Pre-Test

1 Probability (Chance)

The probability of an event has a value between 0 and 1 (inclusively) to measure the possibility of the event to happen. The bigger the value, the highest the likelihood of the event to happen.

Example 1

In a mathematics test, the probability for Joseph to get A grade is 0.2 and the probability for him to get a B grade is 0.5. Then it is more likely for Joseph getting B grade than getting A grade.

1.1 Finding probability by listing possible outcomes

$$\text{probability of an event} = \frac{\textbf{number of outcomes that make such event to happen}}{\textbf{number of all possible outcomes}}$$

* assuming that all possible outcomes are equal likely to happen

Example 2

When a dice is thrown, possible outcomes are 1, 2, 3, 4, 5 and 6. These outcomes are equally likely to happen. Of the six outcomes, the outcomes $\{1, 3, 5\}$ make the event "**get an odd number**" to happen. So

$$\text{Probability of "}\textbf{get an odd number}\text{"} = \frac{3}{6} = \frac{1}{2}$$

This probability could also be expressed as $\mathbf{0.5}$ or $\mathbf{50}\%$.

Similarly,

$$\text{Probability of "}\textbf{get a number >4}\text{"} = \frac{2}{6} = \frac{1}{3}$$

Question (1):
When a dice is thrown, what is the probability of "**getting a multiple of 3**"?

$$P(\textbf{getting a multiple of 3}) = \frac{2}{6} = \frac{1}{3}$$

Example 3

There are two packs of number cards. The first pack has 4 cards with numbers **2, 4, 6** and **10** printed on them respectively. The other pack has 3 cards with **5, 7** and **9** printed on them respectively.

If we draw one card from each pack. There are $\mathbf{4 \times 3 = 12}$ possible and equal likely outcomes.

The event “the sum of two number drawn is greater than 16”only happens when $(\mathbf{10}, \mathbf{7})$ or $(\mathbf{10}, \mathbf{9})$ are drawn. Thus

$$\text{Probability that "}\textbf{sum of the two number drawn}\text{"} = \frac{2}{12} = \frac{1}{6}$$

Question (2):
Using the same packs of cards in example 3. If one card is drawn from each pack, what is the probability that “**the sum of the two numbers drawn is 11**”?

$$P(\mathbf{sum = 11}) = \frac{3}{12} = \frac{1}{4} \quad (or\ 25\%\ or\ 0.25)$$

1.2 Arithmetic Computation on probability

1.2.1

If the probability of an event to happen is $\boldsymbol{p}$, the probability for that event not to happen is $(\mathbf{1} - \boldsymbol{p})$.

Example 4

The chance for “It will rain today” is $\mathbf{20}\%$, then

The chance for “It will not rain today” $= \mathbf{1 - 20}\% = \mathbf{80}\%$

Question (3):

The probability for “**Amy will bring umbrella**” is 0.3.

What is the probability for “**Amy will NOT bring umbrella**”?

$$\mathrm{P}(\textbf{Amy will NOT bring umbrella}) = 1 - 0.3 = 0.7$$

1.2.2

A and *B* are independent events (*the probability of A to happen will not be affected by whether B happens or not, vice versa*). If the probability of A happen is p and probability of B happen is q, then the probability of A and B both happen is $p \times q$.

Example 5

In a mathematics test, the probability for Joseph to get A grade is 0.9. In history, the probability for him to get a A grade is 0.7. The probability for him to get A grades for both subjects is

$$\mathrm{P}(\textbf{get A grades for both subjects}) = 0.9 \times 0.7 = 0.63$$

Question (4a): The probability for "**Amy will bring umbrella**" is 0.3. The probability for "**It will rain today** " is 0.8. What is the probability of "**Amy will not bring umbrella and it rains**"? $P(\textbf{will not bring umbrella } \textit{and} \textbf{ it rains}) = (1 - 0.3) \times 0.8 = 0.56$ (or 56%)
Question (4b): What is the probability of getting 2 "6" if two dices are thrown? $\mathrm{P}(\textbf{getting } 2 \text{ “}\textbf{6}\text{”}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

2 Expected Value

Example 6

In a lucky draw, participants have chances to get coupons of different values. The probability for obtaining different coupons are:

Coupon Value ($)	0	10	50	100
Probability	$\frac{63}{100}$	$\frac{25}{100}$	$\frac{10}{100}$	$\frac{2}{100}$

$0、 $20、 $50、 $100 are all possible events with different probabilities.

We can compute an "expected value" for this lucky draw.

$$\text{E(lucky draw)} = \$0 \times \frac{63}{100} + \$10 \times \frac{25}{100} + \$50 \times \frac{10}{100} + \$100 \times \frac{2}{100} = \$\,9.5$$

We can interpret this "expected value" as the average value of coupons obtained after many times of lucky draws.

Question (5):

(a) The returns of a gambling game will be lose $10, win $10 or win $100. The probability of each event is shown below.

return ($)	−10	+1	+100
probability	0.8	0.15	0.05

Calculate the expected returns of this game.

Answer:

expected return $= -10 \times 0.8 + 1 \times 0.15 + 100 \times 0.05$

$= -\$1.5$

(b) In a game, participant will toss two coins. If both are heads, $5 will be awarded. Otherwise, he/she loses $3. Find the expected return of this game.

$$\text{Probability of "2 heads"} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$\text{Expected return} = \$5 \times \frac{1}{4} + (-\$3) \times \left(1 - \frac{1}{4}\right) = -\$1$$

3 Absolute Value

The **absolute value** of a number is definite to be

$$|x| = \begin{cases} x & if\ x \geq\ 0 \\ -x & if\ x < 0 \end{cases}$$

Therefore,

$|13| =13$

$|-83| = 83$

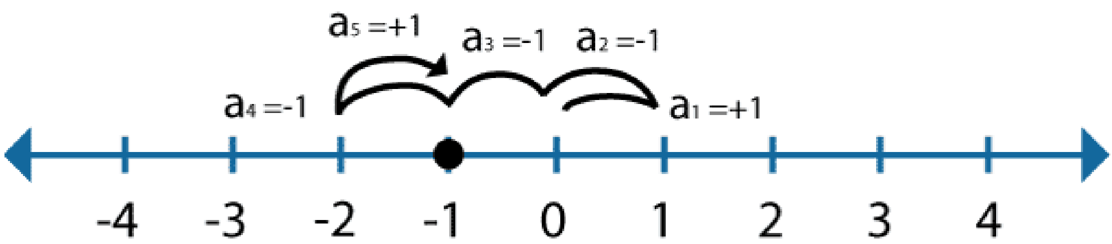
2017/18 The 9th HK Mathematics Creative Problem Solving

Competition for Secondary School

Suggested Solutions

Random walk

Random Walk is a mathematical model that describes a path that consists of a series of random steps on a mathematical space (such as an integer). A basic example of a random walk is a one-dimensional random walk.

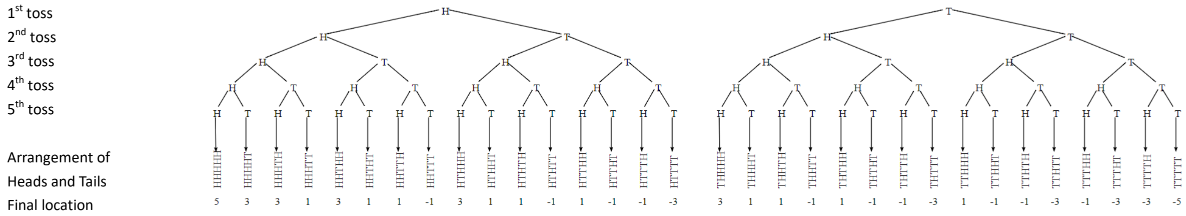


Suppose we set the black spot to 0, then let it take N steps (where N is any number), and the distance for each step is a unit. Now we want to know how far the black spot goes after N walks. Of course, the distance traveled after each repetition of N steps will be different, so what we want to know is that if we repeat the N-step random walk many times, what is the average distance of the black spot from 0. We call the distance between 0 and the black spot to be d.

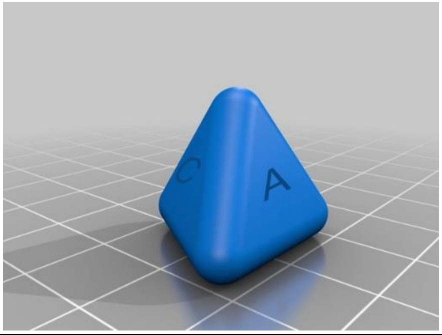
$$d = a_1 + a_2 + a_3 + \ldots + a_N$$

Suppose the location of the black spot is set to zero and use a fair coin for tossing.

If the tossing is a head, the black dot moves one unit to the right (+1). If the tossing is a tail, the black dot moves one unit to the left (-1). After tossing the fair coin for five times, the black spot can land on +1, -1, +3, -3, +5 or -5. With five tosses, 3 heads and 2 tails will land on +1 in any order. There are 10 ways to land on +1 (by tossing 3 heads and 2 tails). There are 10 ways to land on -1 (by tossing 3 tails and 2 heads), and 5 ways to land on +3 (by tossing 4 heads and 1 tail), 5 ways to land on -3 (by tossing 4 tails and 1 head), 1 way to land on +5 (by tossing 5 heads), and 1 way to land on -5 (by throwing five tails). See the figure below for the possible results of 5 tosses. (Appendix 1 is an enlarged figure)



(a) Suppose there is a biased four-sided dice. The probability that A, B, C, or D come up is as follows:



Alphabet	A	B	C	D
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$

Can this dice be used in a one-dimensional random walk? Why or why not?

Yes! [1A]

Define the step = + 1 when getting A or C, otherwise take the step = −1.

Alphabet	A	C	A or C
Probability	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

[1M]

Alphabet	B	D	B or D
Probability	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

[1M]

(b) For one-dimensional random walk, complete the probability for different locations of the black dot after 5 steps.

Step	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$	0	$\frac{1}{2}$				
2				$\frac{1}{4}$	0	$\frac{2}{4}$	0	$\frac{1}{4}$			
3			$\frac{1}{8}$	0	$\frac{3}{8}$	0	$\frac{3}{8}$	0	$\frac{1}{8}$		
4		$\frac{1}{16}$	0	$\frac{4}{16}$	0	$\frac{6}{16}$	0	$\frac{4}{16}$	0	$\frac{1}{16}$	
5	$\frac{1}{32}$	0	$\frac{5}{32}$	0	$\frac{10}{32}$	0	$\frac{10}{32}$	0	$\frac{5}{32}$	0	$\frac{1}{32}$

[2A]

(c) The "Pascal's triangle" is a triangular arrangement of numbers. Observe the following pattern of the Pascal's triangle and complete the line 5.

Line 0						**1**					
Line 1					**1**		**1**				
Line 2				**1**		**2**		**1**			
Line 3			**1**		**3**		**3**		**1**		
Line 4		**1**		**4**		**6**		**4**		**1**	
Line 5	**1**		**5**		**10**		**10**		**5**		**1**

[2A]

(d) (i) Calculate the expected value $\boldsymbol{E(d_5)}$ of the location of the black dot after walking 5 steps in a one-dimensional random walk.

By symmetry, $E(d_5) \;=\; 0$

Or

$$E(d_5) = (-5)\left(\frac{1}{32}\right) + (-4)(0) + (-3)\left(\frac{5}{32}\right) + (-2)(0) + (-1)\left(\frac{10}{32}\right) + 0 \times 0$$

$$+ (1)\left(\frac{10}{32}\right) + (2)(0) + (3)\left(\frac{5}{32}\right) + (4)(0) + (5)\left(\frac{1}{32}\right) = 0$$

[1M+1A]

(ii) Calculate the expected value $\boldsymbol{E(d_N)}$ of the location of the black dot after *N* steps in one-dimensional random walk. (*N* is a positive integer)

$$E(d_N) = (-N)P_N(-N) + (-N+1)P_N(-N+1) + \cdots + (N-1)P_N(N-1) + NP_N(N)$$
$$= 0 \quad (\because \; P_N(-N) = P_N(N), P_N(-N+1) = P_N(N-1), \cdots$$

[2M+1A]

(e) (i) Calculate the expected value of the absolute value of the location of the black dot $|\boldsymbol{d_5}|$ after 5 steps of random walk, $\boldsymbol{E(|d_5|)}$.

$$\boldsymbol{E(|d_5|)} = 2 \times \left(0 \times 0 + 1 \times \tfrac{10}{32} + 2 \times 0 + 3 \times \tfrac{3}{32} + 4 \times 0 + 5 \times \tfrac{1}{32}\right) = 1\frac{7}{8}$$

[1M+1A]

(ii) Calculate the expected value of the square of the location of the black dot $\boldsymbol{d_5}^{\boldsymbol{2}}$ after 5 steps of random walk, $\boldsymbol{E\left(d_5{}^2\right)}$. And then calculate the square root of the expected value of the square of the location of the black dot, $\sqrt{\boldsymbol{E\left(d_5{}^2\right)}}$. This is technically called the Root Mean Square Value.

$$E\left(d_5{}^2\right) = (-5)^2 \times \tfrac{1}{32} + (-4)^2 \times 0 + (-3)^2 \times \tfrac{5}{32} + (-2)^2 \times 0 + (-1)^2 \times \tfrac{10}{32} + 0$$
$$+ 1^2 \times \tfrac{10}{32} + 2^2 \times 0 + 3^2 \times \tfrac{5}{32} + 4^2 \times 0 + 5^2 \times \tfrac{1}{32} = 5$$

[1M+1A]

$$\therefore \sqrt{E\left(d_5{}^2\right)} = \sqrt{5}$$

[1A]

(iii) If we want to calculate the average distance of the black dot from the original location after an N steps random walk, which one should be calculated $\boldsymbol{E(d_N)}$, $\boldsymbol{E(|d_N|)}$, $\sqrt{\boldsymbol{E\left({d_N}^2\right)}}$? Explain your answer.

$E(|d_N|)$.

The average distance from the origin is calculated by absolute values of the distance, no matter the random walker is on the left or right from the origin.

[1A+1A]

(f) Danny said that according to the results of (d)(ii), even N is large, after N steps of random walk, the final distance is close to the original position. Is Danny's argument reasonable?

No! The measure of distance is a positive values, but the d_N in (d)(ii) can take the negative value.

[1A+1M]

(g) Guess or estimate $\sqrt{\boldsymbol{E\left({d_N}^2\right)}}$, where N is a positive integer.

By using (b) and (c), it can be shown or observed that:

$$E\left({d_N}^2\right) = \sum x^2 P_N(x) = N$$

[2M+1A]

$$\therefore \sqrt{E\left({d_N}^2\right)} = \sqrt{N}$$

[1A]

(h) Consider a two-dimensional random walk, each step with equal probability of going up, down, left, and right. That is:

Moving direction	↑	↓	←	→
probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Each team will be provided with a set of 11 dices.

(i) Design an experiment to simulate the two-dimensional random walk by using the dices.

(ii) Simulate **5 rounds** of a two-dimensional random walk with each round having **20 steps**. Record the results in a way you think is appropriate.

Appropriate use of dices to generate random number for the random walk. [1M]
(Optimize the usage?) [1M]
Appropriate assignment of the random walker's move. [1M]
Good record of random walk result showed. [2M]
(How?)

(i) Find the Root Mean Square Value of a two-dimensional random walk using the result of the Root Mean Square Value $\sqrt{E\left({d_N}^2\right)}$ of the one-dimensional random walk.

∵ x-direction and y-direction are independent, the one-dimensional random walk result can be applied independently. [1A]

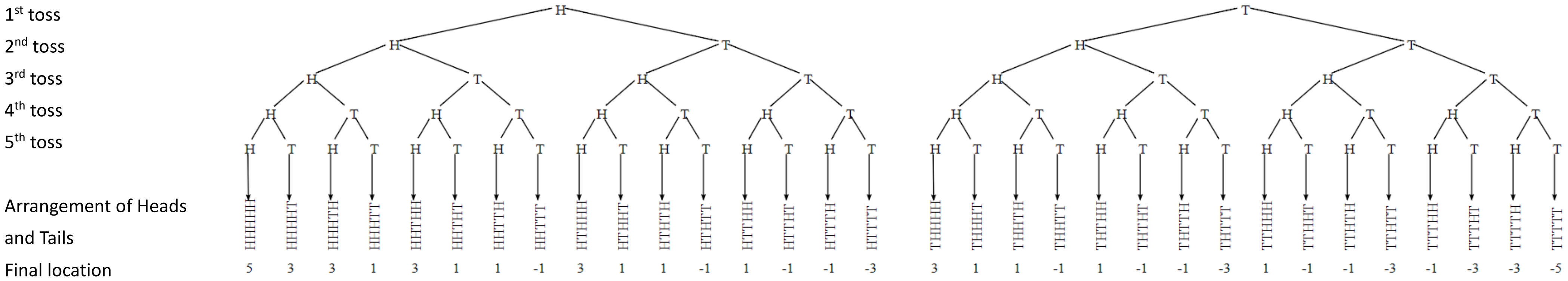
By Pythagoras theorem,

$$\sqrt{E\left({d_N}^2\right)} = \sqrt{\left(\sqrt{\frac{N}{2}}\right)^2 + \left(\sqrt{\frac{N}{2}}\right)^2} = \sqrt{\frac{N}{2} + \frac{N}{2}} = \sqrt{N}$$

[2M+1A]

<End of Paper>

Appendix 1


1st toss
2nd toss
3rd toss
4th toss
5th toss
Arrangement of Heads and Tails
Final location
5 3 3 1 3 1 1 -1 3 1 1 -1 1 -1 -1 -3 3 1 1 -1 1 -1 -1 -3 1 -1 -1 -3 -1 -3 -3 -5