

2018/19

第十屆香港中學數學創意解難比賽

The Tenth Hong Kong Mathematics Creative Problem Solving
Competition for Secondary School

數學解難實驗

Problem Solving Experiment

【均值幾何】

“Quest on Means”

限時 / Time allowed : 50 min

學校編號: School-ID:		匯報序號: Order of Presentation:	
學校名稱: Name of School:			

A. 平均數 / Averages (建議時間 / Time suggested: 5 min)

算術平均數、中位數及眾數均有很多日常用途。

Arithmetic mean, median and mode have many daily uses.

算術平均數常用於成績表上表示同學整學期的學術表現，天文台每月會計算平均溫度及濕度以表示該月份的天氣情況。

Arithmetic mean in the report card is used to indicate the academic performance of the students throughout the semester. The Observatory calculates the mean temperature and the mean humidity monthly to present the weather of that month.

中位數多見於經濟上的統計，例如大學畢業生的入息中位數，樓價中位數等。

Median is more common in economic statistics, such as the median income of college graduates and the median property price.

眾數在日常生活中較少見，有時會於問卷統計結果中顯示：「在數學創意解難比賽的問卷調查中，滿意是項比賽的評分平均分是 4.8(1–6 分評級，6 分為最滿意)，眾數是 5...」。

Mode is rarely seen in daily life, however, it can be used in the statistical results of the questionnaire: "In the questionnaire for the mathematics creative problem-solving competition, the average satisfaction score of the competition is 4.8 (1 - 6 points, 6 points is the most satisfactory), the mode is 5 ...".

1. 「有研究發現，大學畢業生去年的起薪點中位數只有一萬四千三百多元...」

試指出在統計分析薪酬時使用中位數，而非平均數的其中一個好處。 (1 分)

"A study found that the median starting salary of university graduates last year was only \$14,300..."

Please state one of the benefits of using the median, rather than the average, in statistical analysis of salary. (1 mark)

2. 在學生成績表中，很少學校會以「中位數」去取代「算術平均分」來表示同學的綜合學術表現。試舉出使用「算術平均分」的一個好處。 (2 分)

In student reports, schools seldom replace the "arithmetic mean score" with the "median" to indicate the overall academic performance of the student. Please state one of the benefits of using "arithmetic mean score". (2 marks)

B. 「新」的平均數 / “New” average (建議時間 / Time suggested: 10 min)

現時香港的就業人士大多都需要供強積金，而強積金的供款會透過基金的形式進行投資。

Nowadays, most of the employed people in Hong Kong are required to make regular contributions to an Mandatory Provident Fund (MPF) scheme. The MPF contributions are used for investment in funds.

基金的年度回報率($r\%$)是指一年內投資的盈利/虧蝕百分比，當中 $-100 \leq r \leq 100$ 。

Annual return rate ($r\%$) is a profit/loss percentage on an investment over a year, expressed as a proportion of the original investment, where $-100 \leq r \leq 100$.

正值的 r 代表盈利，負值則代表虧蝕。

The positive value of r representing a profit while the negative value representing a loss. i.e.

即/i.e. :

$$\text{年頭基金價值} \times (1 + r\%) = \text{年尾的基金價值}$$

$$\text{Fund price in the beginning of the year} \times (1 + r\%) = \text{Fund price at the end of the year}$$

以下為某基金的例子：

The following shows an example of a particular fund:

	年度回報率 Annual return rate	基金價值 Fund price
1 st Jan 2017	+5%	\$10000
31 st Dec 2017		$\$10000 \times (1 + 5\%) = \10500
1 st Jan 2018	-2%	\$10500
31 st Dec 2018		$\$10500 \times (1 - 2\%) = \10290

若某人在 2017 年 1 月 1 日投資 \$10000 於上述基金。他在 2018 年 12 月 31 日提走基金時可獲得 \$10290。

Someone invested \$10,000 in the fund above on 1st Jan 2017. When he withdrew the fund on 31st Dec 2018, he could get \$10,290.

曾先生現正考慮投資「C 基金」，他發現「C 基金」過去 4 年回報率的算術平均數是 +5%，計算方法是將 4 年的回報率相加後除以 4。

Mr. Tsang is considering an investment in the "C Fund". He found that the arithmetic mean of the annual return rates over the past four years is +5% by dividing the sum of four annual return rates by 4.

他的朋友鍾先生在 4 年前於「C 基金」投資了 \$10000，但最後取回的金額竟然少於 \$10000。已知「C 基金」在該 4 年內沒有出現破產（回報率為 -100%）的情況。

His friend, Mr. Chong, invested \$10,000 in the "C Fund" four years ago, but the amount he got was less than \$10,000 at the end of the fourth year. It is known that the "C Fund" did not go bankrupt (i.e. return rate is -100%) in that period of time.

1. 按照鍾先生的情況，請列出其中一組「C基金」過去4年可能的年度回報率，並驗證最後取回的金額少於\$10000。 (2分)

Base on Mr. Chong's situation, write down a list of possible annual return rates of the "C Fund" over the past 4 years and verify that the final amount was less than \$10,000. (2 marks)

2. 鍾先生氣憤地指出，將每年的回報率直接取算術平均數不是一個合理的方法！他認為較合理的計算是將每年的回報率加上1之後，再計算算術平均數。你贊同他嗎？試解釋你的答案。 (2分)

Mr. Chung criticised that taking arithmetic mean of the annual return rates is not reasonable. He claimed that adding 1 to each annual return rate before calculating the arithmetic mean is more reasonable. Do you agree with him? Explain your answer. (2 marks)

投資專家陳先生認為使用算術平均數去計算投資回報是不合理的。他提出一種新的平均數 (G) 來合理地了解基金的表現。計算方法如下:

An investment expert, Mr. Chan, suggested that using arithmetic mean to calculate the return on investment is also not reasonable. He proposed a new average (G) to get a proper understanding of the fund's performance. The calculation is as follows:

假設另一基金「S 基金」過去 4 年，年度回報率分別為+5%、+6%、+7%和+8%，新的平均數 (G) 的計算如下:

Let the annual return rates of another fund "S Fund" in the past 4 year be +5%, +6%, +7% and +8% respectively. The new average (G) is calculated as follows:

$$\sqrt[4]{(1 + 5\%) \times (1 + 6\%) \times (1 + 7\%) \times (1 + 8\%)} - 100\% \approx + 6.49\%$$

3. 曾先生以新的計算方法去重新了解「C 基金」過去 4 年的表現，得出新的平均年度回報率 (G) 為-3.96%，若曾先生 4 年前在「C 基金」投資了\$10000，現能取回多少？（答案準確至二位小數） (2 分)

Mr. Tsang used this new method to analyse the performance of the "C Fund" over past 4 years. The new average annual return rate (G) was -3.96%. If he invested \$10,000 in the "C Fund" four years ago, how much can he get now? (*Correct your answer to 2 decimal places*) (2 marks)

4. 假設第一年和第二年的回報率分別為 $x\%$ 及 $y\%$ ，其中 $-100 \leq x, y \leq 100$ 。

Let the annual return rates of the 1st and 2nd year be $x\%$ and $y\%$ respectively, where $-100 \leq x, y \leq 100$.

試證明/ Prove that

$$\sqrt{(1+x\%)(1+y\%)} - 1 \leq \frac{x\%+y\%}{2} \quad (2 \text{ 分})$$

(2 marks)

5. 曾先生以新的計算方法去重新了解市場上不同的基金表現，發現新計算方法得出的平均數 (G) 都比算術平均數為小。試證明新的平均數 (G) 必定小於或等於算術平均數。 (3分)
- Mr. Tsang used this new method to analyse the performance of different funds in the market. He found that the new average (G) is always smaller than the arithmetic mean. Prove that the new average (G) must be less than or equal to the arithmetic mean. (3 marks)

(設 $x_1\%$ 、 $x_2\%$ 、 $x_3\%$ 和 $x_4\%$ 分別為過去 4 年的回報率，其中 $-100 \leq x_1, x_2, x_3, x_4 \leq 100$ 。)

(Let the annual return rates in the past 4 years be $x_1\%$, $x_2\%$, $x_3\%$ and $x_4\%$ respectively, where $-100 \leq x_1, x_2, x_3, x_4 \leq 100$)

6. 試解釋為何有些基金會以算術平均數來總結過去回報? (1分)
Explain why do some funds use arithmetic mean to summarise past return? (1 mark)

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C. 離散程度 / Dispersion (建議時間 / Time suggested: 10 min)

比較以下兩班同學的身高(以 cm 為單位):

Compare the heights of students of the following two classes (in cm):

A 班 / Class A : 120, 120, 140, 140, 140, 160, 160

B 班 / Class B : 100, 120, 140, 140, 140, 160, 180

1. 完成下表: (1分)
Complete the following table: (1 mark)

	A 班 / Class A	B 班 / Class B
算術平均數 / Arithmetic mean		
中位數 / Median		
眾數 / Mode		

如果只比較集中趨勢，則會得出錯誤結論，誤以為兩班同學的身高完全一樣。因此除了集中趨勢外，我們亦會討論兩組數據的離散程度，簡而言之，是數據較為集中還是分散的量度。

If we consider the central tendency only, it may draw a wrong conclusion that the heights of the two classes are exactly the same. Therefore, apart from the central tendency, we also consider the dispersion of the two sets of data. In short, it is a measure of whether the data is concentrated or scattered.

常用的離散程度的量度方法有幾種，先介紹最簡單的「全距」。

There are several common methods for measuring the dispersion. The simplest among them is “range”.

「全距」就是組內數據中最大減去最小得出的數值。

“Range” is the difference between minimum value and maximum value in the data set.

A 班的全距 / Range of Class A = $(160 - 120)$ cm = 40 cm

B 班的全距 / Range of Class B = $(180 - 100)$ cm = 80 cm

一般而言，全距越大，數據的離散程度越高，所以我們可以說 A 班身高較 B 班集中。In general, the greater the range, the higher the dispersion. Therefore, we can say that data of class A is more concentrated than that of class B.

2. 試列舉使用全距的好處及壞處。

(2分)

List the advantages and disadvantages of range.

(2 marks)

「標準差」更常用來量度數據的離散程度。其概念為先計算出該堆數據的算術平均數，然後計算每一個數據和此算術平均數的差，再取其平方之算術平均數，將剛才的結果開方，就能得出標準差。例子如下：

"Standard deviation" is commonly used for measuring the dispersion. To calculate the standard deviation, first calculate the arithmetic mean of the data, then calculate the difference between each data and the arithmetic mean, square the difference, and then take the arithmetic mean. Finally, the square root of this result is the **Standard deviation**. For example:

2, 3, 5, 7, 9

算術平均數/ Arithmetic mean = 5.2

$$\text{標準差/ Standard deviation} = \sqrt{\frac{(2-5.2)^2 + (3-5.2)^2 + (5-5.2)^2 + (7-5.2)^2 + (9-5.2)^2}{5}} = \sqrt{6.56} \approx 2.56$$

一個較大的標準差，代表大部分的數值和其算術平均數之間差異較大；

A large standard deviation represents a large difference between most of the data and their arithmetic mean;

一個較小的標準差，代表這些數值較接近算術平均數。

a smaller standard deviation means that data are closer to their arithmetic mean.

例如{4,4,5,5,8}的算術平均數也是 5.2，但標準差就只有 $\sqrt{2.16} \approx 1.47$ 。

For example, the arithmetic mean of {4, 4, 5, 5, 8} is also 5.2, but the standard deviation is only $\sqrt{2.16} \approx 1.47$.

3. 試分別計算 A 班及 B 班同學身高的標準差。哪一班的身高較為集中？（答案準確至二位小數）
(2 分)

Calculate the standard deviations of the heights of the students in class A and class B respectively.
Which class is more concentrated? (*Correct your answer to 2 decimal places*) (2 marks)

兩班同學的身高(以 cm 為單位):

The heights of students of the following two classes (in cm):

A 班/Class A : 120, 120, 140, 140, 140, 160, 160

B 班/Class B : 100, 120, 140, 140, 140, 160, 180

同學可能會覺得計算標準差的方法很繁複。有見及此，數學家找出另一種更容易計算標準差的方法，簡述如下:

Students may think that the calculation of standard deviation is very complicated. In view of this, mathematicians found an easier way to calculate the standard deviation. The statement is as follows:

「平方和的平均」減去「平均的平方」，再取平方根
(Mean of the squares minus the square of means, and then take square root)

即/i.e.

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} - \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2}$$

當中 x_i 為第 i 個數據， n 是數據的總數目， $1 \leq i \leq n$ ， i 和 n 都是正整數。

where x_i is the i^{th} datum, n is the number of data, $1 \leq i \leq n$, i and n are positive integers.

4. 試以上述公式，分別計算 A 班及 B 班同學身高的標準差。（答案準確至二位小數）（2 分）

Using the formula above, calculate the standard deviations of the heights of class A and class B respectively. (Correct your answer to 2 decimal places) (2 marks)

D. 應用 / Applications (建議時間 / Time suggested: 25 min)

同學可以參考 B 部和 C 部的內容，或其他數學知識去解答以下問題。

Students can refer to part B and C, or other mathematical knowledge, to answer the following questions.

某學校會為每一位入學的學生按順序安排一個學生編號。第 1 個學生編號為 0001，第 2 個為 0002。如果該校第一年創校時有 500 位學生，他們獲派的學生證編號就是由 0001 至 0500。

In a school, student ID numbers will be assigned to all students in order. The first student ID is 0001 and the second one is 0002. If there are 500 students in the first school year, their student ID numbers will range between 0001 and 0500.

假設現在是創校的第一年，學校的收生情況沒有對外發放，某報章記者希望估算學校收生人數以作報導。

In the first school year, the admission data was not disclosed to the public. A newspaper reporter wants to estimate the number of students in the first school year for news reporting.

1. 明查暗訪後，記者得知該校所有學生證編號的標準差為 $\sqrt{4760}$ 。問該校第一年共有多少名學生？ (3分)

After investigation, the reporter found that the standard deviation of all student ID numbers in that school is $\sqrt{4760}$. How many students are there in the first school year? (3 marks)

2. 已知標準差為 \sqrt{x} ，試寫出 x 與學生人數 n 之間的關係。 (1分)

Given that the standard deviation is \sqrt{x} , write down the relation between x and the number of students n . (1 mark)

3. 假設記者未能查得標準差、平均數等統計資料，他決定以抽樣調查的方式去估算該校的總人數。由於時間、人力所限，他只能訪問 8 位同學並詢問了他們的學生證編號。

Assuming that the reporter failed to find statistical data such as standard deviation and average, he decided to use a sample survey to estimate the total number of students in the school. Due to the limitation of time and manpower, he can only access 8 students and ask for their student ID numbers.

舉手向監考員示意，監考員會協助同學在紙箱內隨機抽出 8 張號碼紙來代表詢問 8 位學生所得的學生證編號。

Raise your hand to alert an invigilator, the invigilator will assist your team to randomly draw 8 number cards, which represent the student ID numbers of 8 students, from the box.

設 x_i 為第 i 位學生的學生證編號， $1 \leq i \leq 8$ ， i 是正整數。

Let x_i be the student ID number of the i^{th} student, where $1 \leq i \leq 8$, i is an integer.

<u>監考員紀錄用 / For invigilator use only</u>							
							盒 / Box: _____
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8

請嘗試建立不同的方法去估算該校的總人數，並說明原理及分析利弊。 (4 分)

Establish different methods to estimate the total number of students. Explain the principles and analyse the pros and cons of the methods. (4 marks)

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4. 湖中有 N 隻龜，研究人員在一個月前利用編號 1 至 n 把其中 n 隻龜的殼標上編號。梁老師帶領「生物探究小組」進行湖邊考察。組員當天沿湖畔前行，把在湖岸觀察到的龜的數目 (S) 記下，當中有 k 隻龜的背上有編號標記。

There are N turtles in a lake. One month ago, a researcher labeled n of them by marking a number, from 1 to n , on their shells. Last week, Mr. Leung and his “Biology study group” conducted a site visit. They walked along the lakeside and recorded the number of turtles they observed (S). They noticed that k of the observed turtles have labels on their back.

- (a) 試提出一個 n 、 N 、 k 及 S 的合理關係，並加以解釋。 (2 分)

Write down a reasonable relation between n , N , k and S . Explain your answer. (2 marks)

- (b) 已知當天觀察到的龜的數目 (S) 為 32，其中部分的龜背有編號，紀錄如下：

Given that the number of turtles observed (S) is 32, some of them have label on their shells. The labels are shown as follows:

139	78	98	19	96
116	52	54	129	

- 估算有編號的龜的數量 (n) 及湖中龜的總數 (N)。 (4 分)

Estimate the total number of turtles (n) having labels on their shells and the total number of turtles (N) in the lake. (4 marks)

全卷完
End of Paper