2016/17 The Eighth HK Mathematics Creative Problem Solving Competition

for Secondary School – Final

2016/17 第八屆香港數學創意解難比賽(中學) – 決賽

Preparation Test 預備卷

Time allowed: **5 minutes**

A. Tetrahedron (四面體)

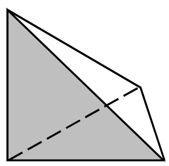
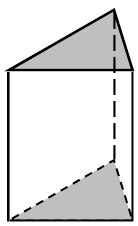
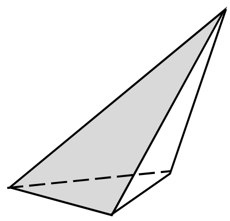
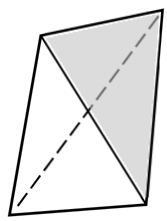
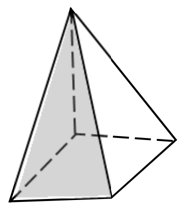
A tetrahedron is a pyramid with a triangular base. It is a polyhedron with 4 vertices, 4 faces and 6 edges.

四面體也就是三角錐體。這個多面體有 4 個頂點、4 個面及 6 條稜邊。

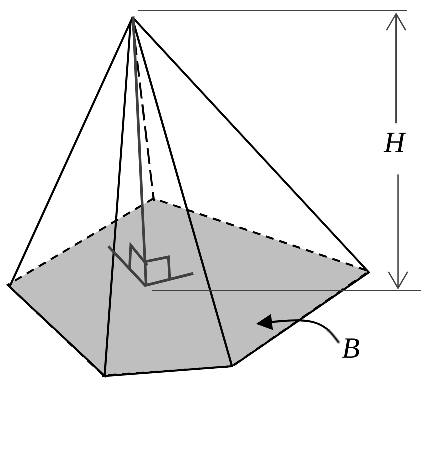
Question 1

In the following, give a ✓ to each figure that represents a tetrahedron.

以下各圖中，以✓選出四面體。

☐ 	☐ 	☐ 	☐ 	☐ 

B. Volume of Pyramid (錐體的體積)



The volume of a pyramid is $V = \frac{1}{3} \times B \times H$.

B is the area of the base and H is the height.

H is the perpendicular distance from the vertex to the base.

錐體體積為 $V = \frac{1}{3} \times B \times H$。

其中，B 為錐體的底面積，H 為錐體的高。

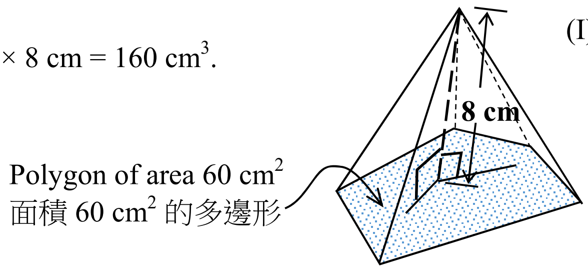
H 是從頂點至底的垂直距離。

Example (例)

The volume of the pyramid (I) in the figure is $\frac{1}{3} \times 60\ \text{cm}^2 \times 8\ \text{cm} = 160\ \text{cm}^3$.

圖中錐體(I) 的體積為 $\frac{1}{3} \times 60\ \text{cm}^2 \times 8\ \text{cm} = 160\ \text{cm}^3$。

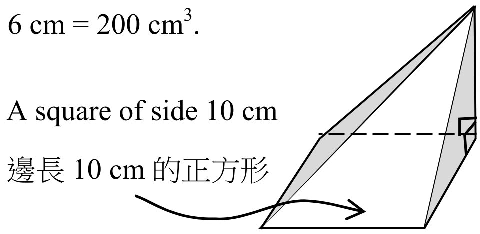
(I)



The volume of the pyramid (II) in the figure is $\frac{1}{3} \times 100\ \text{cm}^2 \times 6\ \text{cm} = 200\ \text{cm}^3$.

圖中錐體(II) 的體積為 $\frac{1}{3} \times 100\ \text{cm}^2 \times 6\ \text{cm} = 200\ \text{cm}^3$。

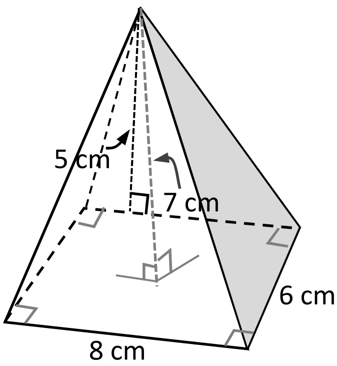
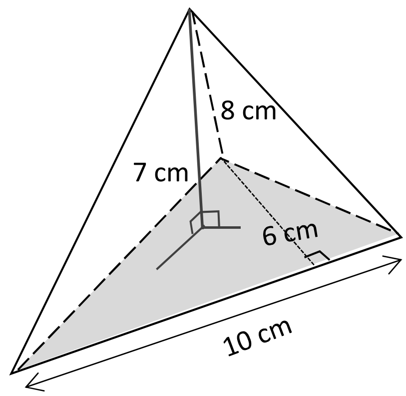
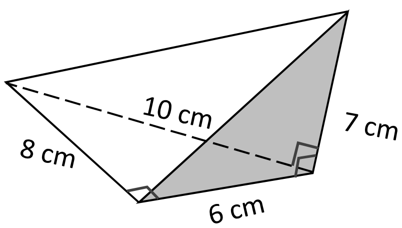
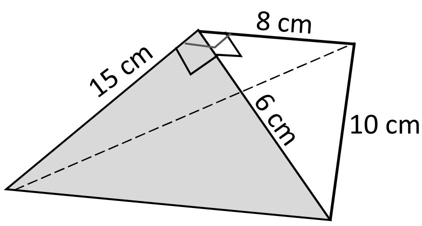
(II)



Question Two

Find the volume of each of the pyramids below.

求下列各錐體的體積。

1. 	2. 
3. 	4. 

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Preparation Test 預備卷

Time allowed: 5 minutes

A. Tetrahedron (四面體)

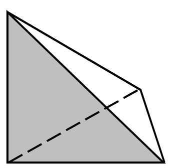
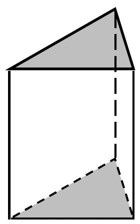
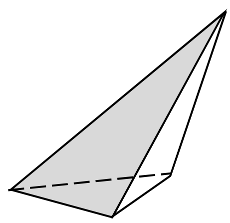
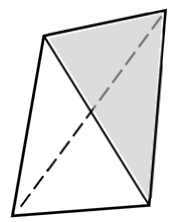
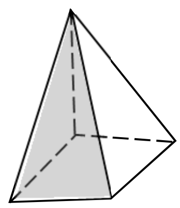
A tetrahedron is a pyramid with a triangular base. It is a polyhedron with 4 vertices, 4 faces and 6 edges.

四面體也就是三角錐體。這個多面體有 4 個頂點、4 個面及 6 條稜邊。

Question 1

In the following, give a ✓ to each figure that represents a tetrahedron.

以下各圖中，以✓選出四面體。

☑	☐	☑	☑	☐
				

B. Volume of Pyramid (錐體的體積)

The volume of a pyramid is $V = \frac{1}{3} \times B \times H$.

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錐體體積為 $V = \frac{1}{3} \times B \times H$。

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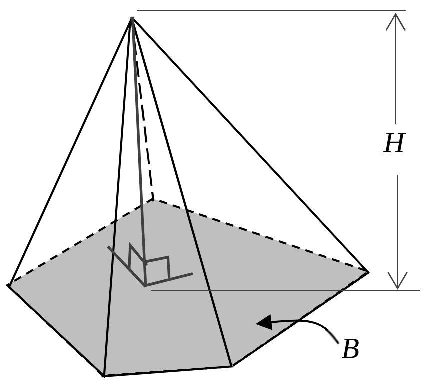

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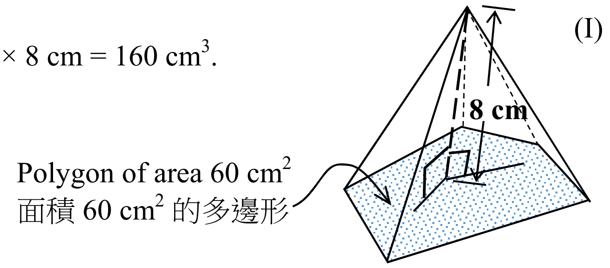

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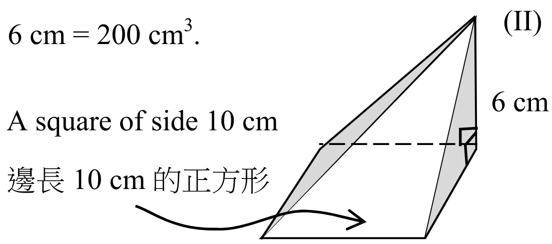
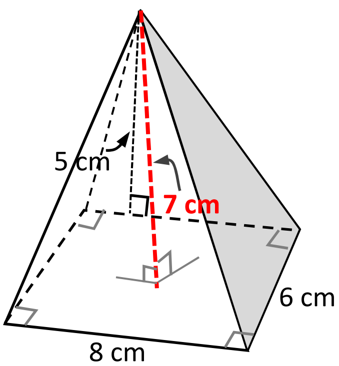
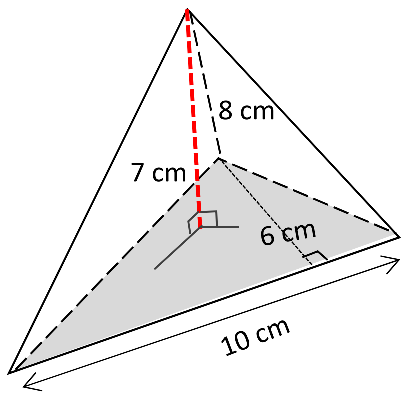
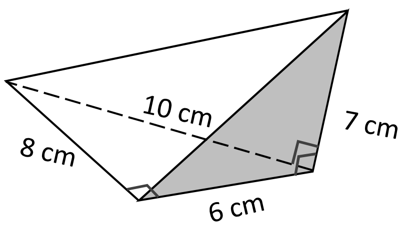
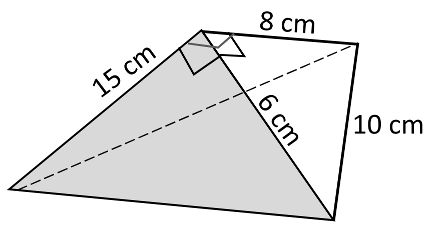
圖中錐體(II) 的體積為 $\frac{1}{3} \times 100\ \text{cm}^2 \times 6\ \text{cm} = 200\ \text{cm}^3$ 。



Question Two

Find the volume of each of the pyramids below.

求下列各錐體的體積。

1.  $\frac{1}{3} \times (8\ \text{cm} \times 6\ \text{cm}) \times 7\ \text{cm}$ $= 112\ \text{cm}^3$	2.  $\frac{1}{3} \times (10\ \text{cm} \times 6\ \text{cm} \div 2) \times 7\ \text{cm}$ $= 70\ \text{cm}^3$
3.  $\frac{1}{3} \times (8\ \text{cm} \times 6\ \text{cm} \div 2) \times 7\ \text{cm}$ $= 56\ \text{cm}^3$	4.  $\frac{1}{3} \times (8\ \text{cm} \times 6\ \text{cm} \div 2) \times 15\ \text{cm}$ $= 120\ \text{cm}^3$

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Time allowed: 45 minutes **限時: 45 分鐘**

Part A ***Silver Rectangles*** **(8 marks)**

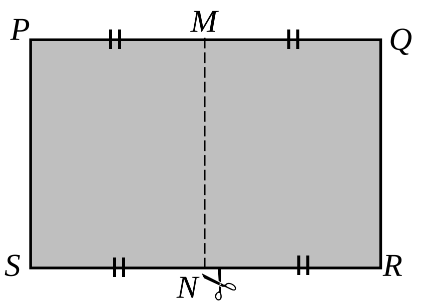
甲部 **白銀矩形** **(8 分)**

The printing papers that we use are mainly in various rectangular forms. There is a type of rectangle in a special shape called ***silver rectangle***. These rectangles have the following property.

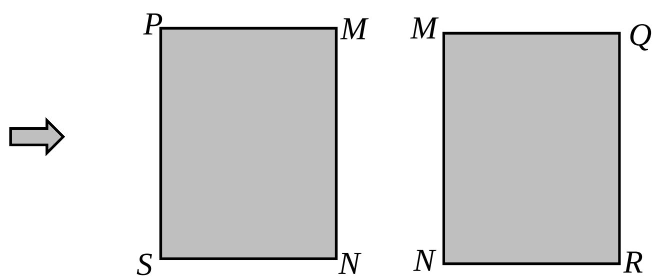
Given that *PQRS* is a piece of paper being a ***silver rectangle.*** If we cut the paper into two equal halves along a line joining the two mid-points *M*, *N* of the longer sides *PQ* and *RS*, then the two rectangles *MNSP* and *QRNM* are both similar to *PQRS*. That is, both the two resulting rectangles are also ***silver rectangles***.

常見的矩形紙張有不同的大小和形狀。其中一種形狀為「**白銀矩形**」。它的性質如下：

設矩形 *PQRS* 為**白銀矩形**。 如果我們沿長邊 *PQ*、*SR* 的中點 *M*、*N* 把矩形分為兩等份，則 *MNSP* 和 *QRNM* 均與 *PQRS* 相似。兩個得出的矩形也為**白銀矩形**。



One ***silver rectangle***
一個白銀矩形



Two smaller ***silver rectangles***
兩個較小的白銀矩形

Part A 甲部

Question 1 問題 1

There is a piece of white paper in the File Pocket (A). Check whether this piece of paper is in the shape of a ***silver rectangle***. Explain your answer.
(A difference of less than 1% from the theoretical shape can be accepted.)

文件套(A)中有一張白紙，試判斷這張紙的形狀是否白銀矩形。並略加解釋。

（可接受與理論少於 1%之差距。）

Question 2 問題 2

There is a piece of colour paper in File Pocket (A). Cut out a silver rectangle from this piece of paper. Explain your steps.

文件套(A)中有一張顏色紙。請從這張顏色紙剪出一個白銀鉅形。解釋你的**步**驟。

Part B ***Folding up a Rectangle to make a Tetrahedron***

乙部 把長方形紙摺成四面體

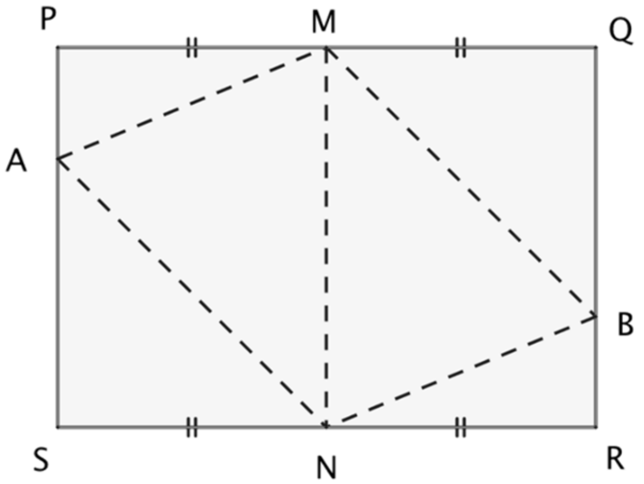
There are pieces of paper in each of the File Pocket (B1) and the File Pocket (B2) for the investigation of Part B. They are in the shape of silver rectangle with one side 210 mm.

文件套(B1)和文件套(B2)內各有數張顏色紙作乙部的探究之用。這些紙均為白銀矩形，其中一邊長 210 mm。

<u>Question 1</u> <u>問題 1</u> **(12 marks /12 分)**

For this question, use a rectangular paper from File Pocket (B1) and fold it to form a tetrahedron according to the following instructions.

這一題，請用文件套 (B1) 中的矩形紙張。按以下的指示摺成四面體。



Let *PQRS* represent the piece of paper where *PQ* is the longer side.

M and *N* are respectively the midpoints on the two longer sides *PQ* and *RS*.

A and *B* are respectively points on the shorter sides *PS* and *QR* such that *AN* and *BM* bisect the angles $\angle SNM$ and $\angle QMN$ respectively.

PQ、*RS* 為矩形紙 PQRS 的長邊。*M* 和 *N* 分別是 *PQ* 和 *RS* 的中點。

A 和 *B* 是 *PS* 和 *QR* 上的點使 *AN* 和 *BM* 分別平分角$\angle SNM$ 和 $\angle QMN$。

a. Fold along the 5 dotted lines *MN*, *AM*, *AN*, *BM* and *BN* to make a tetrahedron so that no paper overlaps. Take picture(s) of the tetrahedron formed.

沿虛線 *MN*、 *AM*、 *AN*、 *BM* 和 *BN* 把矩形紙摺成一個四面體，摺成的立體沒有紙會重疊在一起。拍攝完成的四面體。(可拍多於一張圖片)

Part B **乙部**

Question 1 (Continued) / 問題 1 (續)

b. What is the total surface area of the tetrahedron?

四面體的總表面面積是多少？

c.. By making appropriate measurements, or calculations, or otherwise, find the volume of the tetrahedron formed. Give your answer as accurately as possible. Explain your steps. (You might use text, mathematics expressions, drawings or photographs for your explanations. If more paper space is needed for the working or explanation, use the supplementary papers.)

通過量度、計算或其他方法，找出四面體的體積。答案應力求準確，並請解釋你的步驟。(可以用文字、算式、草圖、相片等協助說明。如需用更多紙張空間，可用補充答題紙。)

Part B **乙部**

Question 1 (Continued) / 問題 1 (續)

c. (continued /續)

d. Comment on the source(s) of error, if any, in your method in finding the answers in (c).

假如你用以求出（c）部分答案的方法中會存在誤差的話，請說明。

Part B　乙部

Question 2　問題 2　　**(12 marks / 12 分)**

For this question, use the papers from File Pocket (B2).

這一題，請用文件套 (B2) 中的矩形紙張。

> If one point is chosen appropriately on each side of a rectangular paper, named *M, A, N* and *B*, the paper can be folded along line segments connecting the points to form a tetrahedron *MANB*, as in a way similar to that in question **B**(1).
>
> 假如在矩形紙的每條邊各適當地選一點，稱它們為 *M*、*A*、*N*、*B*，再加上適當的摺痕，便可用類似題 **B**(1)的方式摺出一個四面體 *MANB*。

a. The teacher asked Johnny to fold up a tetrahedron in the way described above using a piece of paper from the File Pocket (B2). By just writing on his rough work sheet and without folding any paper, Johnny claimed that, "Any tetrahedron folded from that paper should not have a volume exceeding **272855 mm^3**."
Do you agree with Johnny? Explain.

老師請志明用文件夾 B2 的紙張以上述方法摺成一個四面體。志明未有摺起紙張，只在算草上寫寫畫畫，便說道：「從這張紙摺出的四面體的體積不可能起過 **272855 mm^3**。」
你同意志明這說法嗎? 請解釋。

Part B **乙部**

Question 2 **問題 2**

b. Fold a piece of rectangular paper from File Pocket (B2) to form a tetrahedron with the greatest possible volume.
   i. Describe how this tetrahedron can be formed.
   ii. Explain why the tetrahedron formed has the greatest possible volume.
   iii. By making appropriate measurements, calculations, or otherwise, find the volume of the tetrahedron formed. Compare it with the answer in (a).

[You might use writings, drawing and photographs for your description and explanation.]

用一張文件套(B2)中的矩形紙製作體積盡可能大的四面體。

i. 描述四面體如何製成；

ii. 解釋它的體積為何是最大；

iii. 以適當的量度或計算，找出四面體的體積。並跟（a）的結果作比較。

（可以用文字、算式、草圖、相片等輔助解說）

Part B 乙部

Question 3 問題 3 (6 marks / 6 分)

A carpenter is to make a hard solid tetrahedron of exactly the same shape and size as described in part B, question 1. He will then cut a hole in a thin wooden board such that this tetrahedron can pass through.
In the space below, draw the real shape and size of this hole. Draw a hole whose area is as small as possible. Explain your answer.

一位木匠造了有一個和 B 部題 1 相同形狀大少的實木四面體。他想在一塊薄木板上開一個能讓這四面體穿過的洞。

試在下面按真實的形狀和大小，畫出一個面積最少，又能讓這四面體穿過的洞。

請解釋你們的做法。

Part B 乙部

Question 4 問 題 4 **(6 marks / 6 分)**

In the previous two questions, you are instructed to fold a rectangular paper into tetrahedron where there is no overlapping paper area when it is folded up. Let's explore the possibility of doing so when one or two particular point(s) are assigned as the folded tetrahedron's vertices.

在之前的題目中，我們用白銀矩形的紙張摺出四面體（當摺成後沒有紙張重疊）。讓我們探究一下如果某一或兩點已指定作四面體的頂點，可以如何摺。

In each of the following, parts (a) to (f), one or two particular point(s) on a rectangle is/are marked on a rectangle as vertex *A* (or vertices *A*, *B*).

State whether it is possible to fold the rectangle into a tetrahedron where there is no overlapping paper area when it is folded up. The marked point(s) should be one (or two) of the vertices of the tetrahedron formed.

(1) If not possible, put a '×' inside the box ☐.

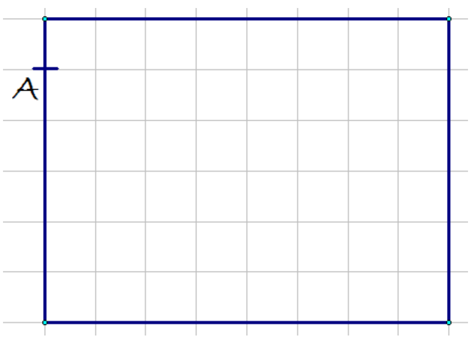
(2) If possible, mark on the rectangle the crest lines along which the rectangle is to be folded.

以下各圖 (a 至 *f*)中，一或兩點 (*A* 或 *A*, *B*) 已標示，請判斷能否把矩形紙在沒有紙張重疊的情況下摺成四面體，並以已標示了的點作為它其中一 (或兩) 個頂點。

(1) 如果認為不可以，請在 ☐ 內加上 × 。

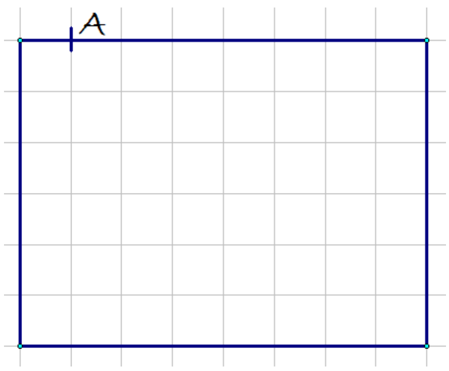
(2) 如果認為可以，請在圖上加上代表摺痕的線。

a. The paper is an 8×6 unit^2 rectangle. *A* is a point on one of a side as shown.
矩形紙為 8×6 平方單位，*A* 點如圖所示在邊上。



☐

b. The paper is an 8×6 unit^2 rectangle. *A* is a point on one of a side as shown.
矩形紙為 8×6 平方單位，*A* 點如圖所示在邊上。



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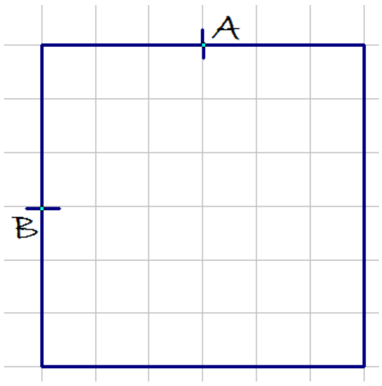
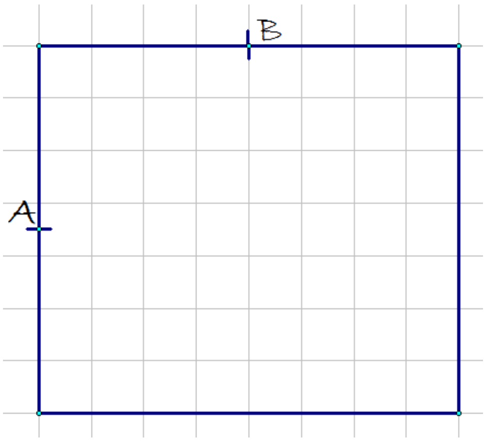
c. The paper is a 6 × 6 unit[2] square. *A* and *B* are respectively the midpoints of two of the sides.
矩形紙為 6 × 6 平方單位，A、B 點如圖所示在兩條邊上。



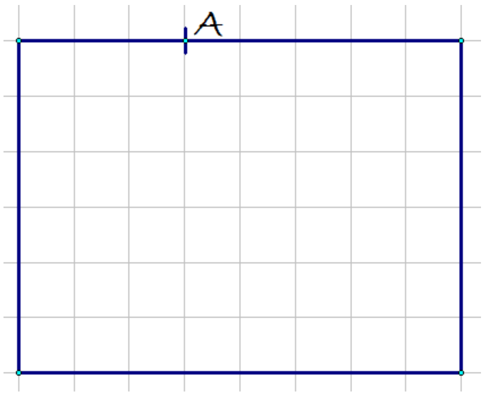
☐

d. The paper is an 8 × 7 unit[2] rectangle. *A* and *B* are respectively the midpoints two of the sides.
矩形紙為8 × 7 平方單位，A、B 點如圖所示在兩條邊上。



☐

e. The paper is an 8 × 6 unit[2] rectangle. *A* is a point on one of the side as shown.
矩形紙為 8 × 6 平方單位，*A* 點如圖所示在邊上。



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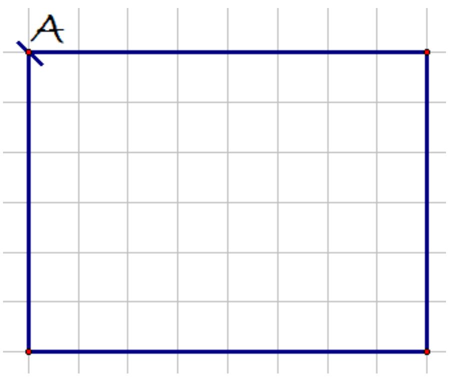
f. The paper is an 8 × 6 unit[2] rectangle. *A* is one of the vertex of the rectangle.

矩形紙為 8 × 6 平方單位，*A* 點如圖所示在紙角上。



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End / 完

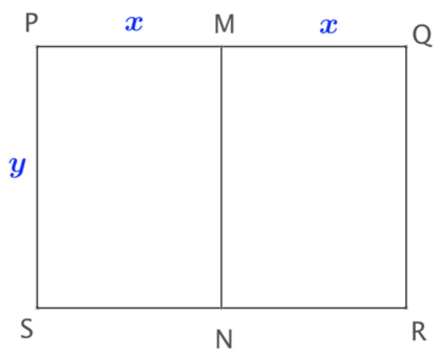
CPS Secondary Final paper 2017 (Solution)

Part A 甲部

Question 1 問題 1

There is a piece of white paper in the Folder A. Check whether this piece of paper is in the shape of a ***silver rectangle***. Explain your answer.
(A deviation of less than 1% from the theoretical shape can be accepted.)

文件夾 A 中有一張白紙，試判斷這張紙的形狀是否白銀矩形。並略加解釋。
（可接受與理論少於 1%之差距。）

	
$\because\ PQRS \sim MNSP$ $\therefore$ $PQ : PS = PS : \frac{PQ}{2}$ $PS^2 = \frac{PQ^2}{2}$ $\frac{PQ}{PS} = \sqrt{2}$ 大會提供的紙張的尺寸為 $279\ mm \times 200\ mm$ $\frac{PQ}{PS} : \sqrt{2} = \frac{279}{200} : \sqrt{2} = 98.6\% : 1$ 和理論的偏差大於 1%	其他方法： 原紙張的長闊比 $\frac{長邊}{短邊} = \frac{279}{200} = 1.395$ 分成兩等分後 $\frac{長邊}{短邊} = \frac{200}{279 \div 2} = 1.43369$ 百分比的改變 $= \frac{1.43369 - 1.395}{1.395} \times 100\%$ $= 2.77\%$ 百分比的改變 > 1%

所以在容許的誤差範圍下，紙張不是「白銀矩形」。

Question 2 問題 2

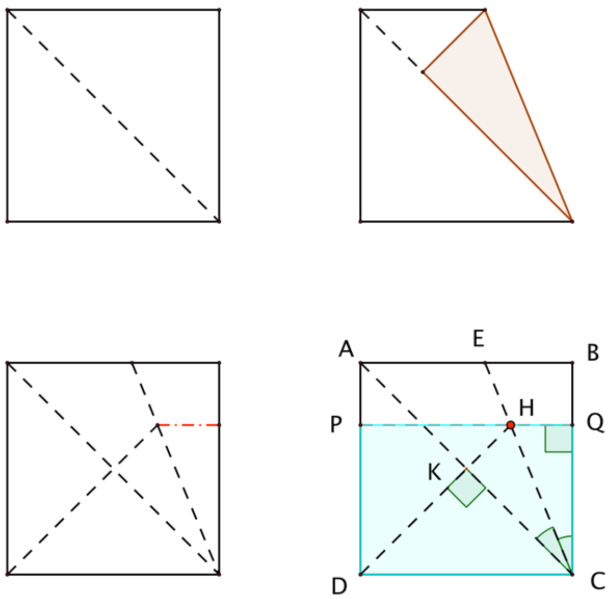
There is a piece of colour paper in Folder A. Cut out a silver rectangle from this piece of paper. Explain your steps.

文件夾 A 中有一張顏色紙。請從這張顏色紙剪出一個白銀矩形。解釋你的部驟。

解：

由上題的結果，我們知道白銀矩形的長短邊比是 $\sqrt{2}:1$ 或 $1:\frac{1}{\sqrt{2}}$ 。

方法一：用摺紙的方法



證明：

$\angle HCK = \angle HCQ$ (由對摺產生)

$\angle HKC = \angle HQC$ (由重疊的部分摺後打開)

$HC = HC$ (公共邊)

$\Delta HKC \cong \Delta HQC$ (AAS)

$$AC = \sqrt{DC^2 + AD^2} = \sqrt{2\,DC^2} = \sqrt{2}DC$$

$$QC = KC = \frac{1}{2}AC = \frac{\sqrt{2}}{2}DC$$

∴ PQCD 是白銀鉅形

方法二：計算和量度

量度正方形紙的長度，得 10 cm。
如果以正方形的邊長作白銀矩形的長邊，則

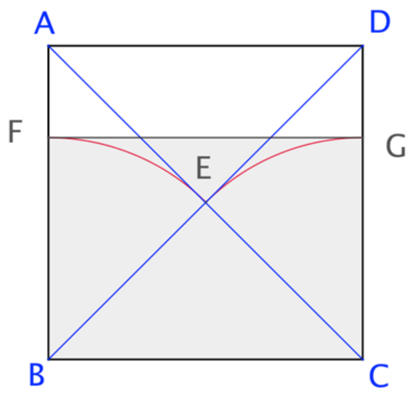
  矩形短邊的長度

$= \dfrac{10}{\sqrt{2}}$ cm

$= 7.1$ cm (準確至兩位有效數字)

依據計算的結果剪下來就可以了

方法三：幾何作圖

	*ABCD* 為正方形，設其邊長為 x。 作對角線 *AC* 和 *BD*。 $AD = BD = \sqrt{x^2 + x^2} = \sqrt{2x^2} = \sqrt{2}x$ 從正方形的性質， $BE = CE = \frac{1}{2}AD = \frac{\sqrt{(2)}x}{2} = \frac{x}{\sqrt{2}}$ 以 *BE* 和 *CE* 為半徑分別作弧 交*AB* 和 *CD* 於 *F* 和 *G*。 則 $BF = CG = \frac{x}{\sqrt{2}}$ 所以 *BCGF* 為白銀矩形。

Part B ***Folding up a Rectangle to make a Tetrahedron***

乙部 把長方形紙摺成四面體

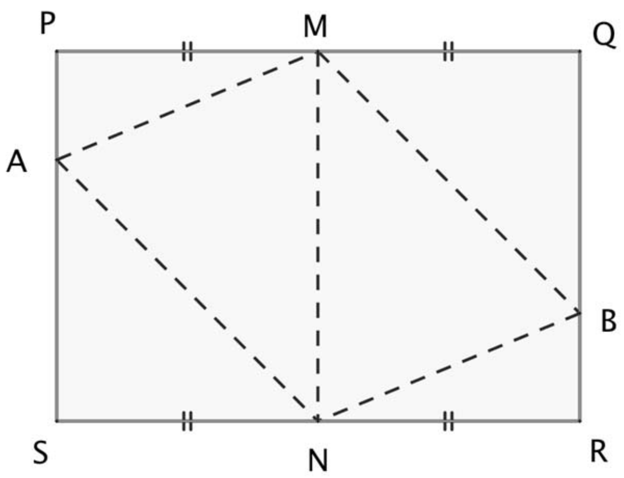
There are 6 pieces of paper in each of the Folder B1 and the Folder B2 for the investigation of Part B. They are in the shape of silver rectangle with the longer side 210 mm.

文件夾 B1 和文件夾 B2 內各有 8 張顏色紙作乙部的探究之用。這些紙均為白銀鉅形，長邊長 210 mm。

<u>Question 1</u> / <u>題一</u> (12 marks /12 分)

For this question, use a rectangular paper from Folder B1 and fold it to form a tetrahedron according to the following instructions.

這一題，請用文件夾 B1 中的矩形紙張。按以下的指示摺成四面體。



Let *PQRS* represent the piece of paper where *PQ* is the longer side.

M and *N* are respectively the midpoints on the two longer sides *PQ* and *RS*.

A and *B* are respectively points on the shorter sides *PS* and *QR* such that *AN* and *BM* bisects the angles $\angle SNM$ and $\angle QMN$ respectively.

PQ 為矩形紙 PQRS 的長邊。

M 和 N 分別是 PQ 和 RS 的中點。

A 和 B 是 PS 和 QR 上的點使 AN 和 BM 分別平分角$\angle SNM$ 和 $\angle QMN$。

a. Fold along the 5 crease lines *MN, AM, AN, BM* and *BN* to make a tetrahedron so that no paper will be overlapped. Take pictures of the tetrahedron formed.

沿虛線 *MN*、*AM*、*AN*、*BM* 和 *BN* 把矩形紙摺成一個四面體，摺成的立體沒有紙會重疊在一起。拍攝完成的四面體。

b. What is the total surface area of the tetrahedron?

四面體的總表面積是多少呢？

由於所有的紙張都用盡，因此，四面體的總表面積

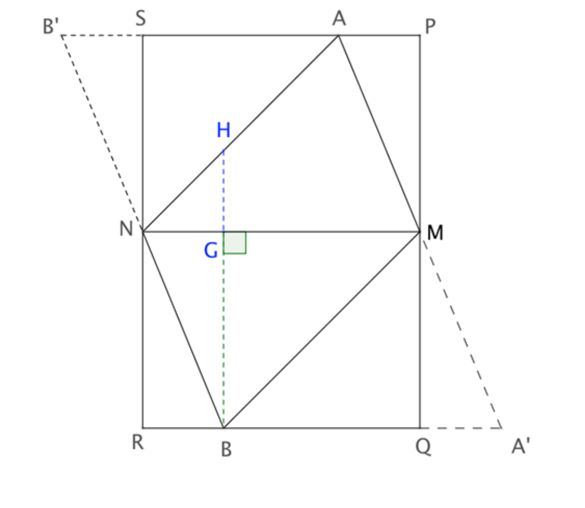
$= \frac{210^2}{\sqrt{2}}\ mm^2$

$= 22050\sqrt{2}\ mm^2$

$= 31183.41\ mm^2$

c.. By making appropriate measurements, or calculations, or otherwise, find the volume of the tetrahedron formed. Give your answer as accurately as possible. Explain your steps. (You might use text, mathematics expressions, drawings or photographs for your explanations.)

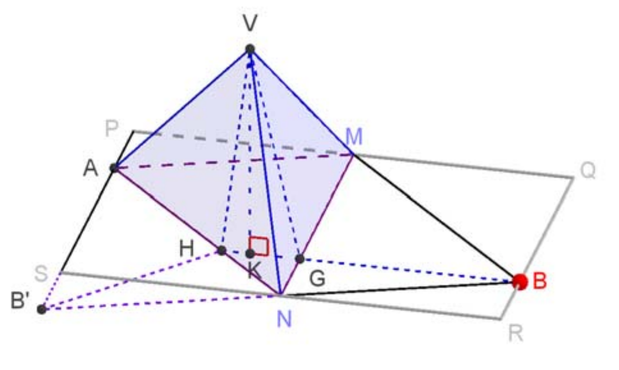
通過量度、計算等方法，找出四面體的體積。答案應力求準確，並請解釋你的步驟。（可以用文字、算式、草圖、相片等協助說明）



設A'和 B' 分別為 A 和 B 以M和 N 作中心旋轉 180° 後的像。

不難發現，以長方形紙 $PQRS$ 摺出來的四面體和以平行四邊形紙$AA'BB'$（以AN, MN 和 MB 為摺痕）所摺出來的四面體是一樣的。

若以 ΔAMN 為底，四面體的高會在通過HG、垂直於 ΔAMN 的平面上。



$GV = GB = MG = MQ$

$= \frac{210}{2} = 105\ mm$

$HG = NG$

$= (148 - 105)\ mm$ (148 mm 由量度得到)

$= 43\ mm$

$HV = HB'$

$= \sqrt{(105-43)^2 + (2 \times 43)^2}$

$= 2\sqrt{2810}\ mm$

設三角形 HGV 的高為 h，HK = x，$HV = a$，$HG = b$，$GV = c$。

$h^2 = a^2 - x^2$ ---- (i)
$h^2 = c^2 - (b - x)^2$ ---- (ii)

(ii) – (i) 得

$$0 = c^2 - a^2 - b^2 + 2bx$$

$$x = \frac{a^2 + b^2 - c^2}{2b}$$

代入 (i)，得

$$h^2 = a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2$$

$$h^2 = \frac{(2ab)^2 - (a^2 + b^2 - c^2)^2}{(2b)^2}$$

$$h^2 = \frac{\big(2ab + (a^2 + b^2 - c^2)\big)\big(2ab - (a^2 + b^2 - c^2)\big)}{4b^2}$$

$$h^2 = \frac{(a^2 + b^2 + 2ab - c^2)\big(c^2 - (a^2 + b^2 - 2ab)\big)}{4b^2}$$

$$h^2 = \frac{(a + b + c)(a + b - c)(c + a - b)(c + b - a)}{4b^2}$$

所以

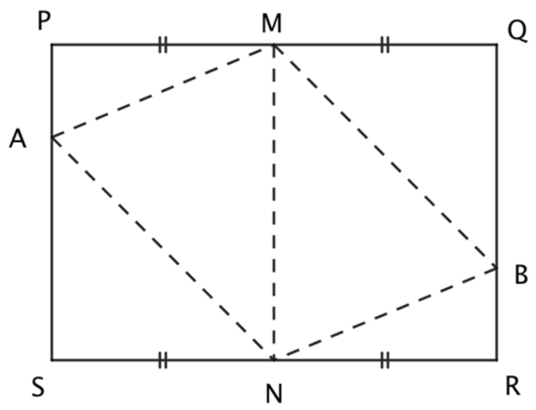
$$h = \frac{1}{2b}\sqrt{(a + b + c)(a + b - c)(c + a - b)(c + b - a)}$$

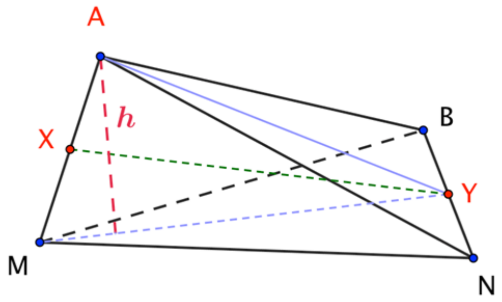
四面體體積

$$= \frac{1}{3}(148\ mm)(105\ mm)h$$

$$= 267460.61 mm^3$$

假如 PQRS 是完美的白銀矩形





設 $MP = 1$ ，則 $PM = MQ = SN = NR = SA = QB = 1$

且 $MB = MN = AN = \sqrt{2}$

$BR = \sqrt{2} - 1$

$$BN^2 = NR^2 + BR^2 = 1^2 + \left(\sqrt{2} - 1\right)^2 = 1 + \left(3 - 2\sqrt{2}\right) = 4 - 2\sqrt{2}$$

設 X 和 Y 分別為 AM 和 BN 的中點

$$BY = \frac{1}{2}BN = \frac{1}{2}\sqrt{4 - 2\sqrt{2}}$$

$$AY^2 = AB^2 - BY^2 = \left(\sqrt{2}\right)^2 - \frac{1}{4}\left(4 - 2\sqrt{2}\right) = 2 - 1 + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2}$$

$$XY^2 = AY^2 - XM^2 = \frac{2 + \sqrt{2}}{2} - \frac{1}{4}\left(4 - 2\sqrt{2}\right) = \sqrt{2}$$

所以， $XY = \sqrt{\sqrt{2}}$

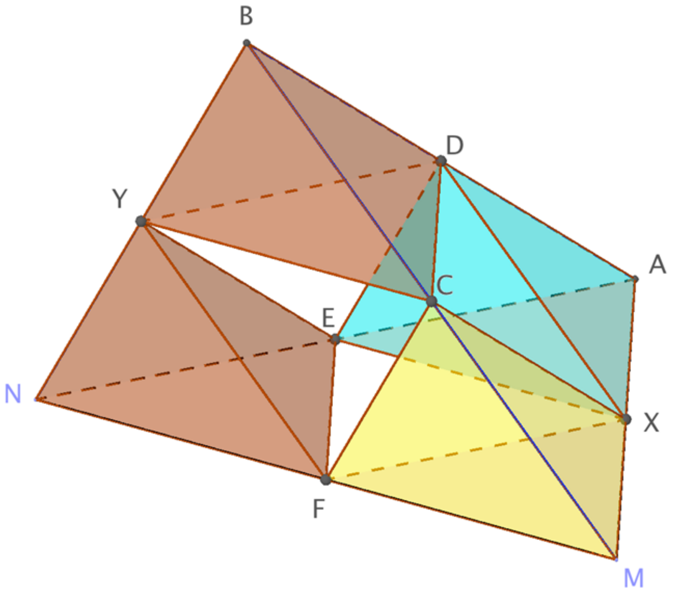
設 h 為四面體的高

$\frac{1}{2}MY \cdot h = \frac{1}{2}AM \cdot XY$ $\sqrt{\frac{2 + \sqrt{2}}{2}} \cdot h = \sqrt{4 - 2\sqrt{2}} \cdot \sqrt{\sqrt{2}}$ $h^2\left(\frac{2 + \sqrt{2}}{2}\right) = \left(4 - 2\sqrt{2}\right)\sqrt{2}$ $h^2 = \frac{4\sqrt{2}\left(2 - \sqrt{2}\right)}{2 + \sqrt{2}}$ $h^2 = 2\sqrt{2}\left(2 - \sqrt{2}\right)^2 = 4\left(3\sqrt{2} - 4\right)$	$h = 2\sqrt{3\sqrt{2} - 4}$ 因此，四面體的體積（$2 \times \sqrt{2}$ 的紙張） $= \frac{1}{3}\left(2\sqrt{3\sqrt{2} - 4}\right)\left(\frac{1}{2}\right)(1)\left(\sqrt{2}\right)$ $= \frac{1}{3}\sqrt{6\sqrt{2} - 8}$

現在用的是 $(105\,mm + 105\,mm) \times 105\sqrt{2}\,mm$ 的紙張，所以體積應為

$$= \frac{1}{3}\sqrt{6\sqrt{2} - 8}(105\,mm)^3$$

$$\approx 268808.7825\,mm^3$$



此外，設 C、D、E、F 為 BM、BA、NA、NM 的中點， 如圖將四個和 ABMN 相似但體積為 ABMN 的 $\frac{1}{8}$ 的四面體移走後，會剩下兩個以正方形 CDEF 為底的椎體。

因此

四面體 ABMN 的體積

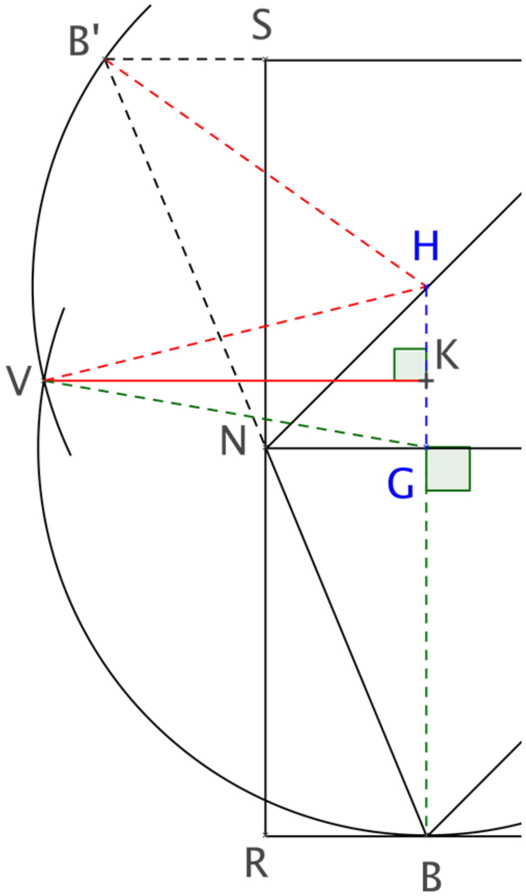
$$= 2 \times \frac{1}{3}CD^2 \times \frac{XY}{2}$$

$$= \frac{1}{3}\left(\frac{BN}{2}\right)^2 XY$$

$$= \frac{1}{3}(2 - \sqrt{2})\sqrt{\sqrt{2}}\,(105\,mm)^3$$

$$\approx 268808.7825\,mm^3$$

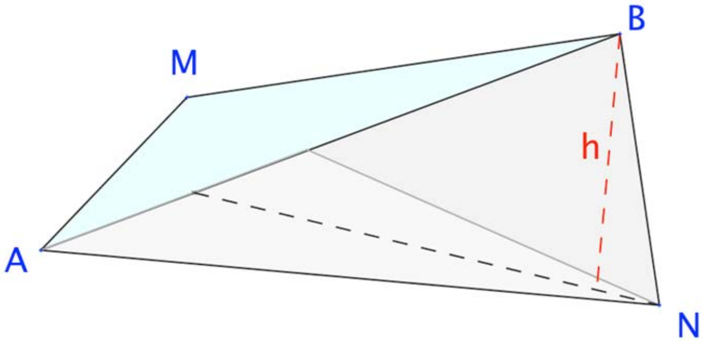
（一）如果不計算四面體的高，可以幾何作圖再量度

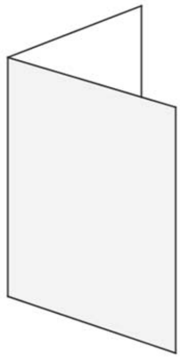


（二）直接量度

重點是如何利用大會提供的物資，量度四面體的垂直高度呢？
以下提供其中一個可行的方案：

做四面體時留空一面，方便量度；B 的投影會落在$\angle MBA$的角平分線上；
把一張紙對摺，打開一些後會得到一個可協助量度垂直高度的工具。





d. Comment on the source(s) of error, if any, in your method in finding the answers in (c).

假如你用以求出（c）部分答案的方法中會存在誤差的話，請說明。

- 如果用計算的方式，誤差只來自最初的量度。 量度上，因為是用刻度準確至 1 mm 的間尺，所以成為誤差的最主要來源
- 當然，如果計算的中間步驟使用小數，則中間步驟出現的數值的有效數字成為誤差的新來源
- 量度模型的話，模型製作的精確度會影響到量度和計算的結果
- 量度的時候，工具使用的正確方法也很重要。例如怎樣看刻度？怎樣得到垂直的高度？

Part B (continued)

Question 2 (12 marks)

For this question, use the papers from folder B(2).

這一題，請用文件夾 **B(2)** 中的矩形紙張。

If four points M, A, N, B are chosen appropriately ~~chosen~~, one on each side a rectangular paper, with some creases added, it can be folded to form a tetrahedron $MANB$ in a way similar to previous question.
假如在矩形紙的每條邊各適當地選一點，稱它們為 M、A、N、B，再加上適當的摺痕，便可用類前題的方式摺一個四邊形 MANB。

a. A tetrahedron is to be folded from a piece of colour paper taken from the folder B(2). Calculate the greatest possible volume of the tetrahedron formed.
請用文件夾 B(2) 中的顏色紙張摺四面體，它的最大可能體積是多少。

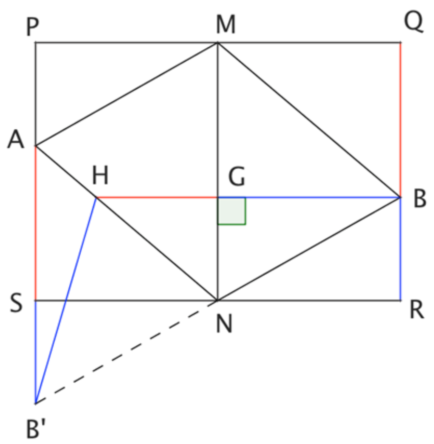
以 ΔMNA 為底，四面體的高不能超過 ΔMNB 沿著 MN 摺起後的最高點，即 105 mm。

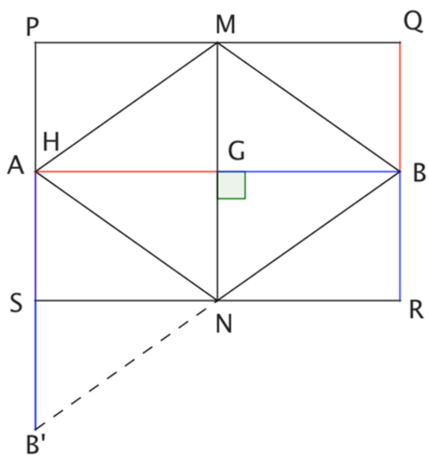
理論上四面體的上界

$$= \left(\frac{1}{3}\right)(105\ mm)\left(\frac{1}{2}\right)(105^2\sqrt{2}\ mm^2)$$
$$\approx 272854.8291\ mm^3$$

或（如果短邊的值是度出來的）

$$= \left(\frac{1}{3}\right)(105\ mm)\left(\frac{1}{2}\right)(105\ mm \times 148\ mm)$$
$$= 271950\ mm^3$$





- 如果要合成冇重疊的四面體，$PA = RB$ 並且摺起後 $AB = MN$，不會因 B 的位置而有的不同。

- 當 B 移到 QR 的中點時，$AG^2 + GB^2 = AB^2$，所以$\angle AGB = 90°$
- 所以理論上的最大體積可以達到。
- 即使 $\frac{RS}{PQ} \neq \sqrt{2}$ ，ΔMNA 和 ΔMNB 的夾角亦會當 B 移到 QR 的中點時最接近 90°
- 同學可透過實驗發現類似的結果

b. Fold a piece of rectangular piece of paper to form a tetrahedron with the greatest possible volume.
   i. Describe how this tetrahedron can be formed.
   ii. Explain why the tetrahedron formed has the greatest possible volume.
   ii. By making appropriate measurements, calculations, or otherwise, find the volume of the tetrahedron formed. Compare it with the answer in (a).

[You might use writings, drawing and photographs for your description and explanation.]

用所提供的矩形紙製作體積盡可能大的四面體。

i. 描述四面體如何製成
ii. 解釋它的體積為何是最大
iii. 以適當的量度和計算，找出四面體的體積。並和（a）的結果比較

（可以用文字、算式、草圖、相片等協助解說）

解釋己在 (a) 提供

Part B (continued)

Question 3 **(6 marks)**

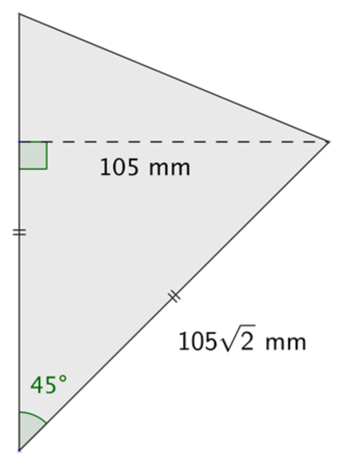
A carpenter is to make a hard solid tetrahedron of exactly the same shape and size as described in question 1, part (B). He will then cut a hole in a wooden board such that this tetrahedron can pass through.

In the space below, draw in the real shape and size of this hole. Draw a hole whose area is as small as possible.

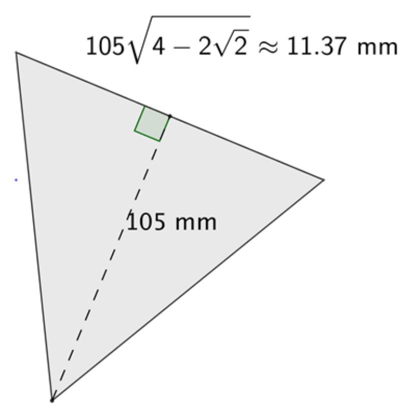
Justify your answer.

一位木匠造了有一個和題一（**b**）相同形狀大少的實木四面體。他想在一塊薄木板上開一個能讓這四邊形穿過的洞。
試在下面按真實的形狀和大小，畫出一個面積最少，又能讓這四邊形穿過的洞。
請解釋你們的做法。

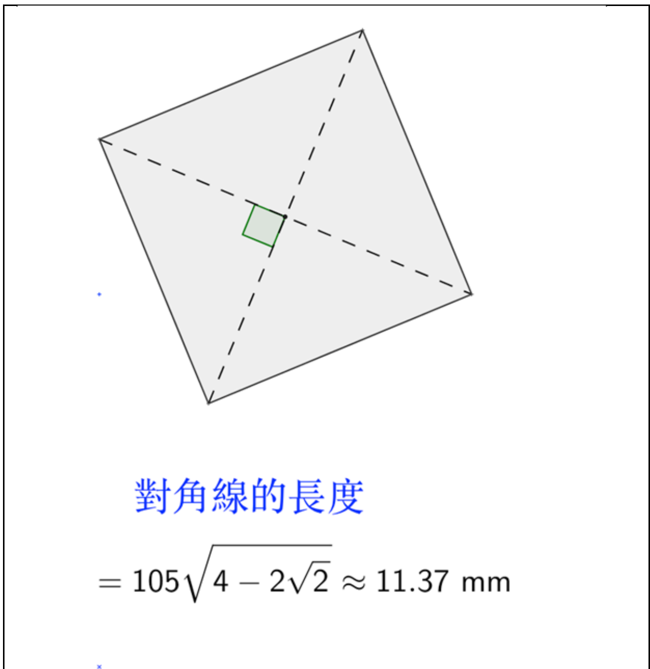
直觀上我們可能以為洞至少要等於其中一面的大小。（圖一）
但細心想想，這是四面體在平面上的投影。
不同擺放方式，四面體會有不同的投影。
（圖二）（圖三）所示的投影都比圖一的小。



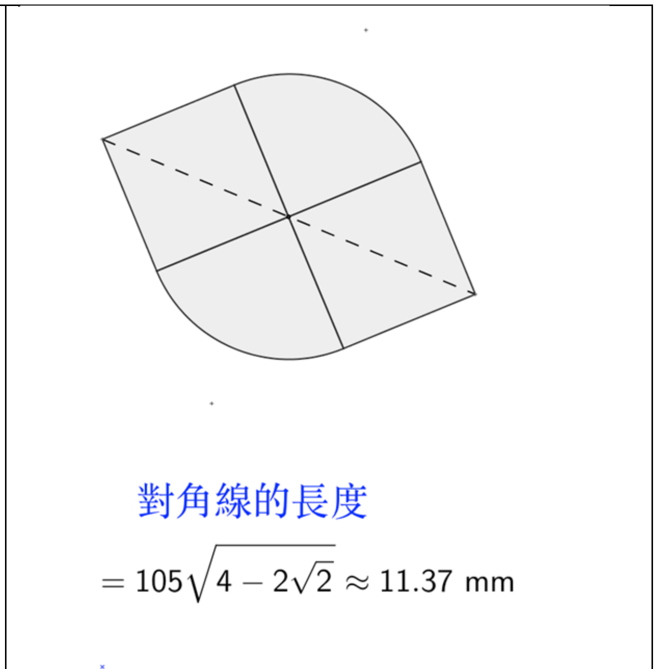
（圖一）



（圖二）



（圖三）



（圖四）

其中圖二在頭三個可能性中是最小的。
假如我們容許四面體在穿過一半後旋轉90° 再前進，則圖四的洞面積最細。

Part B (continued)

<u>Question 4</u> **(6 marks)**

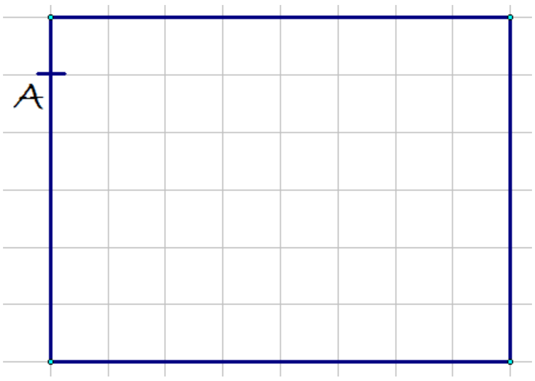
In the previous two questions, you are instructed to fold a rectangular paper into tetrahedron where there is no overlapping paper area when it is folded up. Let's explore the possibility of doing so when some particular points are chosen as the folded tetrahedron's vertices.
In each of the following, parts (a) to (f), one or two particular point(s) on a rectangle is/are marked on a rectangle.
State whether it is possible to fold the rectangle into a tetrahedron ABCD where there is no overlapping paper area when it is folded up. The marked point(s) should be the vertex (vertices) of the tetrahedron formed.

(1) If no, put a × inside the box ☐.
(2) If yes, mark on the rectangle the other vertices and the crest lines along which the rectangle is to be folded.

在之前的題目中，我們用白銀矩形的紙張摺出四面體（當摺成後沒有紙張重疊）。讓我們探究一下如果某一或兩點已指定作四面體的頂點，可以如何摺。以下各圖中，一或兩頂點已標示，請判斷能否把矩形紙在沒有紙張重疊的情況下，摺成四面體。

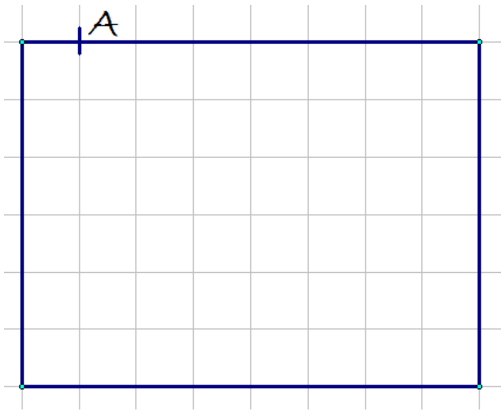
(1) 如果認為不可以，請在☐內加上 × 。
(2) 如果認為可以，請在圖上加上代表摺痕的線。

a. The paper is a in the form of an 8 cm × 6 cm rectangle. *A* is a point on one of a side as shown.
8 × 6 平方單位的矩形紙，A 點如圖所示在邊上。

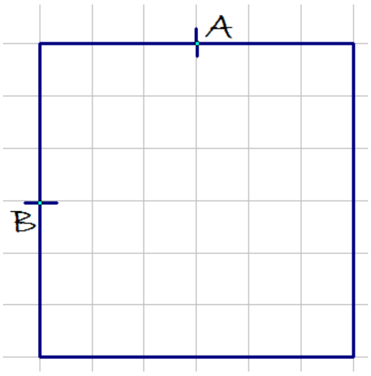
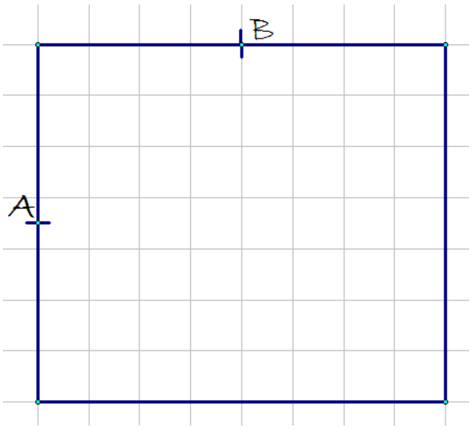
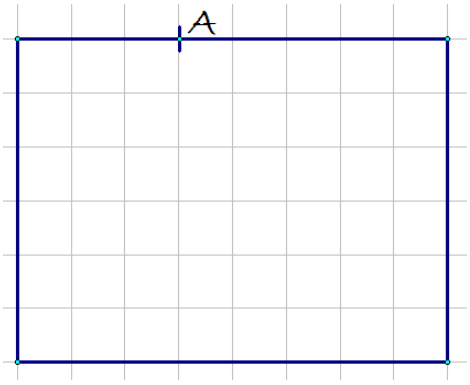
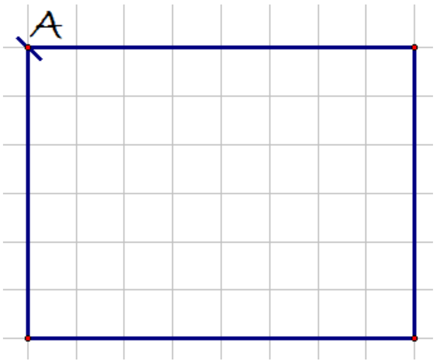


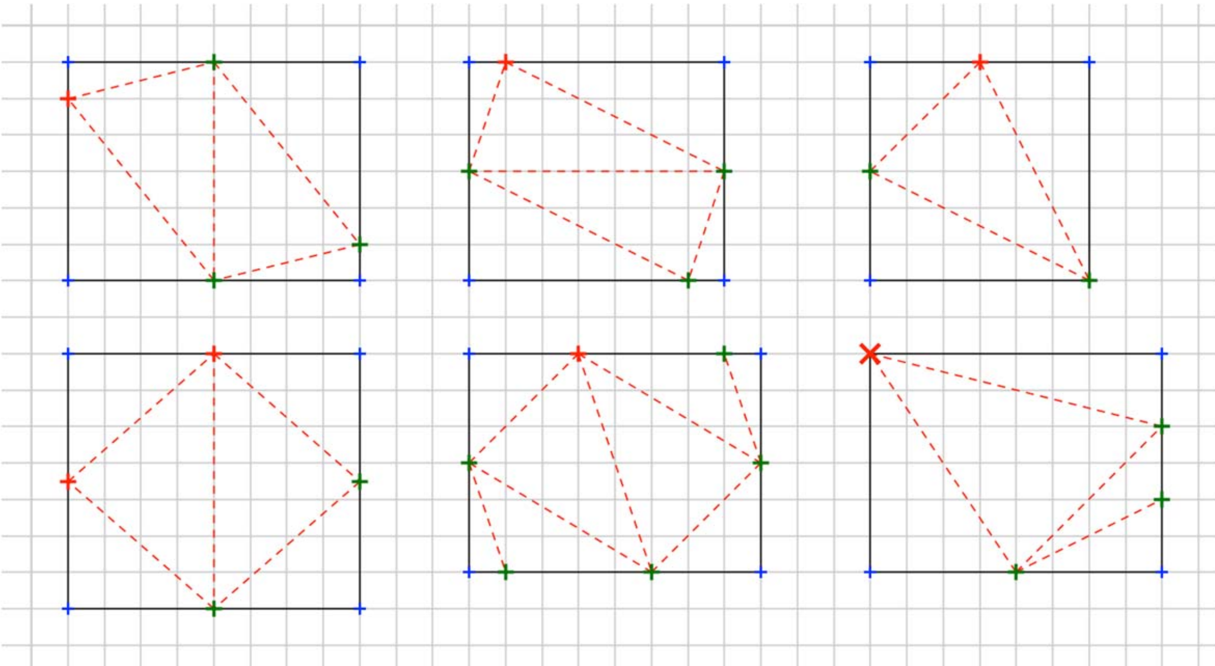
☐

b. The paper is a in the form of an 8 cm × 6 cm rectangle. *A* is a point on one of a side as shown.
8 × 6 平方單位的矩形紙，A 點如圖所示在邊上。



☐

c. The paper is a in the form of a 6 cm × 6 cm square. *A* and *B* are respectively the midpoints of two of the sides. 8 × 6 平方單位的矩形紙，A、B 點如圖所示在兩條邊上。  □	d. The paper is a in the form of a 8cm×7cm rectangle. *A* and *B* are respectively the midpoints two of the sides. 8 × 6 平方單位的矩形紙，A、B 點如圖所示在兩條邊上。  □
e. The paper is in the form of a 8 cm × 6 cm rectangle. *A* is a point on one of a side as shown. 8 × 6 平方單位的矩形紙，A 點如圖所示在邊上。  □	f. The paper is in the form of a 8 cm × 6 cm rectangle. *A* is one of the vertex of the rectangle. 8 × 6 平方單位的矩形紙，A 點如圖所示在角上。  □



End