The 9th Hong Kong Mathematics Creative Problem Solving Competition for Secondary Schools

Pre-Test

1 Probability (Chance)

The probability of an event has a value between 0 and 1 (inclusively) to measure the possibility of the event to happen. The bigger the value, the highest the likelihood of the event to happen.

Example 1

In a mathematics test, the probability for Joseph to get A grade is 0.2 and the probability for him to get a B grade is 0.5. Then it is more likely for Joseph getting B grade than getting A grade.

1.1 Finding probability by listing possible outcomes

$$\text{probability of an event} = \frac{\textbf{number of outcomes that make such event to happen}}{\textbf{number of all possible outcomes}}$$

* assuming that all possible outcomes are equal likely to happen

Example 2

When a dice is thrown, possible outcomes are 1, 2, 3, 4, 5 and 6. These outcomes are equally likely to happen. Of the six outcomes, the outcomes {1, 3, 5} make the event "**get an odd number**" to happen. So

$$\text{Probability of "}\textbf{get an odd number}\text{"} = \frac{3}{6} = \frac{1}{2}$$

This probability could also be expressed as $\mathbf{0.5}$ or $\mathbf{50}\%$.

Similarly,

$$\text{Probability of "}\textbf{get a number >4}\text{"} = \frac{2}{6} = \frac{1}{3}$$

Question (1):

When a dice is thrown, what is the probability of "**getting a multiple of 3**"?

Example 3

There are two packs of number cards. The first pack has 4 cards with numbers **2, 4, 6** and **10** printed on them respectively. The other pack has 3 cards with **5, 7** and **9** printed on them respectively.

If we draw one card from each pack. There are $\mathbf{4 \times 3 = 12}$ possible and equal likely outcomes.

The event "the sum of two number drawn is greater than 16"only happens when $(\mathbf{10}\ ,\mathbf{7})$ or $(\mathbf{10}\ ,\mathbf{9})$ are drawn. Thus

$$\text{Probability that "}\textbf{sum of the two number drawn}\text{"} = \frac{2}{12} = \frac{1}{6}$$

> **Question (2):**
> Using the same packs of cards in example 3. If one card is drawn from each pack, what is the probability that "**the sum of the two numbers drawn is 11**"?

1.2 Arithmetic Computation on probability

1.2.1

If the probability of an event to happen is $\boldsymbol{p}$, the probability for that event not to happen is $(\mathbf{1} - \boldsymbol{p})$.

Example 4

The chance for "It will rain today" is $\mathbf{20}\%$, then

The chance for "It will not rain today" $= \mathbf{1} - \mathbf{20}\% = \mathbf{80}\%$

> **Question (3):**
>
> The probability for "**Amy will bring umbrella**" is 0.3.
>
> What is the probability for "**Amy will NOT bring umbrella**"?

1.2.2

A and *B* are independent events (*the probability of A to happen will not be affected by whether B happens or not, vice versa*). If the probability of A happen is p and probability of B happen is *q*,
then the probability of A and B both happen is $p \times q$.

Example 5

In a mathematics test, the probability for Joseph to get A grade is 0.9. In history, the probability for him to get a A grade is 0.7. The probability for him to get A grades for both subjects is

$$\mathrm{P}(\textbf{get A grades for both subjects}) = 0.9 \times 0.7 = 0.63$$

Question (4a): The probability for "**Amy will bring umbrella**" is 0.3. The probability for "**It will rain today** " is 0.8. What is the probability of "**Amy will not bring umbrella and it rains**"?
Question (4b): What is the probability of getting 2 "6" if two dices are thrown?

2 Expected Value

Example 6

In a lucky draw, participants have chances to get coupons of different values. The probability for obtaining different coupons are:

Coupon Value ($)	0	10	50	100
Probability	$\frac{63}{100}$	$\frac{25}{100}$	$\frac{10}{100}$	$\frac{2}{100}$

$0、 $20、 $50、 $100 are all possible events with different probabilities.

We can compute an "expected value" for this lucky draw.

$$\mathrm{E(lucky\ \ draw)}$$

$$= \$0 \times \frac{63}{100} + \$10 \times \frac{25}{100} + \$50 \times \frac{10}{100} + \$100 \times \frac{2}{100} = \$\ 9.5$$

We can interpret this "expected value" as the average value of coupons obtained after many times of lucky draws.

Question (5):

(a) The returns of a gambling game will be lose \$10, win \$10 or win \$100. The probability of each event is shown below.

return (\$)	–10	+1	+100
probability	0.8	0.15	0.05

Calculate the expected returns of this game.

Answer:

expected return

(b) In a game, participant will toss two coins. If both are heads, \$5 will be awarded. Otherwise, he/she loses \$3. Find the expected return of this game.

3 Absolute Value

The **absolute value** of a number is definite to be

$$|x| = \begin{cases} x & if \;\; x \geq 0 \\ -x & if \;\; x < 0 \end{cases}$$

Therefore,

|13| =13

$|-83| = 83$

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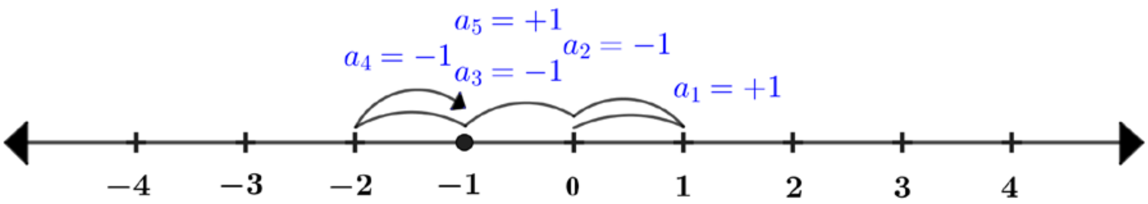
Random walk

Random Walk is a mathematical model that describes a path that consists of a series of random steps on a mathematical space (such as integer). A basic example of a random walk is a one-dimensional random walk.



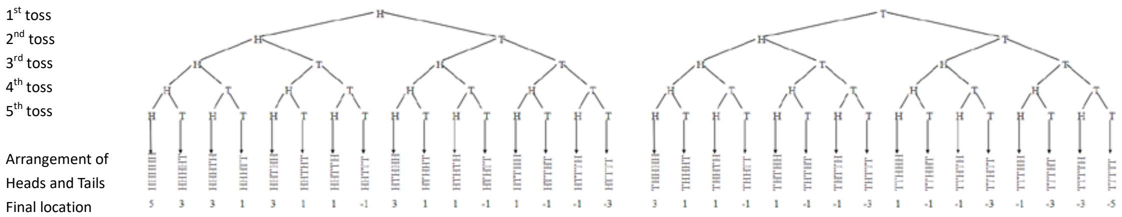
Suppose initially we set the black dot to 0, then let it take N steps (where N is any number), and the distance for each step is a unit. Now we want to know the location of the black dot after N walks. Of course, the location of the black dot after each repetition of N steps may be different. If the "N-step random walk" are repeated many times, we want to know the average final location of the black dot. Let d_N be the location of the black dot after N steps.

$$d_N = a_1 + a_1 + a_3 + \cdots a_N$$

Suppose the initial location of the black dot is set to zero and a fair coin is used for tossing.

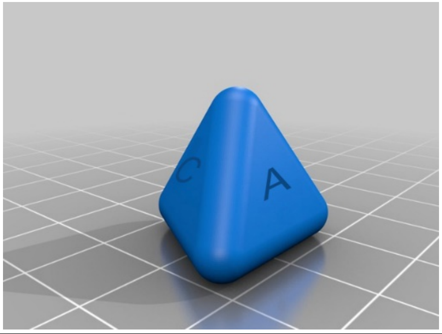
Tail Head

If a head is tossed, the black dot moves one unit to the right ($+1$). If a tail is tossed, the black dot moves one unit to the left (-1). After tossing the fair coin for five times, the black dot can land on $+1, -1, +3, -3, +5$ or -5. For example, if 3 heads and 2 tails are obtained after 5 tosses, the dot will land on $+1$ independent of the order of the heads and tails (There are 10 different ways). There are 10 ways for the dot to land on -1 (by tossing 3 tails and 2 heads), and 5 ways to land on $+3$ (by tossing 4 heads and 1 tail), 5 ways to land on -3 (by tossing 4 tails and 1 head), 1 way to land on $+5$ (by tossing 5 heads), and 1 way to land on -5 (by tossing five tails). See the figure below for the possible results of 5 tosses. (Appendix 1 is an enlarged figure)



**H represents Head, T represents Tails

(a) Suppose there is a biased four-sided dice. The probability that A, B, C, or D come up is as follows:



Alphabet	A	B	C	D
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$

Can this dice be used in the “one-dimensional random walk simulation” mentioned above? Please Explain.

(b) For one-dimensional random walk, complete the probability for different locations of the black dot after 5 steps.

	Final location of dot										
Step	**-5**	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**
0						1					
1					$\frac{1}{2}$	0	$\frac{1}{2}$				
2				$\frac{1}{4}$	0	$\frac{2}{4}$	0	$\frac{1}{4}$			
3			$\frac{1}{8}$	0	$\frac{3}{8}$	0	$\frac{3}{8}$	0	$\frac{1}{8}$		
4		$\frac{1}{16}$	0	$\frac{4}{16}$	0	$\frac{6}{16}$	0	$\frac{4}{16}$	0	$\frac{1}{16}$	
5											

(c) The "Pascal's triangle" is a triangular arrangement of numbers. Observe the following pattern of the Pascal's triangle and complete the line 5.

Line 0 **1**

Line 1 **1** **1**

Line 2 **1** **2** **1**

Line 3 **1** **3** **3** **1**

Line 4 **1** **4** **6** **4** **1**

Line 5 **1** ____________________ **1**

(d) (i) Calculate the expected value $E(d_5)$ of the location of the black dot after walking 5 steps in a one-dimensional random walk.

(ii) Calculate the expected value $\boldsymbol{E}(\boldsymbol{d}_{\boldsymbol{N}})$ of the location of the black dot after *N* steps in one-dimensional random walk. (*N* is a positive integer)

(e) (i) Calculate the expected value of the absolute value of the location of the black dot $|\boldsymbol{d}_5|$ after 5 steps of random walk, $\boldsymbol{E}(|\boldsymbol{d}_5|)$.

(ii) Calculate the expected value of the square of the location of the black dot $\boldsymbol{d}_5{}^2$ after 5 steps of random walk, $\boldsymbol{E}\left(\boldsymbol{d}_5{}^2\right)$. And then calculate the square root of the expected value of the square of the location of the black dot, $\sqrt{\boldsymbol{E}\left(\boldsymbol{d}_5{}^2\right)}$. This is technically called the Root Mean Square Value.

(iii) If we want to calculate the average distance of the black dot from the original location after an *N* steps random walk, which one should be calculated $\boldsymbol{E(d_N)}$, $\boldsymbol{E(|d_N|)}$, $\sqrt{\boldsymbol{E}\left(\boldsymbol{d_N}^2\right)}$? Explain your answer.

(f) Danny said that according to the results of (d)(ii), even *N* is large, after *N* steps of random walk, the distance between the black dot and the origin 0 is still very small. Is Danny's argument reasonable?

(g) Guess or estimate $\sqrt{\boldsymbol{E}\left(\boldsymbol{d_N}^2\right)}$, where *N* is a positive integer.

(h) Consider a two-dimensional random walk, each step with equal probability of going up, down, left, and right. That is:

Moving direction	↑	↓	←	→
probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Each team will be provided with a set of 11 dices.

(i) Design an experiment to simulate the two-dimensional random walk by using the dices.

(ii) Simulate **5 rounds** of a two-dimensional random walk with each round having **20 steps**. Record the results in a way you think is appropriate.

(i) Find the Root Mean Square Value of a two-dimensional random walk using the result of the Root Mean Square Value $\sqrt{\boldsymbol{E}\left(\boldsymbol{d}_{\mathbf{N}}{}^{\mathbf{2}}\right)}$ of the one-dimensional random walk.

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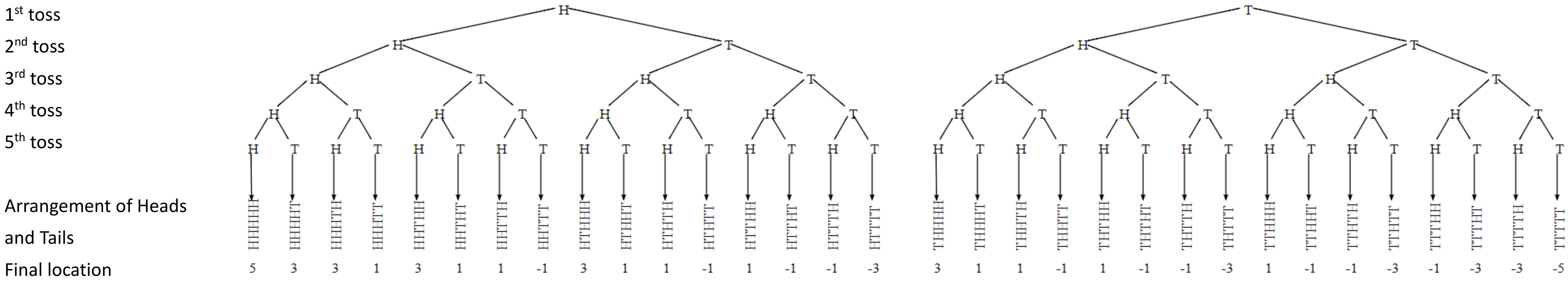
Appendix 1



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