2015/16 第七屆香港中學數學創意解難比賽（決賽）

預備卷

此卷目的為正卷涉及的相關知識作簡介。下列問題雖不作評分用途，其內容對正卷之解答有一定關連，請用心閱讀及解答。各問題之答案印於第 4 頁，請自行核對。

(I) 概率 / 機會率 (Probability / Chance)

一個事件的概率是一個 0 至 1 之間的數值，用以量度這件事情發生可能性。概率的數字越大，這件事情便越有可能發生。

例如:

在一次數學測驗中，志强取得 A 級的機會為 0.2，而取得 B 級的機會為 0.5。

志强取得 B 級的可能性比取得 A 級的可能性大。

1.A 以列出可能結果的方法求概率

某事件的概率 $=\dfrac{\text{其中能使這事件發生的結果的數目}}{\text{所有可能結果的數目}}$。 其中所列舉的結果均有同樣可能。

例 1 投擲一顆骰子，求得出 結果是單數的概率。

當投擲一顆骰子時，結果可能是: 1、 2、 3、4、 5、6。假設這骰子是「公平」的，這 6 個結果均同樣可能。

6 個結果中 1、3、5 這 **3** 個結果使得 「擲出單數」這事件發生，因此:

「擲出單數」的機會 $=\frac{3}{6}=\frac{1}{2}$ (亦可以表示成 50% 機會)

同樣地，「擲出數字大於 4」的概率 $=\frac{2}{6}=\frac{1}{3}$。

問題 (1):

當投擲一顆骰子，得出 3 的倍數的概率是多少? __________

例 2

有兩疊數子卡，其中一疊有四張卡，卡上分別印有數字 2、 4、6 及 10。另一疊有三張卡，分別印有數字 5、7 及 9。

若從每疊都抽出一張數字卡，共有 4×3 = 12 可能結果，所有結果均同樣可能。

可以表格列出所有結果。由於此處關注兩數相加的結果，可於格內計算相加的結果。

	2	**4**	**6**	**10**
5	2 + 5 = 7	4 + 5 = 9	6 + 5 = 11	10 + 5 = 15
7	2 + 7 = 9	4 + 7 = 11	6 + 7 = 13	10 + 7 = 17
9	2 + 9 = 11	4 + 9 = 13	6 + 9 = 15	10 + 9 = 19

事件「抽出的兩數的和大於 16」只發生於 10 + 7 或 10 + 9 這兩種結果之下。

因此，「**抽出的兩數的和大於 16**」的概率 = $\frac{2}{12}$ = $\frac{1}{6}$。

問題 (2):

以例 2 所述的兩疊數字卡，若從每疊抽出一卡，

a. 「兩個抽出數字相加成 11」的機會是多少? ___________

b. 「兩個抽出數字相加大於 10」的機會是多少? ___________

1.B 概率的運算

1.B.1

若一件事件會發生的概率是 *p*， 則這事件不會發生的概率是 (1–*p*)。

例 3

「今天會下雨」的機會是 20%。

則，「今天*不*會下雨」的機會是 = 1 – 20% = 80%。

問題 (3):

a.「志明會帶雨傘」的概率是 0.3。「志明不會帶雨傘」的概率是多少? _____________

b. 陳老師課堂問問題時，問男同學的機會率為 $\frac{3}{4}$ 。問女同學的機會率是 __________。

1.B.2

A 和 *B* 是兩件獨立事 (independent)。若事件 *A* 發生的概率是 *p*、事件 *B* 發生的概率是 *q*，

則 ***A* 和 *B* 都發生**的概率是 $p \times q$。

例 4

志强在數學測驗取得 A 級的機會率是 0.9， 家恩在這測驗取得 A 級的機會率是 0.2。

志强和家恩兩人都在這測驗取得 A 級的機會率 = 0.9 × 0.2 = 0.18。

問題(4):

今天會下雨的概率是 0.6。明天會下兩的概率是 0.2。大明會帶雨傘的概率是 0.3。

a. 今天下雨而大明也帶了雨傘的概率是多少? ________________

b. *今天不下*雨而*明天下*雨的概率是多少? ________________

(II) 期望值 (Expected Value)

例 5

陳先生每次外出早餐都有兩個選擇，一個簡單早餐(消費$22)或一個豐盛早餐(消費$50)。根據陳先生一向習慣，他選簡單餐的概率是 0.8，選豐盛餐的概率是 0.2。

	簡單早餐	豐盛早餐
消費	$ 22	$ 50
概率	0.8	0.2

陳先生的早餐消費中，$ 22 及 $ 50 均有可能出現。而他的「**早餐消費的期望值**」則將這兩個價錢出現的概率均作出考慮，計算如下: **0.8 × $22 + 0.2 × $50 = $27.6**。

這個數也可視為陳先生這長期習慣下，早餐消費的平均數。

雖然陳先生沒有一次早餐會消費 $27.6，但這個期望值比 $22 或 $50 更合適地描述了他的早餐消費。

若某件事情會出現 *n* 個可能的結果，每個結果的概率分別為 P_1 、P_2 、… 、 P_n ，這些結果會帶出 (或涉及) 的價值分別為 V_1 、V_2 、 … 、V_n。

$$\text{期望值} = P_1 \times V_1 + P_2 \times V_2 + \ldots + P_n \times V_n$$

用以綜合各種可能性下這事情可帶出的價值。

例 6

在一個抽獎之中，參加者有機會得到**最高**價值$100 的現金卷。抽得各獎的概率如下:

現金卷價值 ($)	0	10	50	100
概率	0.875	0.124	0.0009	0.0001

獎金的期望值 = $0 × 0.875 + $10 × 0.124 + $50 × 0.0009 + $100 × 0.0001 = $ 1.295

這期望值 $1.295 相比於那「最高」獎金的 $100 更公正地反映參加這抽獎的得益。

問題 (5)

a. 有一個遊戲，結果可以是得 0 分、2 分或 100 分。各種得分的概率如下:

得分	0	2	100
概率	0.8	0.15	0.05

計算這個遊戲的得分的期望值。

答: 得分期望值 = __________

b. 在一個遊戲中，參加者會拋兩個硬幣。若**兩個**硬幣都出了「正面」，參加者會贏得 $50，否則只贏得 $3。這個遊戲的回報的期望值是多少?

兩個硬幣都出了「正面」的率概 = _________

這個遊戲的回報的期望值 = $ __________

若要付出 $15 作為這個遊戲的報名費用，值得嗎?

答案:

(1) [3 的倍數為: 3、6。] 概率 = $\frac{2}{6} = \frac{1}{3}$ 。

(2) a. [符合事件的結果有兩個: 6 + 5, 4 + 7。] 機會率 = $\frac{2}{12} = \frac{1}{6}$ 。

b. [符合事件的結果有 12 – 3 = 9 個] 機會率 = $\frac{9}{12} = \frac{3}{4}$ 。

(3) a. 1 – 0.3 = 0.7。 b. $1 - \frac{3}{4} = \frac{1}{4}$.

(4) a. 0.6 × 0.3 = 0.18 b. (1–0.6) × 0.2 = 0.08

(5) a. 0 × 0.8 + 2 × 0.15 + 100 × 0.05 = 5.3 分

b. 兩個正面的概率 = ½ × ½ = ¼ ，期望值 = 1/4 × ($50) + ¾ × ($3) = $14.75

2015/16 The 7th Hong Kong Mathematics Creative Problem Solving Competition for Secondary Schools (Final)

Preparation Paper

This paper is to introduce some facts and knowledge relevant to the main paper. The questions in this paper will **not** be assessed. Yet, the content is significantly linked to that of the main paper. You are advised to read and work with this paper carefully. Answers for questions are printed on page 5. Please check on your own.

(I) Probability / Chance (概率 / 機會率)

The probability of an event is to measure how likely this event will happen. It is a value between 0 and 1. The greater the probability, the more likely is that the event will occur.

For example:

In the coming mathematics test, the probability for Johnny to get an 'A' in mathematics is 0.2.

The probability for Johnny to get a 'B' is 0.5.

It is more likely for Johnny to get a 'B' than to get an 'A'.

1.A Finding Probability by Listing Possible Outcomes

$$\text{Probability of an event} = \frac{\text{Number of outcomes that make this event occur}}{\text{Number of all possible outcomes}}$$

where all listed outcomes are equally probable to occur.

Example 1 A fair dice is tossed. Find the probability that the result is an odd number.

When a fair dice is tossed, there are 6 possible outcomes: 1, 2, 3, 4, 5, 6. The six outcomes are equally likely to occur.

The **three** outcomes '1', '3', '5', among all the **six** outcomes, make "the tossed number is odd" occur.

Probability of "the tossed number is odd" = $\frac{3}{6} = \frac{1}{2}$. (can also be expressed as 50% chance.)

Similarly, probability of "the tossed number is greater than 4" = $\frac{2}{6} = \frac{1}{3}$.

Question (1):

When a fair dice is tossed, what is the probability that the result is a multiple of 3? __________

Example 2

There are two piles of number cards. One pile has four cards, printed respectively with the numbers 2, 4, 6, 10. The other pile has three cards, printed respectively with the numbers 5, 7, 9.

If one card is drawn from each pile, there are $4 \times 3 = 12$ possible outcomes. All outcomes are equally likely to occur. We can use the table below to list these possible outcomes. Since, in this case, the sum of the two numbers are to be considered, the sums are listed in the table.

	2	**4**	**6**	**10**
5	2 + 5 = 7	4 + 5 = 9	6 + 5 = 11	10 + 5 = 15
7	2 + 7 = 9	4 + 7 = 11	6 + 7 = 13	10 + 7 = 17
9	2 + 9 = 11	4 + 9 = 13	6 + 9 = 15	10 + 9 = 19

The event "the two drawn numbers has a sum greater than 16" occurs only with 10 + 7 or 10 + 9.

Therefore, the probability of "*the two drawn numbers has a sum greater than 16*" = $\frac{2}{12} = \frac{1}{6}$.

Question (2):

One card is drawn from each pile of number cards as described in example 2.

a. What is the chance that "the two drawn numbers add up to be 11"? ____________

b. What is the chance that "the two drawn numbers add up to be greater than 10"? ________

1.B Working with Probability

1.B.1

If p is the probability for an event to occur, the probability for this event ***not to occur*** is $(1-p)$.

Example 3

The chance that "it will rain today" is 20%.

Then, the chance that "it will ***not*** rain today" = 1 – 20% = 80%.

Question (3):

a. The probability of Tommy bringing an umbrella is 0.3.
   What is the probability of Tommy *not* bringing an umbrella? ________________

b. The chance for the teacher to pick a boy to answer his question is $\frac{3}{4}$.
   The chance for him to pick a girl to answer a question is ________.

1.B.2

A and *B* are two events which are independent of one another (互不相干的 / 獨立的).

If the probability for *A* to occur is p and the probability for *B* to occur is q, then the probability for **both *A* and *B*** to occur is $p \times q$.

Example 4

John has the probability of 0.9 of getting an A in his mathematics test. Mary has the probability of 0.2 of getting an A in this test.

The probability that **both John and Mary** get A in the test = 0.9 × 0.2 = 0.18.

Question (4):

The probability that Tommy will bring an umbrella is 0.3. The probability that it will rain today is 0.6. The probability that it will rain tomorrow is 0.2.

a. What is the probability that it rains today and Tommy does bring an umbrella? ________

b. What is the probability that it **will not** rain today and **will** rain tomorrow? __________

(II) Expected Value (期望值)

Example 5

When Mr. Chan has his breakfast outside, he has two choices: a simple breakfast that cost \$22 and a luxurious breakfast that cost \$50. According to his usual habit, Mr. Chan has a probability of 0.8 for buying the simple breakfast and a probability of 0.2 for buying the luxurious breakfast.

	Simple Breakfast	Luxurious Breakfast
Spending	\$ 22	\$ 50
Probability	0.8	0.2

\$22 and \$50 are two possible values for Mr. Chan's breakfast spending. The '**expected value of his breakfast spending**' is a calculated value that take into consideration of the probability of spending each of the two values: **Expected spending = 0.8 × \$22 + 0.2 × \$50 = \$27.6**.

We can take this value as the average spending for many breakfasts bought under this habit of Mr. Chan. He will not spend \$27.6 on any particular morning. Yet, comparing with \$22 and \$50, this expected value \$27.6 can better describe Mr. Chan's breakfast spending.

When there are *n* possible outcomes for some happening, each with respective probability P_1, P_2, …, P_n. These outcomes will bring (or involve) respectively the values . V_1, V_2, …, V_n.

The Expected Value = $P_1 \times V_1 + P_2 \times V_2 + \ldots + P_n \times V_n$.

It describe the value that this happening will bring out considering all its possibilities.

Example 6

In a lucky draw, the participant will be awarded with cash coupons of '*up to* $100' with the following respective probabilities:

Cash Coupon Value ($)	0	10	50	100
Probability	0.875	0.124	0.0009	0.0001

The **expected value** of award from this lucky draw

= $0 × 0.875 + $10 × 0.124 + $50 × 0.0009 + $100 × 0.0001 = $ 1.295

This expected value $1.295, and ***not*** the highest value of $100, is a fair measure of the reward in this lucky draw.

Question (5)

a. There is a game in which the outcomes can be scoring 0 points, 2 points or 100 points with the following respective probabilities:

Scores	0	2	100
Probability	0.8	0.15	0.05

Calculate the expected value of the score from this game.

Answer : The expected value is ___________.

b. In a game, the player is to flip two coins. If ***both*** coins show 'head', he will win $50. Otherwise, he will win $3. What is the expected value of reward from this game?

The probability of both coins showing heads = _________

The expected reward from this game = $ _______________

Is this worthwhile to pay $15 to enter for this game?

Answers:

(1) a. [multiples of 3 : 3 , 6.] Probability = $\frac{2}{6} = \frac{1}{3}$.

(2) a. [For the event to occur: 6 + 5, 4 + 7 ◦] Chance = $\frac{2}{12} = \frac{1}{6}$.

b. [***All but three*** of the outcomes are suitable. 12 – 3 = 9] Chance = $\frac{9}{12} = \frac{3}{4}$.

(3) a. 1 – 0.3 = 0.7 ◦ b. $1 - \frac{3}{4} = \frac{1}{4}$.

(4) a. 0.6 × 0.3 = 0.18 b. (1–0.6) × 0.2 = 0.08

(5) a. 0 × 0.8 + 2 × 0.15 + 100 × 0.05 = 5.3 points

b. Probability for 'both heads' = ½ × ½ = ¼

Expected value = 1/4 × ($50) + ¾ × ($3) = $14.75

2015/16 第七屆香港中學數學創意解難比賽 (決賽)

限時: 40 分鐘

A. 田忌賽馬 (30 分)

昔戰國時，齊田忌數與齊諸公子馳逐重射。孫臏見其馬足不甚相遠，馬有上、中、下輩。於是臏謂忌曰：「君第重射，臣能令君勝。」忌信然之，與王及諸公子逐射千金。及臨質，臏曰：「今以君之下駟與彼上駟，取君上駟與彼中駟，取君中駟與彼下駟。」 既馳三輩畢，而田忌一不勝而再勝，卒得王千金。 <史記:孫子吳起列傳>

上文出自《史記・孫子吳起列傳》，大意是這樣的 :

田忌經常和齊威王賽馬， 馬分上、中、下三等，齊威王的上馬比田忌的上馬略為優勝，中馬和下馬都是這樣子 ; 然而田忌的上馬是可以跑贏齊威王的中馬和下馬的， 而中馬亦可以跑贏齊威王的下馬。

賽制是這樣 :
雙方在上、中、下三等馬中，每個等級揀取一匹參賽， 比賽共分三場，贏得兩場或以上者勝。

每次對壘，田忌總是輸多贏少，後來孫臏給田忌獻計，田忌果然得勝。

孫臏的方法就是用自己的下馬和對方的上馬比試，先輸一場，但之後用上馬對中馬、中馬對下馬，接著連贏兩場，於是總場數二比一，贏了這次對壘。

問題(1)

a. 將上、中、下三等馬，安排在三場比賽中，其中一個安排方法是 :(上中下) ，即第一場出上馬 ；第二場出中馬 ；第三場出下馬。請問田忌安排馬匹出賽，共有多少種安排的方法？
請用(上中下)這種表達形式，把安排的方法全部列舉出來。

b. 下表用以表示田忌與齊威王雙方採用各種的排列方式下賽馬的結果。其中，若田忌以他的「上中下」對齊威王的「上中下」，則齊威王得勝。
請完成下表。

賽馬結果　　(齊: 齊威王勝;　　田: 田忌勝)

		田忌					
		上中下					
齊威王	上中下	齊					

問題(2)

a. 田忌與齊威王對壘，如果大家都是「隨機」安排馬匹的出場次序，也就是雙方均不對任何次序有特別偏好。田忌勝出的機會是多少%？

b. 如果田忌和齊威王一年中都總有 30 次對壘，試估計田忌大約可以勝出多少次。

問題(3)

假設齊威王總是將上馬留在最後一場出賽， 那麼田忌應該怎樣安排自己馬匹的出場次序從而使勝算儘可能增加？這情況下獲勝的機會又是多少% ？

問題(4)

倘若齊威王從不將上馬留在最後一場出賽，田忌獲勝的最大機會又是多少% ？

B.　二人對奕 (44 分)

老師對小明和小芳說 :「想跟兩位玩一個遊戲，你們各自握拳，當我說開始的時候，雙方各把右手伸出，手上豎一隻手指或兩隻手指。
如果大家都豎起一隻，小明得 0 分，小芳得 6 分。
如果大家都豎起兩隻，小明得 2 分，小芳得 4 分。
如果小明豎起一隻而小芳豎起兩隻，則小明 4 分小芳 2 分。
如果小明豎起兩隻而小芳豎起一隻，則小明 5 分小芳 1 分。」

上面的資料可以表列如下 :

遊戲得分 (小明得分, 小芳得分)

		小芳	
		一隻手指	兩隻手指
小明	一隻手指	(0 , 6)	(4 , 2)
	兩隻手指	(5 , 1)	(2 , 4)

他們兩人明白遊戲規則後便經常對玩。

問題(1)

假設小明是隨機地豎起手指，也就是對於兩種選擇均無特別偏好。

a.　若小芳選擇豎起一隻手指，她得分的期望值是多少？

b.　這種情況下，小芳應該豎起一隻還是兩隻手指，方為上算？試詳細解釋。

問題(2)

若小明改變了習慣，他只有 0.25 的機會率豎起一隻手指，卻有 0.75 的機會率豎起兩隻手指。

這種情況下，小芳應該豎起一隻還是兩隻手指，方為上算？請詳細解釋。

問題(3) 遊戲熟習後，小明開始思考策略。

假設小明豎起一隻手指的機會率為 p，豎起兩隻手指的機會概為 $(1-p)$。

a. 若小芳豎起一隻手指，她的得分期望值為 E_1。以 p 表示 E_1。

b. 若小芳豎起兩隻手指，她的得分期望值為 E_2。以 p 表示 E_2。

c. 將 E_1 及 E_2 隨 p 值變化的圖像草繪於圖(1)中。

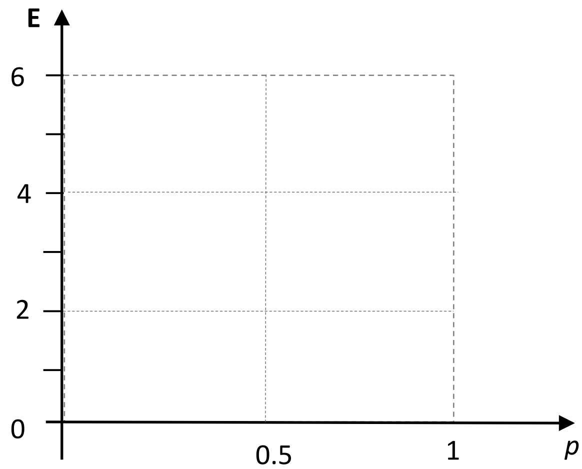

圖 1

d. 從以上所得，小明可否就 p 的值的選取作出一個對他有利的決策?
若有，p 該取值多少？試解釋。

問題(4) 若小芳也作出策略上的思考:

假設小芳豎起一隻手指的機會率為 q。

a. 若小明豎起一隻手指，他的得分期望值為 F_1，若他豎起兩隻手指，他的得分期望值為 F_2。將 F_1 及 F_2 隨 q 值變化的圖像草繪於圖(2)中。

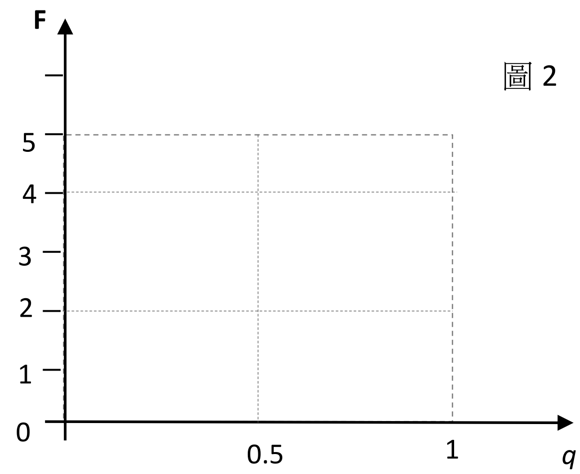

圖 2

b. 從以上所得，小芳該如何就 q 的值的選取作出一個對她有利的決策? q 該取值多少？試解釋。

問題(5)

假如小明和小芳都懂得分別揀取對自己最有利的策略(即題 3 及題 4 中的 p、q 的取值)，這個遊戲對誰較為有利？請詳細解釋。

C. 三人對奕 (26 分)

後來小吉加入了(B)部所描述的小明和小芳的遊戲，計分的方法改變如下：

唯一豎起一隻手指的人可獲 1 分；**唯一**豎起兩隻手指的人可獲 2 分。除這兩個情況外，參與者將不會獲得任何分數。

假設 p、 q、r 分別是小明、小芳和小吉豎起一隻手指的**機會率**。

問題(1)

完成下面兩個評分表，部分空格已填上數值：

三人得分 (小明得分, 小芳得分, 小吉得分)

<table>
<tr><td colspan="2" rowspan="2">如果小吉豎起一隻手指 (r)</td><td colspan="2">小芳</td></tr>
<tr><td>一隻手指 (q)</td><td>兩隻手指 ($1-q$)</td></tr>
<tr><td rowspan="2">小明</td><td>一隻手指 (p)</td><td></td><td></td></tr>
<tr><td>兩隻手指 ($1-p$)</td><td>(2, 0, 0)</td><td></td></tr>
</table>

<table>
<tr><td colspan="2" rowspan="2">若小吉豎起兩隻手指 ($1-r$)</td><td colspan="2">小芳</td></tr>
<tr><td>一隻手指 (q)</td><td>兩隻手指 ($1-q$)</td></tr>
<tr><td rowspan="2">小明</td><td>一隻手指 (p)</td><td></td><td></td></tr>
<tr><td>兩隻手指 ($1-p$)</td><td>(0, 1, 0)</td><td></td></tr>
</table>

問題(2)

已知完成一次遊戲後，小明得分期望值為 $2(1-p)\,qr + p\,(1-q)(1-r)$。
試寫出小芳和小吉的得分期望值。 (答案用 p、q、r 表示)

問題(3)

a. 假設小明、小芳和小吉三人都充分掌握了這遊戲的致勝策略，**各自盤算**，務求令其他兩人得分最少。終會出現有三個特別的數值 a、b 和 c 有以下效果: 小明發覺取 $p = a$、 小芳發覺取 $q = b$、 小吉發覺取 $r = c$ 就可以達到壓制對手得分以增加自己的勝算。

試解釋下式為甚麼會成立：
取任何 0 至 1 之間的數作 p 值，必有 $2(1-a)bc + a(1-b)(1-c) \geq 2(1-p)bc + p(1-b)(1-c)$。

b. 證明： $a\,[(1-b)(1-c) - 2bc] \geq p\,[(1-b)(1-c) - 2bc]$。

c. 小吉最後認為：「如果我們仍然互不合作，誰都不願意讓他人有可乘之機的話，我會把豎起一隻手指的機會調控至約 41.4%。」您同意小吉說法嗎？請詳細解釋。

完

2015/16 The 7th Hong Kong Mathematics Creative Problem Solving Competition for Secondary Schools (Final)

Time allowed: 40 minutes

A. Tian Ji’s Horse Race Game (30 marks)

> 昔戰國時，齊田忌數與齊諸公子馳逐重射。孫臏見其馬足不甚相遠，馬有上、中、下輩。於是臏謂忌曰：「君第重射，臣能令君勝。」忌信然之，與王及諸公子逐射千金。及臨質，臏曰：「今以君之下駟與彼上駟，取君上駟與彼中駟，取君中駟與彼下駟。」
> 既馳三輩畢，而田忌一不勝而再勝，卒得王千金。 <史記:孫子吳起列傳>

The text above was extracted from “Shiji (史記) – Records of the Grand Historians”.

The story was about the *General Tian Ji* and the *King of Qi* of the Warring States Period.

Both *Tian Ji* and the *King of Qi* were horse racing fans. Since the *King*'s horses were better than his, *Tian Ji* lost most of racing games. *Sun Bin*, General *Tian*’s friend gave *Tian* some advice and helped him win.

The rule of the racing game was as follows:

Both side's horses were divided by their speed into three different classes: superior, standard and inferior. For each side, one horse from each of the three classes was chosen to race for one round. The side who won two or more in the three rounds was the winner of the game.

Sun Bin noticed that each of the *King*'s horses was only marginally better than the one of the same class from *Tian*. So, *Sun Bin* suggested to *Tian*: “Arrange your inferior horse to compete with the *King*'s superior, your superior horse to compete with the *King*'s standard one, and your standard one to compete with the *King*’s inferior."

In the three rounds, *Tian* lost one round and won the other two. *Tian* beat the *King* in this race.

Question (1)

a. There are three classes of horses: Superior (Class **A**), standard (Class **B**) and inferior (Class **C**). One way in ordering the three horses is (**A-B-C**): assigning a superior horse for the first round, a standard horse for the second round and an inferior horse for the third round.
How many ways did *Tian* have in ordering the three horses? List all different ways in the form like (**A-B-C**).

b. There were different orders for both *Tian* and the *King* to assign their horses. The table below is to list the results of the races in various cases. For example, in the one when the orders for *Tian* and the *King* were both (**A-B-C**), the *King* won. Complete the table.

Race Results (**K**: the *King* wins; **T**: *Tian* wins)

		Tian					
		A-B-C					
King	**A-B-C**	**K**					

Question (2)

a. If both *Tian* and the *King* assigned their three horses in random order, i.e. with no preference for any particular orders, with what percentage of chance would *Tian* win in a race?

b. If *Tian* and the *King* competed in 30 races in a year, estimate the number of races won by *Tian*.

Question (3)

If the *King* always assigned his superior horse (Class **A**) to the last round, how should *Tian* order his horses so as to make his chance of winning as large as possible? In this case, what was *Tian*'s chance of winning?

Question (4)

If the *King* never assigned his superior (Class A) horse to the last round, what was *Tian*'s best chance of winning?

B. A Two-Person Game (44 marks)

In one lesson, the teacher instructs Albert and Betty to compete in a game. The teacher says, “When I say ‘fire’, both of you should show your moves at the same time. The moves are either ‘**raising one finger**’ or ‘**raising two fingers**’.”

If they both raise one finger, Albert scores 0 mark and Betty scores 6 marks.
If they both raise two fingers, Albert scores 2 marks and Betty scores 4 marks.
If Albert raises one finger and Betty raises two, Albert gets 4 marks and Betty gets 2 marks.
If Albert raises two fingers and Betty raises one, Albert gets 5 marks and Betty gets 1 mark.

The scoring system is shown in the table below.

Scores (Albert’s score, Betty’s score)

		Betty	
		One finger	Two fingers
Albert	One finger	(0 , 6)	(4 , 2)
	Two fingers	(5 , 1)	(2 , 4)

When the two know the rules, they play with this game a lot.

Question 1

If Albert picks his move at random, i.e. no preference for any one of the two moves.

a. If Betty makes the ‘One finger’ move, find the expected value of her score.

b. In this case, is it better for Betty to choose the “One finger” or the “Two fingers” moves? Explain.

Question 2

Then, Albert changes his habit. He raises one finger with the chance of 0.25 and raises two fingers with the chance of 0.75.

In this case, is it better for Betty to choose the ‘One finger’ move or the ‘Two fingers’ move? Explain.

Question 3 When Albert gets familiar with the game, he begins to think of strategy.

Assume that Albert makes the "one finger" move with a probability of p and makes the "two fingers" move with the probability of $(1-p)$.

a. Let $\mathbf{E}_1$ be the expected value of Betty's score when she raises one finger. Express $\mathbf{E}_1$ in terms of p.

b. Let $\mathbf{E}_2$ be the expected value of Betty's score when she raises two fingers. Express $\mathbf{E}_2$ in terms of p.

c. On figure (1), sketch the graphs to show how $\mathbf{E}_1$ and $\mathbf{E}_2$ changes with p.

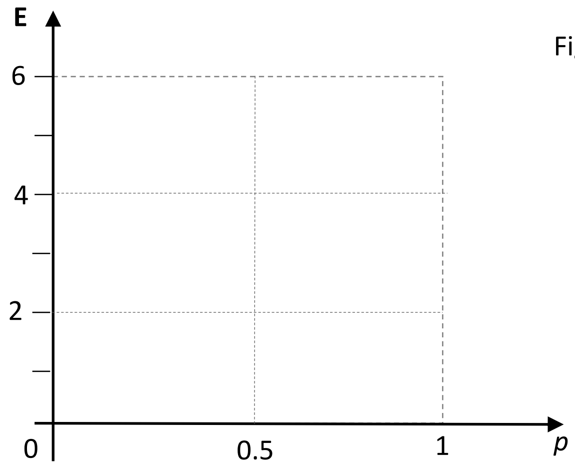

Figure 1

a. With the above results, can Albert have a strategy to benefit himself by choosing a suitable value for p? If yes, what is that value of p? Explain.

Question 4

Betty also begins to think about her strategy.
Assume that Betty makes the "one finger" move with a probability of q and makes the "two fingers" move with the probability of $(1-q)$.

a. Let $\mathbf{F}_1$ be the expected value of Albert's sore when he raises one finger and $\mathbf{F}_2$ be the expected value of Albert's score when he raises two fingers.
   On figure (2), sketch the graphs to show how $\mathbf{F}_1$ and $\mathbf{F}_2$ changes with q.

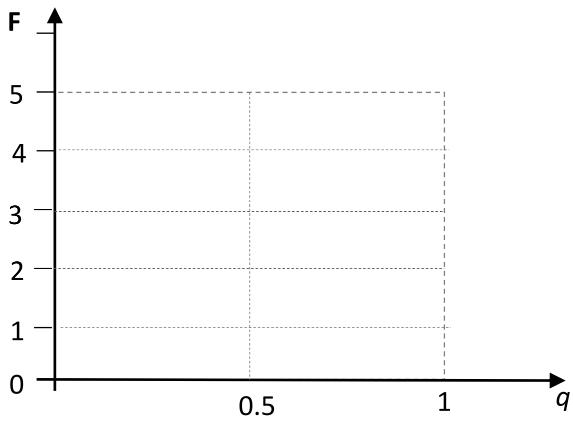

Figure 2

b. With the above results, how can Betty have a strategy to benefit himself by choosing a suitable value for q? What is that value of q? Explain.

Question 5

Assuming that both Albert and Betty know how to play their best strategies (i.e. using the values of p and q obtained in questions 3 and 4), who is in a more favourable position in this game? Explain.

C. The Three-Person Game (26 marks)

Cora joins Albert and Betty in the game described in part (B). The scoring system is changed as below:

The ONLY one who raises one finger will get 1 mark. The ONLY one who raises two fingers will get 2 marks. Otherwise, none of the players will get any mark.

Suppose that p, q and r are respectively the probabilities for Albert, Betty and Cora to raise one finger.

Question 1

Complete the following scoring tables. (Two of the cells are filled.)

Scores awarded to the players. (Albert's score, Betty's score, Cora's score)

When Cora raises one finger (r)		Betty One finger (q)	Betty Two fingers ($1-q$)
Albert	One finger (p)		
Albert	Two fingers ($1-p$)	(2, 0, 0)	

When Cora raises two fingers ($1-r$)		Betty One finger (q)	Betty Two fingers ($1-q$)
Albert	One finger (p)		
Albert	Two fingers ($1-p$)	(0, 1, 0)	

Question 2

The expected value of Albert's score in terms of p, q and r is $2(1-p)\,qr + p\,(1-q)(1-r)$.

Write down the expected values of the scores of Betty and Cora in terms of p, q and r.

Question 3

a. Suppose that Albert, Betty and Cora are all playing their best in this game. They all plan independently for their own good by minimizing the scores of the other players. In their own planning, there are particular values a, b and c that give the following effect: When Albert puts $p = a$, Betty puts $q = b$ and Cora puts $r = c$, they are all convinced that they are doing the best for themselves by minimizing the other players' chance of gaining higher scores.

Explain why, for any values of p between 0 and 1,

$$2(1-a)\,bc + a\,(1-b)(1-c) \geq 2\,(1-p)\,bc + p\,(1-b)(1-c).$$

b. Show that $a\,[\,(1-b)(1-c) - 2bc\,] \geq p\,[\,(1-b)(1-c) - 2bc\,]$.

c. Cora then says, “If we insist that we *do not cooperate*, each guarding against the chance of others gaining benefit, I shall be setting my probability for my ‘one finger’ move to about 41.4%.”

Do you agree with Cora? Explain.

End