正如馮振業(2004)提出：「雖然『折扣』是一個小學數學課題，歸入百分數應用中，但它卻並非一致地被視為數學術語」，所以這個意義亦隨時代的更替有所轉變，所謂怎樣寫，一般尋常百姓從了解、共識的角度去看是可以接受的，就正如當八折寫成8折時所造成的影響也不大。

有關正午12時的問題，很多文章已經討論過了(雖然教科書都把這個問題避而不談，只以中文的"正午"、"午夜"來表示)。最初a.m.及p.m.都不會用在十二時的，因為a.m.的意思是正午之前(ante meridiem)，p.m. 的

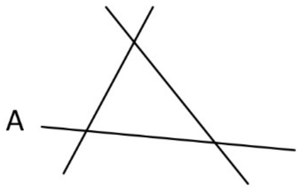
意思是正午之後(post meridiem). 所以正午十二時既不是a.m.也不是p.m.。但後來不同國家為了方便，就開始使用12.00a.m.及12.00p.m.。而多數地區及國家會用12.00p.m.代表正午，12.00a.m.代表午夜。但有些國家，如日本則剛剛相反 (https://bit.ly/2z9vqfA 日本標準時網)。所以，為了避免誤會，有人提出正午十二時以12.00 noon or 12.00n.n.表示；午夜十二時即以12.00 midnight or 12.00 m.n.來表示了。

而在教學上的建議，我認為在小學應以最“正確”而“簡單”的寫法教導學生，如折扣寫成“八五折”；12時即以12.00a.m.及12.00p.m.先開始，並讓學生從日常生活的例子中討論記錄的方法，培養學生明辨性思維，相信會更為合適。

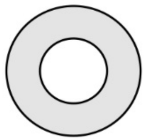
2. 概念運用

編排數學教學設計的背後，應以該項目的基本概念為基礎，才能引導學生欣賞數學美，和探求數學的動機，否則一般好像很容易明白的教學內容，深入探討一回便會出亂子。就以上述情境一為例，閉合圖形看似簡單，但出現一些不常見的例子時，老師能否解困就取決於教師本身的概念是否清晰了。

有關閉合圖形，在界定上到如今仍有討論空間。有學者曾為以折線圍成的封閉平面圖形作以下定義：「如果一折線中的任意一端點都不會只位於一條線段之上，則稱它為封閉圖形」(文耀光，2007，《幾何與度量》頁 26)。這個說法在一般簡單平面圖形上大部分情況也是適用的。我們在小學教學上仍然可以這個說法進行引入，一些特殊例子還是留待學生掌握後進一步探究是再作深入討論。例如情境一的例子即容易解答了。



當然還有些例子容易引起討論，例如：這個中空的圖形是否閉合圖形？引申到它的周界又如何找出？我們在小學階段是否要與學生討論？相信也要在課程設計上加以深思了(中空圖形的問題黃毅英教授已有討論 (https://bit.ly/2PsHWkB 現代數學網專業互動平台)，也有其他文章探討，在此不作討論)。



上述問題旨在喚醒教師自身的裝備學習，並非要在日常課堂上展示"展示實力"或"賣弄高深"，只要裝備好自己，便能設計出有利學生探究發展的教學策略，讓學生有所得著。然而在教學的建議上，我還是認為以最簡單/常見的圖形作為教學例子，讓學生容易掌握，再從探索的過程中了解背後用於四海皆通的概念，讓學生更有系統及效率地學習。

3. 通用共識

小學教育需要讓學生有系統地學習，當提出一些數學的寫法、表示方法時，無需引入太多不同的見解，「咁又得、咁又得」的說法容易讓學生混淆。我們需要以學生的學習階段進程來決定所謂的「對與錯」，學生於初小時皆以情境應用方式學習乘法，那時在列寫算式中我們會強調被乘數與乘數的關係；但到高小學習方程時，我們又會強調表示方式（數字在前，字母在後，如$a\times5$會寫成$5a$）被乘數與乘數位置又會變得模糊。

同樣有關四邊形的名稱，當學生累積了相關經驗及知識時，我們便可要求學生回應更精準的答案。通過學生不同學習階段時的答題共識，我們便有一個較清晰的回應答題標準了。曾經聽聞有一位視導教師對一位實習教師說，設計排水法活動時最適合就是用石頭作為排水的例子，比較生活化。但不禁要問，石頭是生活化的例子麼？有多少小學生在平日生活中接觸石頭？以學生接觸較多的橡皮擦為量度例子是否較為合適？這些問題讓我深思我們對學生的「數學世界」的認識有多少？如果我們不認識學生的學習模式，又如何能設計配合他們學習需要的學與教活動？

黃毅英教授提到數學教育：「是搭建由兒童周遭經驗到正規數學之路。要做到這點，老師要返回兒童對數學那種粗糙理解的層面，由他們的認知世界作起步點。教師則對數學內容本身不只要「熟」，還要「通」。透過種種教學手法讓學生建立起數學意識和關係性理解。」(黃

毅英，2007。數學化過程與數學理解。頁15)

本文非要嚴格處理上述問題,過往已有不少材料討論相關課題,可供有心人作參考。討論這些問題旨在藉此反思我們作為數學老師的任重道遠,期望我們都能好好裝備自己,作育英才,好讓學生在生活中進入奇妙的數學世界。

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8. The role of Mathematics in STEM education

TONG Man-ling
Sha Tin Government Secondary School

Introduction

'What is the use of learning this?' Many students have asked the same question when the teacher taught them Mathematics. Like learning basic trigonometry in S2, the teacher asks the students to use the ratios to find the side or angle of a right-angled triangle. Many students think that it is so boring and not useful, just for the purpose of assessment. While when the teacher asks the students to find the height of the school building, the students may find that it is interesting and the knowledge learnt is useful in our daily life. Nowadays, all schools promote STEM education, making connections across Science, Technology, Engineering and Mathematics, or at least two of them. The main aim in implementing STEM education is to support students learning in the traditional content and concepts from the subjects to solve unfamiliar problem or even create some new knowledge and learning outcomes.

Mathematics is vital in STEM education

Students are innately curious in an environment. When the teacher introduce the context like 'making a water rocket that can fly with a longer distance'. The context of 'water rocket' is used to motivate the students, while the design of lessons involves knowledge of different scopes involved and uses lots of mathematics. Mathematics acts as an auxiliary tool during the learning process of STEM activities, while most of the time none of new mathematics knowledge is deduced. When you ask the students what you have learnt after several lessons, the students may say they have made a water rocket, but will not mention they have used mathematics during the process. That's why most of the teachers think that mathematics is not valued in STEM education. In my view, a good foundation of mathematics is vital in implementing STEM education. Mathematics is mentioned as the underpinning of the other disciplines of STEM because it serves as a language for Science, Engineering and Technology. STEM education learning opportunities provide the context for enhancing the development of mathematical skills. It is a good time to let our students to consolidate what they have learnt and appreciate the uses of mathematics in our surroundings.

From my observation

As a seconded teacher this year, it was a precious time for me to visit many schools, both primary and secondary, and be an observer in their STEM education. I have joined many lesson collaboration meetings, lesson observations and post-lesson discussion meetings. Many schools didn't know how to start implementing STEM education, the school authority just assigned different subject panels to form a committee, including many subjects like Physics, Chemistry, Biology, Science, Mathematics and Computer Literacy and even

Physical Education, etc, that involved lots of teachers who didn't know how to start. In most of the lesson collaboration meetings, it was not difficult to see that the teachers have many ideas and the ambition to try. Owing to the activity with cross-subjects nature, good collaboration between teachers is an important factor to succeed. The teachers need to design the lesson plan and worksheets, they should know their students well to determine what kind of activity will be done. I have observed a S2 Science lesson, the activity used a small home swimming pool, about 2.5m×1m. Mixture with water was poured into it before doing the activity. In order to calculate the volume of mixture in the pool, the Mathematics teacher has revised with the students how to calculate the volume of a cuboid and measure errors before the lesson. When the lesson proceeded, we found that the students didn't know how to use the measuring tape to measure the length of the pool! Two students just chose any point on the opposite sides of pool and measured the length of it (Figure 1). Poor measurement made large errors in calculation. So in designing the lesson, we should make sure all the parts were well prepared include the prerequisites of our students, not only the knowledge but also the skills involved.

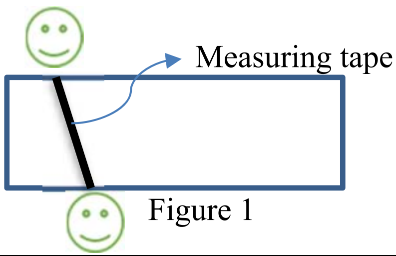


Figure 1

What should be prepared?

I would like to share a good practice after visits. A school forms a core STEM committee at the beginning of term and decide which levels and subjects will be developed in this academic year, for example focusing on S1 Integrated Science, S2 Computer Literacy, etc. The members should have basic understanding about the curriculum of different subjects. It is suggested that the activity chosen is related to what the students have learnt in order to consolidate their knowledge. List out all prerequisites before conducting the activity and create a working schedule. Different subjects must have enough input to the students according to the working schedule, and remember that the starting point of different classes may be different, like the S2 students mentioned above who didn't know how to do measurement by using measuring tape, some students needed more help! Teachers may ask if the students forgot most of the knowledge and skills they have learnt before, how we start doing the activity since we have to spend lots of time in retrieving their memories of prerequisite knowledge. To solve this problem, I suggest that the teacher may ask the students to revise that part by means of reading a web article or watching a short video clip on the internet, etc. The students may revise again and again if they really need to. Ask the students to finish a simple worksheet in order to know whether they were well equipped to learn. In fact, it is part of the process of self-directed learning that will also be used during the learning process. Decide how many lessons will be used in different subjects, the learning sequence must also be considered since some information used must be learnt from the other subject first. Designing worksheet is to guide the learning process. The ideal activity is learner-centred to provide a chance for students to do their own investigations. Try to conduct the activity in one class first. Observe the effectiveness of lesson plan and the learning outcome, refine the design and try again in another class. A good design could be used as a level-based activity in the coming academic years.

Mathematics can be used ……

During the learning process, the students should decompose the problem and make use of mathematics, some examples of mathematics will be used in different situations like:

- Simple measurements, using formula to find the dimensions of 2D figures and 3D solids

- Finding errors

- Data analysis

- Graph-plotting to show the relationship between variables and predict the trend

- Observe any patterns and make generalisations

- Make a mathematical model or use of algorithm in advanced level

Conclusion

STEM education is the worldwide trend. More and more STEM related careers arise that need its knowledge and skills. Implementation of the STEM education with the formal curriculum systematically and effectively is the concern of educationists and teachers. We have the vision that our students will be lifelong learners who will learn actively and solve different kind of problems.

9. 自主學習的反思 — 興趣為先

吳專色老師

記得自己是由小學開始喜歡數學科，因為當時的數學老師公開地表揚我的數學能力比班裡考第一名的同學還要好，小小年紀、懵懵懂懂的我便被這個肯定、這個虛榮感帶領著我踏上喜歡數學的路途，但有幾多學生可以遇到賞識他的伯樂？

其實回想自己求學時亦不見能理解所有課題的講解，只是經過不斷的操練，即使不能完全理解亦可以成功處理有關題目，但當中的我偏愛數學，有成功感推動我繼續操練下去，才可以做到熟能生巧，但對於討厭數學、害怕數學的學生，有甚麼理由可以推動他們去堅持操練？

記得曾經任教過一個中一級的學生，一個無論是她本人，還是她家人或小學老師都很擔心她的數學成績，一個對數學失去信心的學生，一個因為數學唔合格而哭的小朋友，在輕鬆的課堂氣氛下、偶然進行的小比賽、同學之間的良性競爭，但更多更多是她自己的堅持及努力，由補底班的一員，變成中一級數學科期終考的全班第三。這個令我感動的例子讓我明白，決定學生數學科的成功不是所謂的天資，而是她對數學科的接受及投入。要學生投入，要學生堅持，我相信先要令他們保有對數學的興趣，消除他們認為計數是沉悶的誤解，只要用對的方法，學生一定可以從中明白數學的樂趣，只有喜歡或感興趣才可以令學生持之以恆地學習。

經過三年的認識及試行，我相信自主學習是可以幫助提升學生學習興趣，本文只是自己個人對自主學習的小小反思及分享。

自主學習的課堂設計流程：

1. 預習影片：視乎課堂設計，可以是重溫先備知識，或者是簡單的基礎引入。影片形式很多，可由學生自行製作，亦可利用互聯網上的教學影片，但謹記片長不可多於三分鐘，千萬不要高估學生的耐性，以免浪費大家的心血。

2. 預習工作紙：按預習影片設計的檢視工作紙，精簡即可。老師不宜批改，可留給學生課後作自我檢視，但可以稍為提點學生的常犯錯誤。

3. 課堂活動：建構概念或鞏固知識，最好安排不同程度的活動照顧學生差異。

4. 課後工作紙：檢視學生能否掌握及明白新概念；如活動目標是鞏固知識，學生可重做預習工作紙，從而作自我檢視。

自主學習的好處：

- 透過不同的課堂活動，吸引學生學習興趣，提升動機。

- 課堂活動可按學生能力靈活安排不同程度的活動，既

照顧學生學習差異，亦可提升學生信心。

- 探究活動可以啟發學生思考，引發學生興趣，推動課後的延伸學習。

很高興我任教的學校參加了兩年由教育局推廣的自主學習「種籽」計劃，透過科組內同事們的合作，我們合共設計了兩個中一及兩個中二級的課堂活動。我們選擇了一些學生較難掌握的概念為課題，期望透過不同的課堂經歷幫助學生學習。我們付出很多時間作準備，過往我們用五分鐘所教授的內容，現在要用兩個連續的課堂或更多的時間去完成，教師難免抱怨浪費時間。可是，我們都不會否定這些活動的成效，雖然效果未必長久，但課堂內學生積極投入於活動中，表現比我們所預期的更佳。但如何改變學生的學習習慣？如何長期推行？如何減輕老師的工作量？都是大家要考慮的問題。

合適課題

有不少參加自主學習計劃的老師都會強調挑選合適課題的重要性，因為大家時間有限，大家都希望精心設計的課堂活動可以達到最高效益。但經過三年接觸自主學習的經驗，我發覺其實很多課題都可以推行自主學習，因為自主學習的目標可以是建構新知識，可以是鞏固知識，可以是知識的應用，等等。選擇課題可以考慮學生難處、學生能力、學生興趣，但切記內容不宜太多，重點處理，學生會較容易掌握。以下是一些參與學校處理不同課題的方法：

- 統計、幾何等課題：可以設計一些學生動手操作的活

動，加深印象。

- 生活有關的課題： 可以透過學生的切身經驗引發共鳴，強化課題的實用性。

- 數與代數的課題：可透過分組遊戲、比賽，用輕鬆的氣氛去鞏固技巧。

團隊的合作

老師習慣於現有的課堂安排，要大家走出自己的舒適區，願意破舊立新已屬難得，但在繁重工作及緊張課時的阻撓下，老師要孤軍抗戰，實在是困難重重。很高興見證不少參與計劃的學校都能安排整級科組的同工投入討論、課堂設計及準備的工作，大家分工合作。這樣不但能大大減低老師的工作負擔，合眾人之力所設計出的活動，考慮更全面，成效更佳。當然團隊合作並不是未然的，但既然大家都清楚明白團隊合作的優勢，科組領導要好好發揮自己魅力，建立一個齊心協力的團隊。

教案資料庫

記得今年的參與學校中，有老師印製一張五米乘五米並印有坐標系統的橫額，讓學生親身站在坐標系統內的體驗，有學校善用 3D 打印技術準備不同教具協助課堂學習，有學校安排動手量度的課堂活動，有學校予操練於比賽中，有學校善用電子平台的回饋功能檢視學生學習成效。每個設計都見證著老師的用心及團隊的合作。這些現存的課堂設計，如果可以組成資料庫供老師參考，要知道學生們常

犯的錯誤都是大同小異，只是各學校校情略有不同，學生能力不同，但老師可以從中得到啟發，可以節省不少的預備時間，相信都是推動及延續自主學習的一大助力。

個人展望

我不是一個十分熱愛數學的人，我亦相信無法強迫大家都喜歡數學。但對於我的學生，我期望他們不要討厭數學，因而抗拒數學，這個是我作為數學老師的起碼要求。曾經成功，亦曾經迷失，希望未來的自己仍然有心有力去發掘可以幫助學生的方法。希望各位同工亦可保存自己成為一個老師的初心。共勉之！

10. Implementation of STEM Education in Secondary School

CHENG Po-chun
Tak Nga Secondary School

Introduction

In this era of science and technology globalization, STEM education that integrates the four interdependent disciplines of Science, Technology, Engineering, and Mathematics, is one of the focal points in the ongoing renewal of the school curriculum. In STEM education, Mathematics serves as a discipline that equips students with knowledge and skills on algebra, geometry, data handling and logical reasoning that facilitate students to integrate and apply knowledge and skills across disciplines in solving real-life problems with practical solutions and innovative designs (Curriculum Development Council, 2017). Stating that mathematics underpins the other disciplines sets mathematics up in a supporting role in integrative STEM education contexts. Ideally, mathematics should be given more standing and be considered as an enabler or imperative for the advancement of understanding of concepts in other disciplines (Fitzallen, 2015). Implementation of STEM education is therefore a new challenge for most teachers. In practice, Mathematics is one of the core subjects and is taught in isolation. The majority of time engaged in teaching and learning mathematics are teacher's direct instruction and students' independent calculations.

Challenges in Promoting STEM Education

To promote STEM education in school, the design of learning tasks based on a topic in Mathematics to solve real-life problems employs relevant topics from science, engineering, or technology. However, most teachers lack both content and pedagogy knowledge of other disciplines. On the other hand, students need to work with each other and actively engage in the discussion and be equipped with a strong foundation of the knowledge to facilitate themselves to apply the skills across the other disciplines.

Support from School to Promote STEM

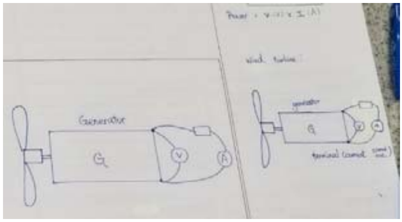
To respond closely to the ongoing renewal of the school curriculum and changing needs of parents, students and society, my school has fully supported the collaboration of Mathematics, Integrated Science and Computer Literacy to develop our school-based STEM learning task across the disciplines in S2. I was offered the invaluable opportunity to be seconded in the Mathematics Education Section of Curriculum Development Institute of the Education Bureau in 2017/18. On-the-job trainings give me exposure and involvement in two 'seed' projects, 'Exploration and Development of Effective Strategies for Promoting and Implementing STEM Education in Secondary Mathematics' and 'Exploration and Development of Effective Self-directed Learning Strategies in Junior Secondary Mathematics'. In this article, I would like to share the experience in implementing STEM education in my school, which was one of the participating schools.

STEM Education

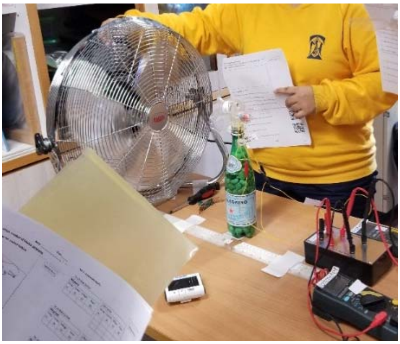
In the Science Curriculum at Key Stage 3, students should recognise the need for developing alternative energy sources (e.g. solar energy, biomass energy, nuclear power, wind power and hydroelectric power) and calculate the cost of electricity from the amount of electrical energy consumed in the Integrated Science lesson. For every unit of electricity the household consumed, the students could further investigate how much carbon dioxide was emitted to the air daily in the Mathematics lesson. The KLA in Science provides the context for developing hands-on activities to deepen understanding of conservation of the environment through the calculations of the rate of carbon emissions daily in the Mathematics lesson.

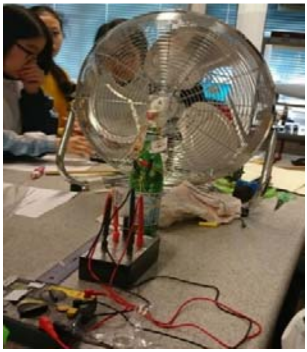
Integrated Science: Self-made Wind Turbine

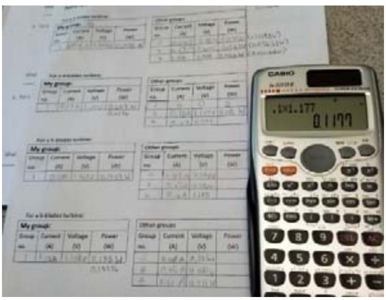
The objective of the STEM task in Integrated Science was to find the number of blades that generated the highest power of a self-made wind turbine. The students were divided into groups of four and a video was shown to demonstrate how a 3-blade of the wind turbine was made from a waste plastic bottle. Each student would make one of the blades for her group: 3-blade, 4-blade, 5-blade and 6-blade, and hand them to the teacher for carrying out the experiment in the next lesson. Good questioning techniques were important to guide students to investigate the experimental set-up and they were encouraged to draw diagrams of their circuit.



The current and the voltage generated by their self-made wind turbines with 3 to 6 blades were found. Power was calculated from the equation, Power (W) = Current (A) x Voltage (V).





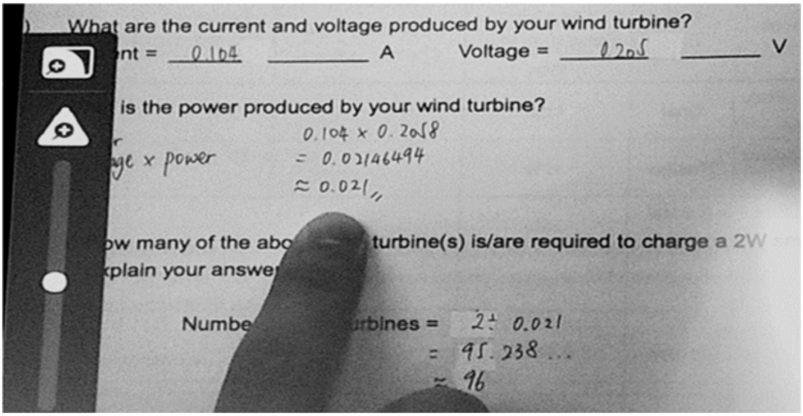


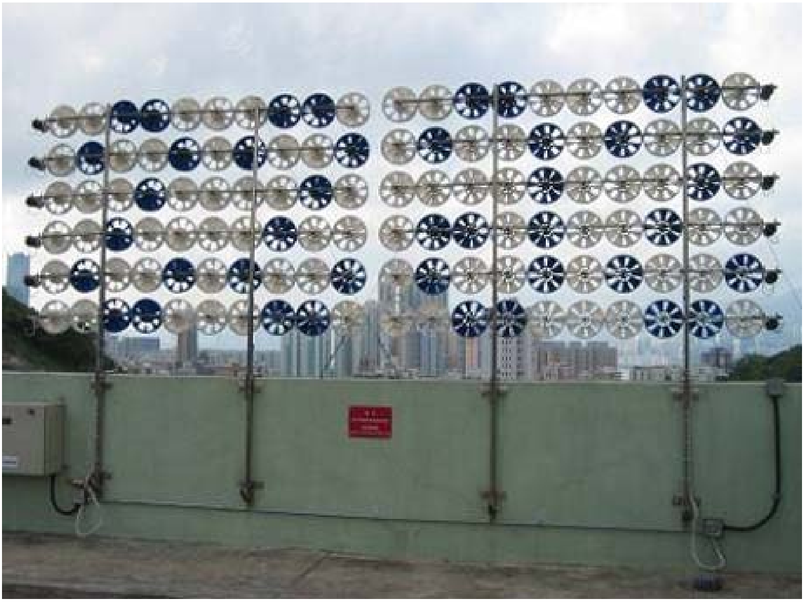
Mathematics: Rate of Carbon Emission

The objectives of the STEM task in Mathematics lesson were to find the average rate of carbon emissions per unit of electricity (kg/kWh) and the maximum and minimum average daily carbon emissions of their household electricity bills.

Lead-In

The students were grouped in the same way as in the Integrated Science lesson. The lesson started a lead-in task to associate the findings of the self-made wind turbine to the real-life situation. The power output collected in the Integrated Science lesson was used to find the number of 2W smart phones that could be charged. From the calculation, 96 self-made wind turbines made in Integrated Science were required to charge a 2W smart phone. The students thought it was unlikely to happen in real life. Thus, a 100W micro wind turbine system of Shau Kei Wan Government Secondary School was introduced to them and they were asked to calculate the number of 2W phones that could be charged by the system.





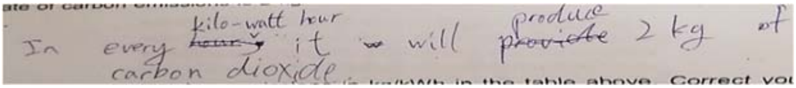
100W micro wind turbine system on the roof of the school building of Shau Kei Wan Government Secondary School

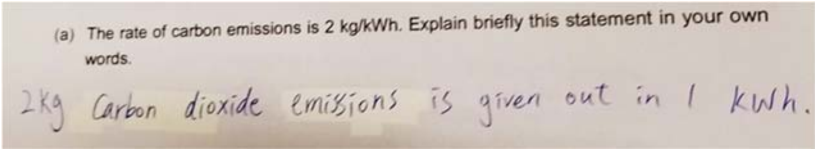
Find the Average Rate of Carbon Emission Per Unit of Electricity Sold

From the table below, students were aware that about 70% of the electricity output was generated by burning coal, natural gas and oil. Development of sustainable energy could reduce the carbon intensity in the air. 'Rate' in Mathematics was a tool to help students analyse the data and interpret what the findings represent.

It was observed that students could find out the rate of carbon emissions but could not work out the average rate of carbon emissions per unit of electricity sold in 2016. After the evaluation of the effectiveness of the STEM tasks, amendments were made before the tryout lesson of the other class.

The students were asked to explain the statement 'The rate of carbon emissions is 2kg/kWh.' in their own words. Their work was posted on the screen for discussion. The statement was well explained by most of the groups and they showed understanding of the concept of 'Rate'. It was found that the students could easily derive the answer of 0.54 kg/kWh, the average rate of carbon emissions per unit of electricity sold in 2016.


In every kilo-watt hour it will produce 2 kg of carbon dioxide


(a) The rate of carbon emissions is 2 kg/kWh. Explain briefly this statement in your own words.

2kg Carbon dioxide emissions is given out in 1 kWh.

Carbon emissions refer to carbon dioxide emissions. Carbon dioxide is a type of greenhouse gas that contributes to climate change. According to the figures of CLP Power Hong Kong Limited in 2016, 34442×10^6 kWh of electricity were sold. (1 unit of electricity = 1 kWh)

The percentage of electricity consumed by each fuel type and their carbon emissions are as follows:

Power Stations	Fuel Type	Electricity Output by Generation Fuel Type	Amount of Electricity Sold ($\times 10^6$ kWh)	Carbon Emissions ($\times 10^6$ kg)	Rate of Carbon Emissions (kg/kWh)
Castle Peak Power Station	Coal	41%	14121.22	14737	1.04
Daya Bay Nuclear Power Station	Nuclear	32%	11021.44	0	0
Black Point Power Station	Natural Gas	26%	8954.92	3745	0.42
Penny's Bay Power Station	Oil	0.003%	1.03	1.2	1.17
Guangzhou Pumped Storage Power Station	Hydro	0.997%	343.39	0	0
Total		100%	34442	18483.2	

1. (a) The rate of carbon emissions is 2 kg/kWh. Explain this statement briefly in your own words.

For 1 kWh of electricity sold/consumed, 2 kg carbon dioxide emits to the air.

(b) Calculate the rate of carbon emissions in kg/kWh in the table above. *Correct your answer to* ***2 decimal places***.

(c) What do you notice about the rate of carbon emissions in kg/kWh of coal and natural gas?

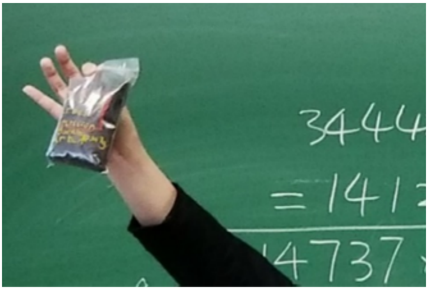
$$\frac{1.0436}{0.4182}$$

$= 2.4955$

≈ 2.50

$\therefore$ The carbon emissions of the electricity generated by coal is about 2.5 times more than that of natural gas. (Calculation is not required. Students understand why natural gas was used to generate electricity as opposed to coal.)

The complete combustion of the carbon in the bag below would emit 0.54 kg of carbon dioxide to the air.



2. Assume that carbon emissions by nuclear power and hydro power are zero, calculate the average rate of carbon emissions per unit of electricity sold in kg/kWh in 2016. *Correct your answer to* ***2 decimal places****.*

   The carbon emissions per unit of electricity sold

   $$= \frac{(14737+3745+1.2)\times 10^6}{34442\times 10^6} \text{ kg / kWh}$$

$$= \frac{18483.2}{34442} \text{ kg / kWh}$$

$$= 0.5366 \text{ kg / kWh}$$

$$\approx 0.54 \text{ kg / kWh}$$

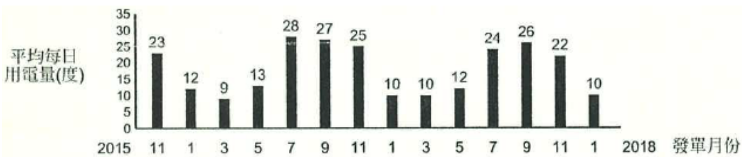
Find the Average Daily Carbon Emissions from the Electricity Bill

The electric bill of the teacher was disclosed to the students. The average daily electricity consumption was 9 units in March and 28 units in July in 2016. The maximum and minimum average daily carbon emissions attributed to his household were 15.12 kg and 4.86 kg respectively. Two bags of corresponding weights of carbon consumed in summer and winter were passed around. The students could experience the amount of carbon that was burnt to produce electricity for the daily consumption of the teacher. They were astonished to find the amount of carbon dioxide that was released into the atmosphere by each household every day.

3. According to the chart of average daily electricity consumption in your electricity bill of my house, calculate the maximum and minimum average daily carbon emissions attributed to your household in past years.

   My electricity bill:



According to my electricity bill,
The maximum average daily carbon emission
= (28 × 0.54) kg
= 15.12 kg

The minimum average daily carbon emission

= (9 × 0.54) kg

= 4.86 kg

4. Collect energy labels of different models of air conditioners. Investigate the carbon emissions of an 8-hour operation of the household air conditioners in summer night by the information on the *energy labels* of different air conditioners.

According to the energy label of an air conditioner, the determination energy consumption is based on 1200 hours/year operation.

Annual energy consumption = 966 kWh

If an air conditioner operates 8 hours per night in summer, the carbon emissions for a night

$(\frac{966}{1200} \times 8 \times 0.5366)$kg

≈ 3.46 kg

A questionnaire was conducted after the tryout lesson. 89% of students agreed that they understood more about how the renewable energy sources help to reduce the amount of carbon dioxide in the air.

Reflection on 'Seed' Project on STEM Education

Close and collegial collaboration with experienced EDB officers and many other front-line school teachers in designing school-based STEM tasks and teaching aids, discussing the expected envisaged misunderstandings and difficulties of the students before conducting peer lesson observations, evaluating the effectiveness of the STEM tasks on students' learning and proposing refinements have all enriched my subject and pedagogical knowledge in STEM education in Mathematics.

Personal professional growth was developed by providing opportunities to expose to printed reference (such as library books, journals and materials developed by EDB) and e-resources (such as Desmo, GeoGebra, Plickers, Kahoot and Scratch).

Context-based teaching and learning of STEM activities enhance both the teachers' and the students' awareness of the interconnections among mathematics, science, technology and the society. Our teachers have gained new teaching experience through trials in STEM education. To build on this new experience, the "Seed" project will definitely provide us with expertise and resources to further develop our capacity in implementation STEM education in future.

Vote of Thanks

I would like to express my sincere gratitude to my supervisor Mr CHAN Sau-tang, the Curriculum Development officers, Dr NG Yui-kin, Mr LEE Kin-sum, HO Yee-hung and Mr CHENG Sze-man and other officers, who have offered guidance and supports by giving me lots of invaluable suggestions and informative resources. I would also like to express my deep thanks to my school Principal and Vice Principal who have provided internal and external training courses and arranged common free periods for Science and Mathematics Panels for collaboration. Last but not the least, our colleagues who have devoted much of their precious time in designing and preparing the STEM tasks. All of these give me useful insight, helping me brainstorm ideas, formulate plans and discuss the feasibility of implementing the STEM and Self-directed Learning activities in schools.

Through dissemination of materials and sharing of good practices of other schools, more schools and students will be benefited as learning and teaching effectiveness will increase with the implementation of a viable STEM curriculum.

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11. 李善蘭 尖錐術

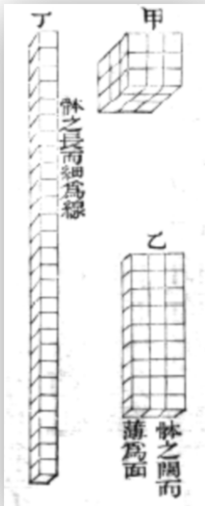
鄧廣義老師
元朗商會中學

李善蘭(1810—1882),字壬叔,號秋紉,中國清朝數學家。自幼喜愛數學,10 歲即通《九章算術》,15 歲通習《幾何原本》6 卷。35 歲,撰《方圓闡幽》、《弧矢啟秘》、《對數探源》,在三角函數、對數函數的冪級數展開式的研究上取得比前人更大的成就,他創造的尖錐術提出了幾個定積分公式,在接觸西方微積分之前,獨立地跨進了微積分的門檻。1852 年,離開家鄉到上海,與英國傳教士偉烈亞力合譯《幾何原本》後 9 卷、《談天》、《代數學》、《代微積拾級》、《圓錐曲線說》、《奈端數理》、《重學》、《植物學》等書。

《方圓闡幽》

命題一

當知世人所謂點、線、面皆不能無體。點者,體之小而微者也;線者,體之長而細者也;面者,體之闊而薄者也。



劈頭第一個命題就與《幾何原本》大相逕庭,在李善蘭的概念中,點、線、面竟然是三維的!

命題二

當知體可變為面，面可變為線。
盈尺之書由疊紙而得，盈丈之絹由積絲而成也。

由於李善蘭家中藏有古籍九章，可能熟知祖暅原理(即卡瓦列尼原理)，這種積絲成絹的比喻猶如微積分思想。

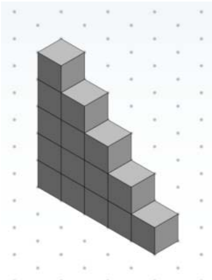
命題三
當知諸乘方有線面體循環之理。
方而因之則長，長而因之則匾，匾而因之則復方。
命題四
當知諸乘方皆可變為面，并皆可變為線。

由於點線、線、面皆是三維，所以可以重新拼合成為任何形狀。

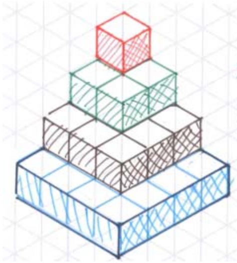
這四條命題曾經被人猛烈批評，與幾何原本中的點、線、面定義大相違背。因為徐光啟先譯幾何原本前六卷而李氏續譯，其實李善蘭自幼熟讀幾何原本，當然熟知其中的抽象定義；這四條與幾何原本相違背的命題，正正說明了李善蘭的數學思想不同於傳統的歐氏幾何觀念。

命題五
當知平立尖錐之形。

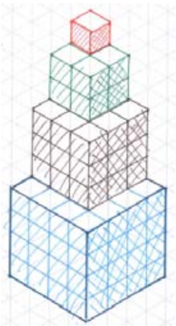
命題六
當知諸乘方皆有尖錐。



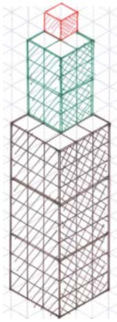
元垛(乘方數 p=1)



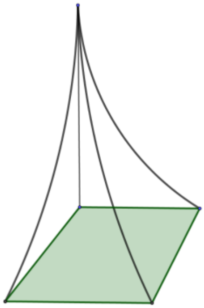
一乘方垛(乘方數 p=2)



二乘方垛
(乘方數 p=3)



三乘方垛
(乘方數 p=4)



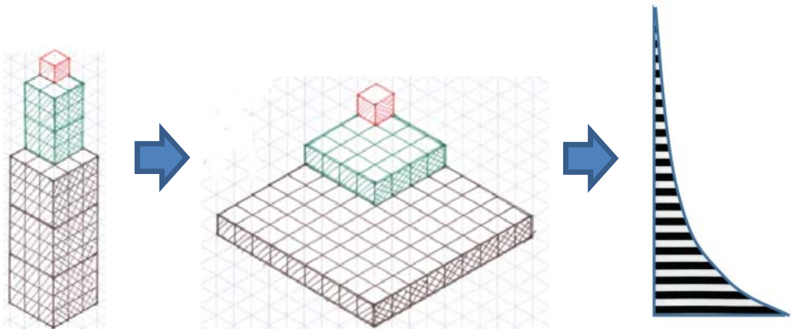
(p=n)

(元垛(p=1)由 $1^1+2^1+3^1+\ldots+n^1$ 個方塊所組成；

一乘方垛(p=2)由 $1^2+2^2+3^2+\ldots+n^2$ 個方塊所組成；

二乘方垛(p=3)由 $1^3+2^3+3^3+\ldots+n^3$ 個方塊所組成；

三乘方垛(p=4)由 $1^4+2^4+3^4+\ldots+n^4$ 個方塊所組成。)



(上圖以三乘方垛為例示範命題四：當知諸乘方皆可變為面，并皆可變為線。)

命題七

當知諸尖錐有積疊之理。

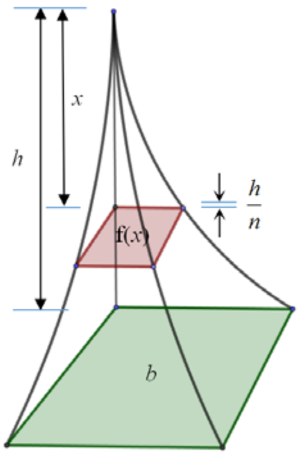
命題八

當知諸尖錐之算法。

以高乘底為實，本乘方數加一為法，除之，得尖錐積。

此定理說明：如果尖錐之乘方數為 p ，則體積$=\dfrac{高\times底}{p+1}$；相當於定積分 $\int_0^h x^p dx=\dfrac{1}{p+1}h^{p+1}$ 。

李氏並沒有提供證明，由後世的數學家補充如下：



設尖錐之乘方數 p，層數為 n，底部面積為 $b=n^{p}$，高度為 h，則每層的厚度是 $\frac{h}{n}$，第 k 層和頂點的距離為 $x=\frac{kh}{n}$，所以第 k 層的面積就是 $\mathrm{f}(x)=k^{p}=(\frac{nx}{h})^{p}=b(\frac{x}{h})^{p}$ 。

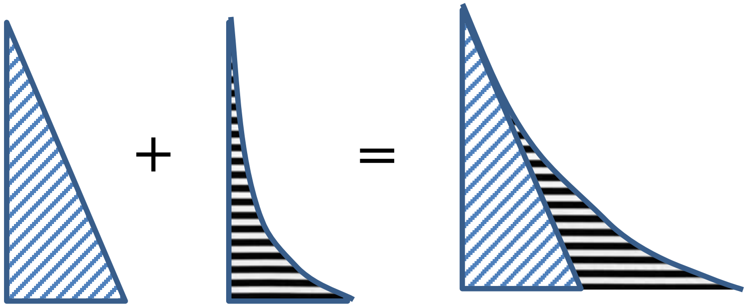
因此第 k 層的體積等於 $\mathrm{f}(x)\cdot\frac{h}{n}=b(\frac{x}{h})^{p}\cdot\frac{h}{n}=\frac{bh}{n^{p+1}}k^{p}$ ；最後，所有層數的體積總和得出尖錐體積為 $\frac{bh}{n^{p+1}}(1^{p}+2^{p}+3^{p}+\ldots+n^{p})\rightarrow\frac{bh}{p+1}$ (當 $n\rightarrow\infty$)。

這裏牽涉到自然數冪和公式

$1^p+2^p+3^p+\ldots+n^p=\dfrac{1}{p+1}n^{p+1}+\ldots$，李善蘭使用了獨特的垛積術求得，限於篇幅所限，從略。

命題九

當知二乘以上尖錐，其所疊之面，皆可變為線。



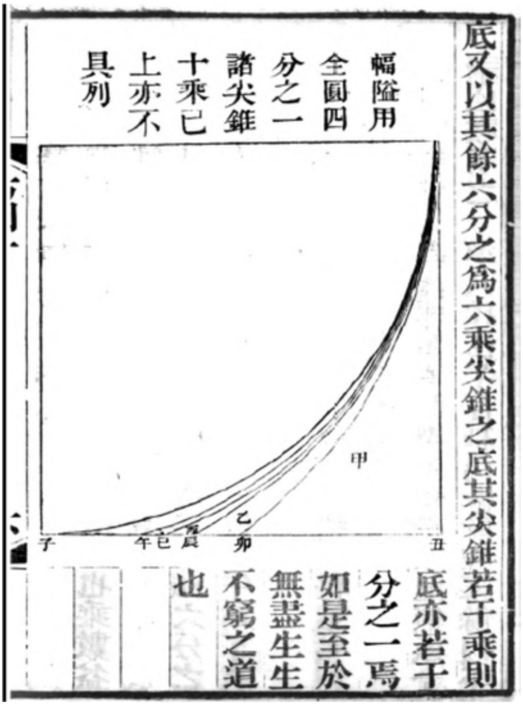
因為點線面可以互通，三維的體積就可以轉化為平面的面積，並加以處理。。

命題十

當知諸尖錐既為平面則可併為一尖錐。

這相當於定積分

$\int_0^h(\mathrm{f}_1(x)+\ldots+\mathrm{f}_n(x))dx=\int_0^h\mathrm{f}_1(x)dx+\ldots+\int_0^h\mathrm{f}_n(x)dx$ 。



以尖錐術割圓

在列出了十個命題作為尖錐術的基本理論之後，李氏立即應用此理論求出圓周率，他使用了無窮級數：$1-\sqrt{1-x^2}=\frac{1}{2}x^2+\frac{1}{2\cdot 4}x^4+\frac{1\cdot 3}{2\cdot 4\cdot 6}x^6+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^8+\ldots$。但此時微積分等尚未傳入中國，李善蘭是如何得知此公式呢？根據原著提及，他使用開方術求得

$$\sqrt{1-(0.0001)^2}=\sqrt{0.99999999}=0.9999999949999999874999999374999996093749997265624 9\ldots$$

所以

$$1-\sqrt{1-(0.0001)^2}=0.000000005000000012500000062500000390625000273437 50...$$

注意到在眾多 0 位之間出現的數字：

$0.5=\frac{1}{2}$(再乘以10^{-8})

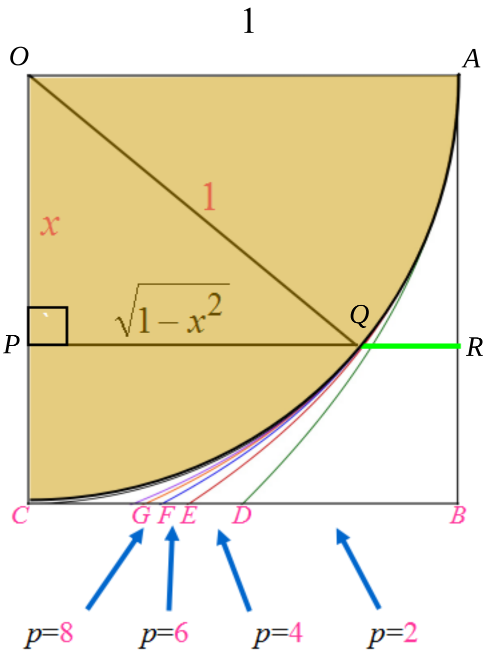
$0.125=\frac{1}{8}=\frac{1}{2\cdot 4}$(再乘以$10^{-16}$)

$0.0625=\frac{1}{16}=\frac{1\cdot 3}{2\cdot 4\cdot 6}$(再乘以$10^{-24}$)

$0.0390625=\frac{5}{128}=\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}$(再乘以$10^{-32}$)

$0.02734375=\frac{7}{256}=\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}$(再乘以$10^{-40}$)

他使用$1-\sqrt{1-2\times(0.0001)^2}=\sqrt{0.99999998}$再驗證一次，證實該式確實無誤。(超強的觀察力啊！)



設 $OABC$ 為邊長 1 的正方形，以 OA 作半徑畫四分一個圓，橫線 PR 與圓相交於 Q 點，所以

$$QR = 1-\sqrt{1-x^2} = \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^8 + \ldots$$

，因此

第一個尖錐應為平方數 2，配以底部長度為 $BD = \frac{1}{2}$，

第二個尖錐應為平方數 4，配以底部長度為 $DE = \frac{1}{2\cdot 4}$，

第三個尖錐應為平方數 6，配以底部長度為 $EF=\frac{1\cdot 3}{2\cdot 4\cdot 6}$，如此類推。

把上述(無限個)尖錐重疊，求得白色面積為

$$\int_0^1 \left(\frac{1}{2}x^2+\frac{1}{2\cdot 4}x^4+\frac{1\cdot 3}{2\cdot 4\cdot 6}x^6+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^8+\ldots\right)dx$$

，

$$=\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2\cdot 4}\cdot\frac{1}{5}+\frac{1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1}{9}+\ldots$$

所以 $\frac{\pi}{4}=1-\left(\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2\cdot 4}\cdot\frac{1}{5}+\frac{1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1}{9}+\ldots\right)$。

筆者嘗試以 excel 演算，結果如下：

項數	π(約近值)
56	3.142477113
500	3.141626240
50000	3.141593717
100000	3.141593030

以上圖形筆者以 GeoGebra 檔案繪製，搜尋關鍵字為 Li Shan Lan，有興趣的讀者可嘗試下載：[https://www.geogebra.org/m/DJmMWgJw]。

結語

在求圓周率這個問題上，自劉徽的割圓術直至明安圖的割圓連比例法，鮮有突破。李善蘭另闢途徑，使用了尖錐術及無窮級數，實在令人耳目一新。

作者電郵:
tangkwongyee@yahoo.com.hk

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12. 「STEAM 教育」的推行

李永佳老師、黃碧瑤老師、楊承峻老師
天主教領島學校

- **STEAM 教育的由來**

自教育局於 2015 年推行 STEM 教育，希望學生透過不同領域的跨學科學習和價值教育提升自學和解決問題的能力，從而激發學生的創意潛能，為社會培育多元人才，提升香港在國際的競爭力。

比對世界上先進的國家，TIMSS 世界測試的結果均顯示香港在 STEM 發展上必須急起直追，除了教育局在課程上作出變革更新學習元素外，學校同時也應就著校本的需要發展出切合學校的 STEM 學習元素和特色。

- **推行 STEAM 教育的計劃**

要讓學校順利推展校本 STEAM 教育，課程的規劃和行政的安排同樣重要，如核心小組的成立、教師教擔和時間表等。此外亦需要得到家長和全校師生的認同和支持，STEAM 教育才能順利推展。

學校在推行 STEAM 教育時，初時以培養學生興趣和加強教師培訓為首目標，以發展編程和科探雙軌並行，隨後加強各科的配合發展，相關計劃如下：

- 於 2016-2017 年度成立 STEAM 小組，籌劃各科組在 STEAM 教育的發展和協調

- 參與 QEF：From Coding to STEM 和 CoolThink@JC 計劃，加強教師專業培訓和擴闊網絡聯繫

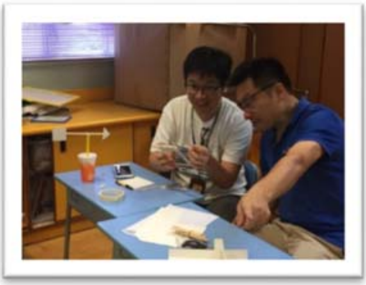
- 舉辦 STEAM 學習週和 STEAM 嘉年華，展示學生在 STEAM 學習的成果

- 參與校外不同的 STEM 體驗活動和比賽

- **如何在數學科推行 STEAM 教育**

本校舉辦的 STEAM 嘉年華，以往都是以常識科、電腦科為主導。近兩年開始嘗試以數學科為主導，設計相關的 STEAM 主題活動。為了令活動順利進行，於 STEAM 嘉年華舉辦前，我們會有一連串的鋪排。分別為教師工作坊、STEAM 課堂及 STEAM 嘉年華。

首先，我們會為教師準備了一個相關的工作坊。於工作坊內，讓教師先了解 STEAM 在數學科扮演的角色以及體驗學生的 STEAM 課堂內容。我們邀請了出版社協助，讓老師能在工作坊內體驗 STEAM 課堂，實行「落手落腳去做」。



第一年(2016-2017 年)，由於是試驗的階段，STEAM 課堂以每兩級進行。(如下圖)

一、二年級	三、四年級	五、六年級
小富翁	風繼續吹	低頭一族

隨著第一年的嘗試，第二年(2017-2018 年)逐漸變成以一級進行一個主題，並嘗試扣緊各級所學的數學課題。(如下圖)

	主題	數學理念
一年級	家庭樹相架	平面圖形
二年級	風繼續吹	四個方向
三年級	空氣炮	長度
四年級	低頭一族	圖形的穩定性
五年級	電動風扇	距離及面積
六年級	橡皮放映機	速率

推行 STEAM 數學教育，我們的理念是每個 STEAM 主題都以一個設計誘因引入。教師在學生進行體驗活動前引入設計誘因，提問學生一些有趣的社會現象，並刺激學生設計出相關產品以解決社會現象所衍生的問題。

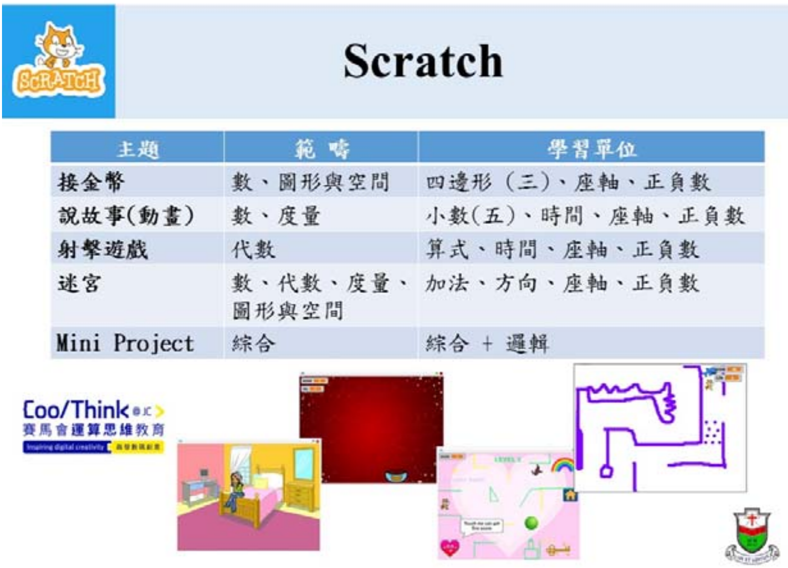
以四年級 STEAM 主題為例—社會現象是：人們現今使用電子器材的時候，經常地需要低下頭，成為了低頭一族！教師提出可以怎樣設計及製作一個承托平板電腦的裝置，以減輕低頭使用平板電腦而產生的頸椎問題呢？

教師需要引導學生設計裝置；從設計所使用的圖形，學生需要聯想所使用圖形的特性，例如，三角形具有穩定性、四邊形具有不穩定性等，如何影響裝置成功與否。從課堂中經歷多次的嘗試，最後找出成功設計裝置的原因。教師

於 STEAM 課堂後，再挑選優秀的作品於 STEAM 嘉年華作展覽。

- **編程中的數學**

學校一至六年級均會接受不同程度的編程教育，其中四至六年級會學習 Scratch、App Inventor 及 mBot。其中，我們發現大部份數學能力較好的同學在編程上亦有較好的表現，而且他們都具有運用計算思維解決問題的能力。



主題	範疇	學習單位
接金幣	數、圖形與空間	四邊形（三）、座軸、正負數
說故事(動畫)	數、度量	小數(五)、時間、座軸、正負數
射擊遊戲	代數	算式、時間、座軸、正負數
迷宮	數、代數、度量、圖形與空間	加法、方向、座軸、正負數
Mini Project	綜合	綜合 + 邏輯

其實，在學習編程的時候涉及不少有關數學的課題，以公式為例，我們讓學生以 App Inventor 設計一個計算身高體重指數(BMI)的流動應用程式(簡稱：App)。當中，他們需要了解計算身高體重指數的公式，更涉及四則運算、重量、高度、平方和代數等數學課題。

App Inventor

主題	範疇	學習單位
矩形的周界和面積	1. 度量 2. 代數	1. 周界、面積 2. 公式
BMI APP	1. 數 2. 代數	1. 四則運算、平方 2. 重量、高度 3. 公式
彈球遊戲 PinBall	度量	座軸、方向、速率
加法遊戲	數	座軸、加法、隨機
擲骰子	數據處理	隨機數、概率

另外，每一次編程教育都需要一個主題，再讓學生運用相關的編程技巧設計出一個與主題相關的程式。在行政安排上，教授四至六年級編程教育的老師均是數學科老師，所以我們亦會構思一些與數學相關的主題，例如：以 Scratch 製作出四邊形的過關遊戲，讓學生在遊玩及製作時對各種四邊形的特性更深刻。

特性 \ 四邊形	正方形	長方形	平行四邊形	菱形	梯形
兩組對邊分別互相平行	✓	✓	✓	✓	
兩組對邊長度分別相等	✓	✓	✓	✓	
兩組對角分別相等	✓	✓	✓	✓	
四條邊長度相等	✓			✓	
四個角都是直角	✓	✓			
只有一組對邊平行					✓

總括而言，編程教育與數學知識密不可分，很容易地與電腦科形成一個橫向連繫，只要我們在編寫相關進度時有良好的溝通，學生又多一次運用數學知識的機會，使學習更具成效。再者，我們在構思編程教育的主題時，花多一點心思與數學課題扣連起來，我們所教授的數學科便能全年實踐 STEM 教育。

- **STEAM 教育的展望**

STEAM 教育推行有賴不同學科和組別的同工配合和通力努力，透過不斷的嘗試汲取經驗和優化，推行時仍不忘引導學生應有的態度和精神，為我們的孩子裝備好，迎接廿一世紀的挑戰。

13. Design Rationale and Implementation of Summer Gifted Programs for Mathematically Gifted Students

KWAN Cheuk-kuen, Anderson
Associate Member, Centre for Advancement in Inclusive and Special Education (CAISE), Faculty of Education, The University of Hong Kong
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andersonkwanck@gmail.com

Introduction

Hong Kong students often obtain sound results in international mathematics tests and surveys such as the OECD Programme for International Student Assessment (PISA) (Patton, 2011) and Trends in International Mathematics and Science Study (TIMSS) (Leung, 2009) respectively for many years before.

Also, it is always encountered by most math teachers that there are a few students in their math classes who could often learn different topics very quickly on their own pace even without teacher's teaching and guidance. They could also do all questions on their own for any chapters especially for some harder problems even the chapter is newly taught.

Besides, those students would actively and regularly ask for some challenging questions or searching some harder problems from other resources. It is a common practice for some math teachers to find some harder problems for different topics in advance or to hold after-school advanced math courses for them to do or attend. The other common practice is that math teachers would nominate those students to join advanced math courses held by outside organizations. Those widely known organizations include the Hong Kong Academy for Gifted Education, the Center for the Development of the Gifted and Talented of Hong Kong University of Science and Technology, and the Program for Gifted and Talented offered by the Faculty of Education at Chinese University of Hong Kong, etc.

Besides, the Academy for the Talented of University of Hong Kong (https://aal.hku.hk/talented/about/membership-system) will invite academically elite students from nomination of top ranking students by selected secondary schools in Hong Kong and overseas.

Over the years, the demands for the gifted programs in Hong Kong such as talks, courses, courses, etc. have been eventually raising indeed. Many parents are willing to spend much money for their children to join gifted programs.

In 2013, the author and an associate professor of Faculty of Education at the University of Hong Kong jointly published a paper in Gifted and Talented International (GTI) which is a journal of the World Council for Gifted and Talented Children and the main focus of the paper was to mainly see the implementation of a summer gifted program the author held it in 2012 and the analysis from participants' feedbacks for the program. Besides, the paper also mentioned the current development of gifted education in Hong Kong.

In this paper, the author would also discuss the mathematically gifted education from different perspectives. He would raise some questions and reflections about mathematically gifted students first that most mathematics teachers or anyone who are interested in mathematically gifted education should concern. Then, he would have literature reviews about mathematically gifted education in response to the first three questions/reflections he raised. He would also share his own experiences of designing and implementing some gifted programs at HKU for mathematically gifted students in response to the last two ones. At last, the latest development of the gifted education in the world would also be mentioned as well.

It is expected that readers could have a clearer and more comprehensive picture of mathematically gifted education as well as get some insights respectively from the paper. It is, of course, welcome for readers to send their own experiences or other resources for nurturing mathematically gifted students to the author for ideas exchange or further discussion.

Before the author starts to raise the questions and reflections below, let him herewith introduce himself as well as the background of the courses for mathematically gifted students first so that readers may understand more why and how he would design and implement the courses in that way.

Anderson Cheuk-kuen Kwan is a part-time lecturer and course designer of the summer gifted programs held by the Centre for Advancement in Inclusive and Special Education (CAISE) at Faculty of Education of HKU since 2013. Besides, he is also one of the founding members of the Special Interest Group of Gifted Education, Creativity and Talented Development (SIG-GECD) of CAISE at Faculty of Education of HKU which was launched in 2012.

Then, CAISE launched summer programs such as seminars, courses and workshops for primary and secondary gifted students since 2013 and also started to hold programs for kindergartens, parents as well as teachers since 2016 so that more and more people could also benefit from the programs.

For gifted summer courses, they covers a variety of subjects such as math, IT, business, affective education, languages and so on and course participants may learn the subject contexts beyond their normal school curriculum during the courses so that the course participants would widen their horizons in the course(s) they enroll. Additionally, the contexts of those courses not only satisfy course participants' learning need and interests, but also give course participants more opportunities to raise their communication skills, critical skills as well as interpersonal skills during participating the course activities. Much research findings (Neihart, Pfeiffer, & Cross, 2015) also indicate that gifted students have social and emotional problems, therefore the captioned skills are very crucial to their better growth.

The main served groups for those courses would be for gifted students coming from Hong Kong, Macau as well as mainland China. If any students would like to enroll the gifted courses, they may find a nominator who is their school teacher, parents, etc. to download a form from CAISE's website and fill out the form. In the form, a nominator must simply describe why the students are suitable for the course(s) they enroll with providing evidences.

In general, the aims of those programs for gifted students are as follows:

- To enhance course participants' creativity, higher-order thinking skills, personal and social skills, and career and talent development

- To help course participants recognize problems and explore approaches to their solution

- To help course participants become more autonomous learners

- To help course participants appreciate individual differences and human diversity

(Adapted from the CAISE's website: https://caise1.wixsite.com/hku-caise-summer)

Questions or Reflections for Mathematically Gifted Students raised by the author

1. Whether may students good at math assessment results of inside or outside school only be identified as a mathematically gifted student?

2. What are the specific learning needs of mathematically gifted students?

3. What is the three-tier implementation model in Hong Kong for gifted students and how it can satisfy different gifted students' learning needs?

4. How are gifted courses designed and implemented for mathematically gifted students in the courses the author designs and holds so that the elite students could benefit from them?

5. What are the feedbacks for the course participants upon

completion of the courses? What could we be inspired from their feedbacks?

In the following, the author would try to answer the captioned questions or reflections in a comprehensive way.

1. Definition and Characteristics of Gifted Students

In the past, many parents or teachers generally perceived that students would be identified as gifted if they could only obtain a high IQ score after taking IQ test or demonstrate high intellectual ability, say higher scores in some school subjects.

However, in the first decade of 21^{st} century, some gifted authorities in US advocate that it does identify students as gifted with more comprehensive elements or multiple criteria, not only counting students' scores obtained in IQ test (Borland, 2003; Pfeiffer, 2003; Van Tassel-Baska, Feng, & Evans, 2007). Besides, the Education Bureau of the Hong Kong Special Administrative Region (HKSAR) also adopted a broad definition according to the Education Commission Report No.4 (1990).

4.3 MEASURES TO PROVIDE FOR STUDENTS WHO ARE ACADEMICALLY GIFTED

4.3.1 In this section we consider means to help gifted students develop their potential more fully.

(a) Definition 4.3.2 Gifted children are those who show exceptional achievement or potential in one or more of the followings

[1] - (i) a high level of measured intelligence;

(ii) specific academic aptitude in a subject area;

(iii) creative thinking - high ability to invent novel, elaborate and numerous ideas;

(iv) superior talent in visual and performing arts such as painting, drama, dance, music etc;

(v) natural leadership of peers - high ability to move others to achieve common goals; and

(vi) psychomotor ability - outstanding performance or ingenuity in athletics, mechanical skills or other areas requiring gross or fine motor coordination

(Adapted from Education Commission Report No.4 (1990), p.47)

Besides, the following points are also some of the common

[1] This definition is based on that used in Marland's report to the Congress of the United States in 1972 on "Education for the Gifted and Talented".

characteristics of gifted students.

- Have excellent memory
- Remember a large amount of vocabularies
- Enjoy solving problem especially numbers
- Ask probing questions
- Be highly sensitive
- Have longer attention span
- Be strong sense of justice and idealism
- Have high curiosity
- Put ideas and things together that not in a traditional way

(Adapted from Webb, Gore, Amend, and DeVries, 2007)

Many educators also advocate that gifted students are not defined only with high IQ scores they obtained. Each gifted student should have his or her own strengths in different categories such as leadership skills, mathematics, languages, etc. and they must have higher achievement in the near future if their strengths would be nurtured more fully.

2. Characteristics and Learning Needs of Mathematically Gifted Students

In point 2, the author will just focus on discussing the characteristics and learning needs of mathematically gifted students.

Mathematically gifted students can interpret relationships among different topics, concepts and ideas even without their teacher's instruction and guiding given before (Heid, 1983). Because of those gifted students' intuitive understanding of mathematical functions and processes, they may skip over steps to find out answers but can't explain how to find out the answers (Greenes, 1981).

Besides, they would additionally demonstrate some special features. They could also demonstrate in solving different math questions in a greater depth and breadth way (Sheffield, 1994). Mathematically gifted students would always want to know that "how" and "why" the mathematical ideas come up with rather than how to compute the respective mathematical problems only (Sheffield). Therefore, let us see what mathematically gifted students could exclusively demonstrate while learning mathematics as follows:

Highly able mathematics students should demonstrate to

- display mathematical thinking and have a keen awareness for quantitative information in the world around them.
- think logically and symbolically about quantitative, spatial, and abstract relationships.
- perceive, visualize, and generalize numeric and non-numeric patterns and relationships.
- reason analytically, deductively, and inductively.
- reverse reasoning processes and switch methods in a flexible yet systematic manner.
- work, communicate, and justify mathematical concepts in creative and intuitive ways, both verbally and in writing.
- transfer learning to novel situations.
- formulate probing mathematical questions that extend or apply concepts.
- persist in their search for solutions to complex, "messy," or "ill-defined" tasks.

- organize information and data in a variety of ways and to disregard irrelevant data.

- grasp mathematical concepts and strategies quickly, with good retention, and to relate mathematical concepts within and across content areas and real-life situations.

- solve problems with multiple and/or alternative solutions.

- use mathematics with self-assurance.

- take risks with mathematical concepts and strategies.

- apply a more extensive and in-depth knowledge of a variety of major mathematical topics.

- apply estimation and mental computation strategies.

(Adapted from Sheffield, 1999)

From the above mentioned, we also found that it is quite fantastic for mathematically gifted students could learn and solve mathematics problems so brightly.

Most mathematics teachers would also select some harder problems from textbooks or other resources to let their mathematically gifted students get higher sense of satisfactory of learning mathematics. By the way, how can mathematics teachers let those mathematically gifted students learn or solve mathematics problems more fully?

That is why some educators advocate that students, who are highly gifted in mathematics, need a separate or supplementary mathematics programme, tailored to their needs and abilities (Gavin, et. al., 2009; Gavin & Sheffield, 2010).

The implementation modes for gifted education in Hong Kong would be introduced and discussed in points 3 & 4 below and how the author implements the gifted courses. It is hoped that readers would understand more how to nurture mathematically gifted students more fully and divergently.

3. Implementation of Gifted Education in Hong Kong

   The policy on gifted education for schools in Hong Kong was first recommended by the Education Commission Report No.4 (1990) and it also suggested guiding principles for the implementation of gifted education in Hong Kong. In accordance with the guiding principles, the three-tier gifted education was adopted in 2000 as shown below.

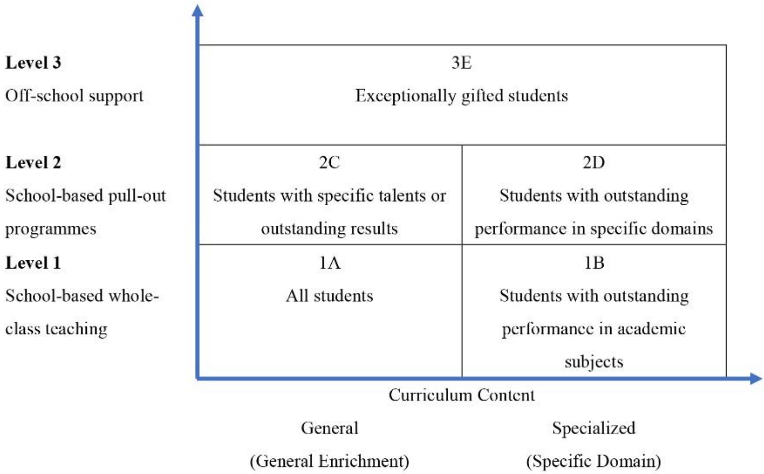

Figure 3.1: The Three-tier Implementation of Gifted Education in Hong Kong

(a) Level 1 refers to using pedagogies that could tap the potential of students in creativity, critical thinking, problem solving or leadership in the regular classroom;

(b) Level 2 refers to offering pull-out programmes in disciplinary or interdisciplinary areas for the more able students within the school setting; and

(c) Level 3 refers to provision of learning opportunities for the exceptionally gifted students in the form of specialist training outside the school setting.

(Figure 3.1 and 3-level Descriptions are adapted from EDB's

Website: https://www.edb.gov.hk/en/curriculum-development/major-level-of-edu/gifted/index.html)

About top 2% of qualified students in each school may be nominated by their school to join the off-school support programs of level 3.

Therefore, if gifted students are qualified to join the off-school support programs, school teachers should encourage them to enroll the programs.

4. Design and Implementation of Summer Gifted Courses for Mathematically Gifted Students

a) Program Framework

The courses held by the author including different topics such as "Math used in the Workplace", "Math Wonders", "Beautiful Geometry" and so on are specially designed for mathematically students between 8 and 12 years old who are studying in Primary 3 to 6 in Hong Kong Government Subsidy Schools, Direct Subsidy Scheme Schools, International School. Because of preventing from interest conflict, the author who is a secondary mathematics teacher only designs and holds gifted courses for primary mathematically gifted students.

The design of all courses he designs and holds would mainly be based on a model of academically talented youth (Stanley, 1991).

- situations related to mathematics in context
- a need to apply knowledge and skills to solve unique problems
- opportunities to be a creative and critical learner
- experiences to foster appreciation of the beauty and utility of mathematics
- exploration of mathematics as it applies to daily life
- opportunities for gifted learners to increase their understanding of the role of mathematics in the workplace

Besides, problem-solving tasks, collaborative group work and in-depth discussions among course participants would have to be specially emphasized during different learning activities in all courses so that the course participants' communication skills, intrapersonal skills, problem solving skills as well as higher order thinking skills would be enriched.

Conceptually-oriented themes and open-ended exploration in each session of each course are also deliberately embedded.

b) Selection of Course Participants

For those who are interested in joining the summer courses, they may be nominated by their class teacher, subject teacher, mentor, social worker or parents and then their nominators must fill out a nomination form to be downloaded from the center's website. In the nomination form, their nominator must simply describe what talents the applicant equips with. On the other hand, it is also welcoming to provide some additional information by a nominator how well the applicant is suitable for taking course(s) they enroll. At last, the center would consider case by case what course(s) an applicant would be offered.

c) Topic Selections for Summer Courses

Before designing summer gifted courses, the author would also take so many references of local and other countries' professional organizations for gifted education what mathematically gifted courses they offer or do not offer. Besides, the course participants and their parents would be also asked upon the completion of the courses what courses they expect that CAISE would hold next year.

In 2008, National Mathematics Advisory Panel proposed that three clusters of concepts and skills that included as follows:

- Fluency with whole numbers,
- Fluency with fractions, and
- Particular aspects of geometry and measurement.

Wheatley (1983, 1988), Johnson (1994) and Sheffield (1994) also suggested essential topics for mathematically gifted elementary students as shown below.

- Arithmetic and algebraic concepts
- Computer programming and robotics
- Estimation and mental math
- Geometry and measurement
- Math facts and computation skills
- Probability and statistics
- Problem solving

- Ratio, proportion, and percent
- Spatial visualization
- Structure and properties of the real number system

In 1994, Sheffield also suggested many topics for mathematically gifted students and some of the topics are shown below.

- Fractals and chaos
- The Pythagorean theorem and Pythagorean triples
- Fibonacci numbers
- Finite differences
- Pascal's triangle
- Figurate numbers

Johnson and others at the Center for Gifted Education at the College of William and Mary (2004) have developed criteria for curriculum and resources for mathematically gifted students as follows:

- Learning Materials must contain a high level of sophistication of ideas.

- The design of the learning materials have to be very challenging to the mathematically gifted students.

- The selection of learning materials must contain different levels, interests and backgrounds and the level of learning materials should have no upper boundary.

- Higher order thinking skills and problem-solving skills must be involved in projects.

- Learning materials must have less emphasis on basis skills.

- Opportunities for student exploration must be based on students' interest

Based on the topics and the criteria suggested by different scholars for mathematically gifted education, the author has been designing his summer courses in that way.

d) Learning Activities of Each Lesson during Courses

In general, there will be also a main theme for each session of a course. Each session's theme must be closely related to the course. Course participants would not only grasp knowledge in the course they seldom learn them in school normal curriculum, but also appreciate the beauty of the topic of mathematics. It is expected that the course would inspire the participants that learning mathematics is more meaningful to their lives and would motivate some of them to choose Mathematics as their undergraduate study in future in university.

Besides, each session of any courses at least involves three main elements called 3C and 3C stands for "Critical Thinking Skill", "Communication Skill" and "Creativity". 3C would make courses more challenging and interesting to course participants. Furthermore, the design and selection of learning tasks would provide several opportunities for course participants to enrich their higher order thinking skills.

The design and selection of learning activities during each session is also to expect that the course participants would be highly motivated to think about how to solve the questions first with their prior knowledge. The tutor expects that the course participants may think about how and why to solve the mathematical problems in divergent ways.

e) Examples of Some Learning Activities during Courses

Below were some learning activities to be conducted during the courses over the years. In Figure 4.1, the course participants would be asked to put off their shoes and then put their shoes like a shape "L" on the floor. Then, they would be asked to guess that how many additional shoes were necessary to put on the floor to form a triangle with the 2 lanes of shoes. Of course, the course participants generally felt free to give their answers, by the way, the main purpose of the activity was to arouse course participants' interests in learning the topic first.

In addition, the course participants would be guided by the tutor that how and why they could find the correct answer at last. In most learning activities throughout different courses, the course participants would have to be guided how to solve the learning tasks creatively and critically on their own first.

Figure 4.1: Course participants suggested different approaches to estimate how many shoes that were needed to form a right-angled triangle.

In Figure 4.2, course participants would be asked how to use geometrical approach to solve some algebraical identities by using different sizes of paper.

Figure 4.2: Course participants tried to find out algebraic identities by using geometrical approach.

In Figure 4.3, course participants tried to find a diagonal of a cuboid after they learned Pythagoras' theorem. Even some course participants had learned the theorem on their own before, they also could not solve the problem in a short period of time because they only learned and applied the theorem in a plane before, but not in a solid instead. Therefore, a same similar problem would be more challenging if it was applied in another scenario.

Figure 4.3 A group of course participants worked together to find out the length of diagonal of the cuboid by using Pythagoras' theorem.

The author learned how to use origami for learning mathematics especially geometry in 2012 from Mr. TAM who is famous in Hong Kong because origami lets students learn geometry in a more interesting and effective way. Figure 4.4 was an example that course participants attending a course of "Math Wonders in Daily Life" tried how to find out some specific angles by using folding a piece of paper.

Then, the tutor would also explain why and how they could find some specific angles with folding a piece of paper. During the activity, most course participants were very concentrated on the tasks.

Figure 4.4: A participant tried to fold a piece of paper to find some specific angles the tutor asked.

In a course on "Math Wonder in Daily Life", course participants were first guided to learn fractal geometry such as Sierpinski triangle and perimeter of korch snowflake. Then, all groups would make a model of Sierpinski tetrahedra shown in Figure 4.5 to know the three equations for Siepinski tetrahedra guided by the tutor.

Let N_n be the number of tetrahedra, L_n the length of a side and A_n the fractional volume of tetrahedra after the n^{th} iteration. Then, the three equations for Siepinski tetrahedra are as follows:

$$N_n = 4^n$$

$$L_n = \left(\frac{1}{2}\right)^n$$

$$A_n = L_n^3 N_n = \left(\frac{1}{2}\right)^n$$

Most course participants were very concentrated on making the model and never wanted to stop the activity even the lesson had finished for long time.

Figure 4.5: A group of course participants displayed their completed model with a very high sense of satisfaction.

5. <u>Course Participants' General Feedbacks obtained upon the Completion of a Program on "Math in the workplace" in 2012 for Mathematically Gifted Primary Students</u>

Below are 4 tables to list out students' degree of satisfaction with the program, which lesson(s) they like most, which lesson(s) they like least and students' suggestions for the program.

	Strongly disagree	Disagree	Agree	Strongly agree
1. I certainly understand the objectives of this programme.	0.0%	5.0%	45.0%	50.0%
2. The programme content was easy.	10.0%	45.0%	40.0%	5.0%
3. The programme content was difficult.	5.0%	45.0%	40.0%	10.0%
4. Contents and activities of this programme were comprehensive.	0.0%	5.0%	60.0%	35.0%
5. Exercises and activities in this programme were useful for my learning.	0.0%	10.0%	35.0%	55.0%
6. This programme enabled me to acquire new knowledge.	0.0%	10.0%	20.0%	70.0%
7. This programme enriched my knowledge about its main theme.	5.0%	5.0%	45.0%	45.0%
8. This programme has raised my interests in its main theme.	5.0%	5.0%	35.0%	55.0%
9. I perceive this programme to be meaningful.	5.0%	5.0%	40.0%	50.0%
10. Tutor's presentation was clear and easy to understand.	0.0%	5.0%	60.0%	30.0%
11. Tutor performed with great enthusiasm.	0.0%	0.0%	60.0%	40.0%
12. Tutor provided many opportunities for us to apply in-depth thinking.	0.0%	10.0%	35.0%	55.0%
13. Tutor encouraged us to present our own ideas.	0.0%	5.0%	50.0%	45.0%
14. I actively participated in this programme.	5.0%	10.0%	45.0%	40.0%
15. I got along well with other course members.	5.0%	10.0%	40.0%	45.0%
16. I actively participated in all discussions.	5.0%	5.0%	45.0%	45.0%
17. I am able to apply the knowledge I gained to my daily life.	10.0%	0.0%	45.0%	45.0%
18. I am satisfied with the arrangement of the programme.	5.0%	20.0%	35.0%	40.0%
19. I am satisfied with the location and facilities.	0.0%	0.0%	55.0%	45.0%
20. In conclusion, I am satisfied overall with his programme.	5.0%	5.0%	30.0%	60.0%

Table 1: Students' degree of satisfaction with the program (Adapted from Kwan, A.C.K. & Yuen, M, 2013)

Which lesson(s) did you like most, and why?
Lesson 1 Typical comments:
There are many sources of information and formulas. Guest speaker is able to inspire me to engage in more

thinking. [3 students rated this lesson as the best.]
Lesson 3 Typical comments:
I can assemble “Math Cubes”. The topic on “3 side views” is very interesting. I love Mathematics and Medicine. [4 students rated this as the best lesson]
Lesson 4 Typical comments:
Since my Dad is a surveyor, I now have more understanding about his job [1 student rated this lesson as the best].
Lesson 5 Typical comments:
I have learned about investment before, and found it very interesting to learn more. [2 students rated this as the best lesson]
Lesson 6 Typical comments:
I think the task is very interesting and challenging. I love learning and applying formula. It like this session because a big gift is given! It is because the application of mathematics on that day is very interesting. [4 students rated this as the best lesson]
Other comments
Students also expressed their preferences for different categories of concepts or content. Geometry was most popular, with statistics rating second.

Additionally, 2 students reflected that they liked all lessons because they were able to learn new mathematics knowledge, and because the course tutor performed well in teaching.

Table 2: Students' perceptions of the lesson they appreciated most (Adapted from Kwan, A.C.K. & Yuen, M, 2013)

Which lesson (s) you like least, and why?
Overall Typical comments:
I loved all lessons. Mr. Kwan performs very well. Each lesson is very good. Every lesson is so sound. Each lesson was very interactive and well taught. [8 students reflected that there was no lesson they disliked]
Lesson 1 Typical comments:
It is because I need to handle many statistics graphs. It is boring (4 students) I do not have much interest in banking.
Lesson 3 Typical comments:
I do not understand much of the material.
Lesson 4 Typical comments:
I did not understand some things and questions (2 students). It is because the challenging task on that day is very difficult.
Lesson 5 Typical comments:
I found learning of compound interest too difficult to understand (2 students).

I found it boring.
No guest speaker (1 student):
It is boring.
Medical treatment facilities
I don't understand. Interest (1 student)

Table 3: Students' perceptions of the lesson they liked least (Adapted from Kwan, A.C.K. & Yuen, M, 2013)

What are your suggestions for improving this programme?
No comment from 3 students
Sample comments from individuals:
Hope for more guest speakers and each lesson to last one hour. Shorten duration of each lesson. Add more gifts (rewards), and each of us also has gifts. (A view expressed by three students). The venue should not be too far away. (A view expressed by several students). More encouragement to motivate our thinking. Conduct this programme in summer vacation Don't be so boring. Change the time to 1.5 hour. May increase number of lessons and duration for each lesson. Increase duration for each lesson so as to have more discussion. Contents of this programme should be made very difficult

indeed. Add more elements in this programme. Longer duration for each lesson.

Table 4: Students' suggestions for improving this programme (Adapted from Kwan, A.C.K. & Yuen, M, 2013)

General speaking, most course participants were satisfactory with the course held by the author in 2013. Then, the author keeps on designing and conducting the gifted program for primary mathematically gifted students to enroll after 2013. Besides, he also takes a reference of the data summarized in the 4 tables as shown above to modify the following courses he holds. Some course participants have enrolled the author's courses for the consecutive three years. The author also thinks that mathematics teachers could also take a reference of the data as shown above when they design some learning materials for their mathematically gifted students.

Updated Development of Gifted Education in the World

In August 2017, the University of Erlangen-Nuremberg signed a comprehensive contract with the Hamdan Award for Distinguished Academic Performance concerning the development and joint establishment of a World Giftedness Center (WGC) (http://worldgiftednesscenter.org/). The WGC will open its doors for the first time during the Expo 2020 in Dubai, United Arab Emirates and will commence operations in close collaboration with the United Nations Educational, Scientific, and Cultural Organization (UNESCO).

Key outcomes of the project include:

- Initiation of globally-facing campaigns and research projects

- Establishment of a worldwide mentoring program

- Establishment of regular conferences that bring together the foremost authorities in gifted education and research

- Establishment of international awards in the field of gifted education

- Creation of a webinar channel with an expressly global reach

Hong Kong and other countries' students, teachers, educators and parents will benefit a lot in the days to come after the launch of the World Giftedness Center in Dubai in 2020.

Conclusion

It is expected that the gifted students would actually benefit from the courses the author held before and they may widen their horizons in mathematics knowledge, appreciate the beauties of mathematics as well as perceive learning mathematics more meaningful to their life upon the completion of the courses.

The author expects that readers could understand more about mathematically gifted students' learning needs and get some sights in the paper how to nurture their mathematically gifted students who could develop their ultimate potentials more fully.

Gifted student may join several gifted programmes such as courses, seminars and workshops held by local universities and the Hong Kong Academy for Gifted Education throughout the year via two main learning modes such as face-to-face and online learning. Besides, they may also join some online programmes held by overseas universities such as GiftedandTalented.com guided by ongoing research at Stanford University (https://giftedandtalented.com/) and Johns Hopkins Center for Talented Youth: Gifted and Talented Programs (https://cty.jhu.edu/) in which they have held the relevant programmes for many years.

In 2020 onwards, the launch of World Giftedness Center (WGC) in Dubai will give gifted students coming from different cities or countries a large variety of new and sound choices of gifted programmes to enroll via either online distance learning or face-to-face learning mode.

At the same time, educators, parents and teachers would also have more chances to share their experiences and obtain updated research findings among different people or organizations throught different conferences held by WGC so as to have closely collaboration among different cities and countries to push the sustainable development of gifted education in the world.

Last but not least, it is very meaningful that the World Giftedness Center will establish awards in the field of education. The awards would be possibly for gifted students, teachers and educators so that their contributions to the gifted education would be highly and wisely recognized in the world.

It has no doubt that the sustainable development of gifted education is very crucial to a better development of different countries including their economy, technology and so on. That is why that the launch of the World Gifted Center is very meaningful and important.

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