2020/21 第十一屆香港中學數學創意解難比賽

2020/21 The 11th Hong Kong Mathematics Creative Problem Solving Competition for Secondary Schools

題解 Solutions

甲部 Section A

1.

將以下各數由小至大排列。

Rearrange the following numbers in ascending order.

$$2^{1190}, 3^{850}, 4^{680}, 5^{510}$$

建議答案 Suggested solutions:

$125 < 128 < 243 < 256$

$\therefore\ 5^3 < 2^7 < 3^5 < 4^4$

$\therefore\ 5^{510} < 2^{1190} < 3^{850} < 4^{680}$

2.

若 x_1，x_2，x_3，x_4，x_5 是等差數列，即 $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = x_5 - x_4$ 。

圖一顯示了一塊 5 × 5 正方形板。每行和列的數字都組成一條等差數列。已知板上的一些數字，求 x。

If x_1 , x_2 , x_3 , x_4 and x_5 is an arithmetic sequence, then

$x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = x_5 - x_4$.

Figure 1 shows a 5 × 5 square board. The numbers in each row and column form an arithmetic sequence. Some of the numbers were given. Find the value of x.

				0
	4			
		10		
x			14	

圖一 / Figure 1

建議答案 Suggested solutions:

設第 i 行 和第 j 列 的公差分別為 m_i 及 n_j，當中 $1 \le i \le 5$ and $1 \le j \le 5$。

Let the common differences for the i th rows and j th columns be m_i and n_j respectively, where $1 \le i \le 5$ and $1 \le j \le 5$.

Common diff. / 公差	n_1	n_2	n_3	n_4	n_5
m_1		C	D		0
m_2		4			
m_3			10		
m_4					
m_5	x	A	B	14	

$A = 4 + 3n_2$, $B = 10 + 2n_3$;

A, B, 14 組成一等差數列。

A, B, 14 form an arithmetic sequence.

$$\therefore \quad B - A = 14 - B$$

$$10 + 2n_3 - (4 + 3n_2) = 14 - (10 + 2n_3)$$

$$-3n_2 + 4n_3 = -2 \qquad (1)$$

同理 / Similarly

$C = 4 - n_2$, $D = 10 - 2n_3$

$$\frac{0 - C}{0 - D} = \frac{3m_1}{2m_1}$$

$$\frac{0 - (4 - n_2)}{0 - (10 - 2n_3)} = \frac{3}{2}$$

$$2n_2 - 6n_3 = -22 \qquad (2)$$

(1) and (2)

=> $n_2 = 10$ and $n_3 = 7$

$A = 34$ and $B = 24$

$\therefore \quad x = A - (14 - B) = 44$

Common diff. / 公差	13	10	7	4	1
2	−8	−6	−4	−2	0
−1	5	4	3	2	1
−4	18	14	10	6	2
−7	31	24	17	10	3
−10	44	34	24	14	4

3.

寫出一個利用四個「**3**」組成的數值最大的數。

Write down the largest number formed by using four "**3**"s.

建議答案 Suggested solutions:

四個 3 能夠組成的數值/Possible value formed by four 3s

$3^{3^{33}}$, $3^{3^{3^3}}$, 3^{33^3} 33^{33} , 33^{3^3} , 333^3 , 3333…

$$3^{3^{33}} > 3^{3^{3^3}} = 3^{3^{27}}$$

$\because$ $3^{33} = 27^{11} > 33^3$

$\therefore$ $3^{3^{33}} > 3^{33^3}$

明顯地 Obviously,

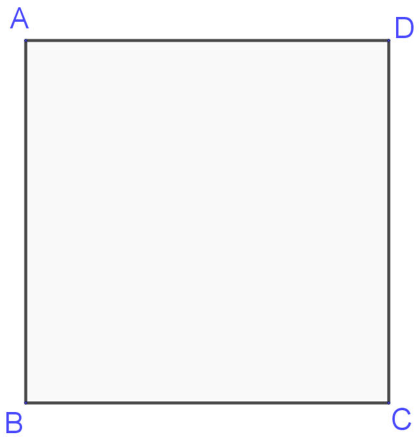
$33^{33} >$ 33^{3^3} $> 333^3 > 3333$

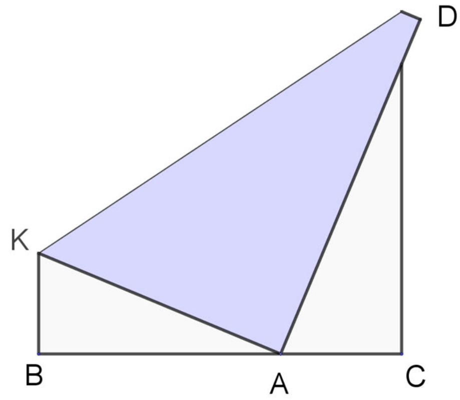
$$\begin{aligned}
3^{3^{33}} &= 3^{3^{(3\times11)}} = 3^{27^{11}} \\
&= 3^{27} \times 3^{27} \ldots \times 3^{27} \quad (11 \ \text{times}) \\
&= (3^{27} \times 3^{27}) \times (3^{27} \times 3^{27} \times 3^{27} \times 3^{27}) \times \ldots \\
&= 3^{54} \times (27^9 \times 27^9 \times 27^9 \times 27^9) \times \ldots \\
&= 3^{54} \times 27^{36} \times \ldots \\
&> 3^{33} \times 11^{33} = 33^{33}
\end{aligned}$$

$\therefore$ $3^{3^{33}}$ is the largest number.

$\therefore$ 數值最大的數是$3^{3^{33}}$。

4.





圖二 / Figure 2 **圖三 / Figure 3**

圖二的正方形紙ABCD按以下方式摺一次。(見圖三)

(I) A 點在 BC 上；

(II) BA：AC = 2：1。

在圖三中，求 KB：BA：AK。

A piece of square paper ABCD in Figure 2 is folded one time (see Figure 3) so that

(I) Point A lies on BC;

(II) BA：AC = 2：1。

In Figure 3, find the ratio of KB: BA: AK.

建議答案 Suggested solutions:

Let / 設

$AB = 1, KB = x$.

$KA = 1 - x$ ， $BA = \frac{2}{3}$.

By Pythagoras Theorem,

按勾股定理，

$$x^2 + \left(\frac{2}{3}\right)^2 = (1 - x)^2$$

$$x^2 + \frac{4}{9} = 1 + x^2 - 2x$$

$$2x = \frac{5}{9}$$

$$x = \frac{5}{18}$$

$$KB: BA: AK = \frac{5}{18} : \frac{2}{3} : 1 - \frac{5}{18}$$

$$KB: BA: AK = 5: 12: 13 \quad *$$

* 答案未能約至最簡但符合 5 : 12 :13 ，得一分。
Answers not in simplest form but satisfy 5 : 12 : 13 will give 1 mark

5.
已知 x，y 及 z 為非零實數，求 x。
Given that x, y and z are non-zero real numbers. Find the value of x.

$$\begin{cases} xy = 3(x + y) \\ yz = 6(y + z) \\ xz = 9(x + z) \end{cases}$$

建議答案 Suggested solutions:

$$\begin{cases} xy = 3(x + y) \\ yz = 6(y + z) \\ xz = 9(x + z) \end{cases}$$

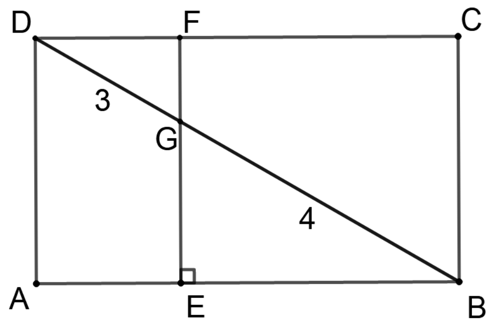
$$\begin{cases} \frac{1}{3} = \frac{1}{x} + \frac{1}{y} \\ \frac{1}{6} = \frac{1}{y} + \frac{1}{z} \\ \frac{1}{9} = \frac{1}{z} + \frac{1}{x} \end{cases}$$

$\therefore \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{\frac{1}{3}+\frac{1}{6}+\frac{1}{9}}{2} = \frac{11}{36}$

$\therefore \quad \frac{1}{x} = \frac{11}{36} - \frac{1}{6} = \frac{5}{36}$

$\therefore \quad x = \frac{36}{5}$

6.



圖四 / Figure 4

圖四顯示一個長方形 ABCD。 E 和 F 分別是 AB 和 DC 上的點，使得 FE 垂直 AB。 對角線 DB 與 FE 相交於 G。

已知 DG = 3 及 GB = 4 ，求 FG × GE + AE × EB。

Figure 4 shows a rectangle ABCD. E and F are points on AB and DC respectively such that FE is perpendicular to AB. The diagonal DB cuts FE at G.

Given that DG =3 and GB = 4, find FG × GE + AE × EB.

建議答案 Suggested solutions:

$\because$ ΔDFG ~ ΔBEG (AAA)

$\therefore \quad \frac{\mathrm{DG}}{\mathrm{GB}} = \frac{\mathrm{AE}}{\mathrm{EB}} = \frac{\mathrm{FG}}{\mathrm{GE}} = \frac{3}{4}$

$\therefore \quad \mathrm{GE} = \frac{4}{3}\mathrm{FG} \quad$ and $\mathrm{EB} = \frac{4}{3}\mathrm{AE}$

$\mathrm{FG} \times \mathrm{GE} + \mathrm{AE} \times \mathrm{EB}$

$= \frac{4}{3}(\mathrm{FG})^2 + \frac{4}{3}(\mathrm{AE})^2$

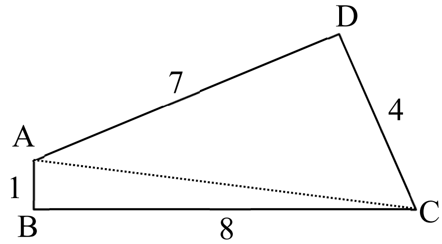
$= \frac{4}{3}(\mathrm{DG})^2$

$= 12$

7.
四邊形 ABCD 的四條邊為 AB=1, BC=8, CD=4 及 DA=7。求該四邊形的最大面積。

The 4 sides of quadrilateral ABCD are AB=1, BC=8, CD=4 and DA=7. Find the maximum area of the quadrilateral.

建議答案 Suggested solutions:



沿 AC 切開四邊形 ABCD。
Cut the quadrilateral along AC.

The area of △ABC is the maximum when AB⊥BC

Maximum area of △ABC is $\frac{1\times 8}{2}$ and $AC=\sqrt{1^2+8^2}=\sqrt{65}$

The area of △CDA is the maximum when CD⊥DA

Maximum area of △CDA is $\frac{4\times 7}{2}$ and $AC=\sqrt{4^2+7^2}=\sqrt{65}$

Hence the area of ABCD is maximum when AB⊥BC and CD⊥DA

The maximum area is $\frac{1\times 8}{2}+\frac{4\times 7}{2}=18$

8.

有 36 名學生參加田徑選拔賽，選拔賽中首 4 名學生將代表學校出賽。現時運動場有 6 條跑道，即每次只能讓 6 名學生同時較量，每場的先後排名會被紀錄。在沒有計時器的幫助下，最少要進行多少場跑步比賽才能選出 4 名代表?

There are 36 students in a preliminary selection of running race. The top 4 winners in the preliminary selection will be the school representatives. The running track has 6 lanes. It only allows 6 students run at the same time. What is the minimum number of races required to find the 4 representatives without using a timer?

建議答案 Suggested solutions:

平均分為 6 組, A, B, C, D, E and F。每組各自比賽。得出以下次序：

Evenly divide the students into 6 groups, A, B, C, D, E and F. Hold a race for each group. We have the following order:

$A_1 > A_2 > A_3 > A_4 > A_5 > A_6$ (1)

$B_1 > B_2 > B_3 > B_4 > B_5 > B_6$ (2)

…

$F_1 > F_2 > F_3 > F_4 > F_5 > F_6$ (6)

A_i , B_i … F_i 為學生 i 為 1 至 6 的整數；「 > 」代表 「比… 快」

where A_i , B_i … F_i are the students for all i from 1 to 6 ; “ > ” means “faster than”

每組首名再作第七場比賽

Hold the seventh race for each champions in the group

WLOG

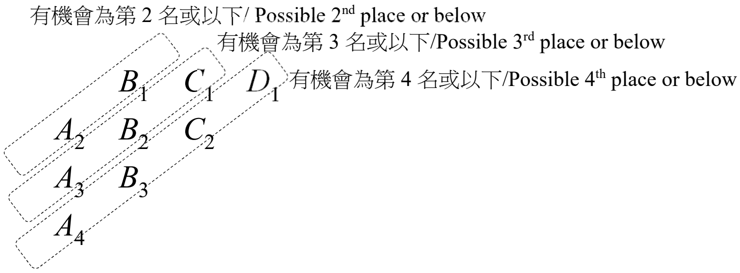
$A_1 > B_1 > C_1 > \ldots > F_1$ (7)

明顯 A_1 是在 36 人中的第一名

Obviously, A_1 is the champion among 36 of them.

下列的學生均有機會為首 4 名

The following students still have a chance to be the top 4.



第八場比賽 / The eighth race

$\boxed{B_1 , A_2}$, $\boxed{C_1 , B_2 , A_3}$, $\boxed{D_1}$

情況一： B_1 是第八場的首名 / Case I: B_1 is the 1st in the eighth race.

$B_1 > C_1 > X > X > X > X$ => 第九場 / ninth race ($A_2 , B_2 , C_2 , D_1 , X , X$)

$B_1 > B_2 > X > X > X > X$ => 第九場 / ninth race ($A_2 , B_3 , C_1 , X , X , X$)

$B_1 > A_2 > X > X > X > X$ => 第九場 / ninth race ($A_3 , B_2 , C_1 , X , X , X$)

情況二： A_2 是第八場的首名 / Case II: A_2 is the 1st in the eighth race.

$A_2 > B_1 > X > X > X > X$ => 第九場 / ninth race ($A_3 , B_2 , C_1 , X , X , X$)

$A_2 > A_3 > X > X > X > X$ => 第九場 / ninth race ($A_4 , B_1 , X , X , X , X$)

∴ 共需至少九場比賽。

At least 9 races are needed.

9.

$16^{2019\cdot2020} = a^b$

a 和 b 是正整數，求 a 的可能值的數目。

a and b are positive integers. Find the number of possible values of a.

建議答案 Suggested solutions:

$$16^{2019\cdot2020} = 2^{4\times2019\times2020} = 2^{2^4\times3\times5\times101\times673}$$

No. of factors of /因子數目

$2^4 \times 3 \times 5 \times 101 \times 673$ is $(4 + 1)(1 + 1)^4 = 80$.

$\therefore$ number of possible value of a = No. of factors = 80

a 的可能值的數目 =因子數目= 80

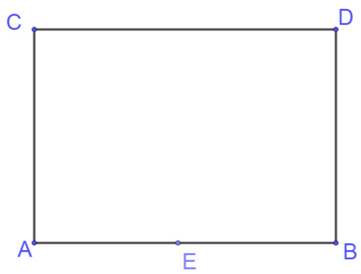
10.

圖五顯示了一張長方形紙 ABDC ，其中 $AB: AC = \sqrt{2}: 1$。E 是 AB 的中點。沿着 CE 摺，再沿着∠AEB 的角平分線對摺（圖六）。角平分線與 DB 上的 F 點相交。將紙攤開，連接 CF（圖七）。

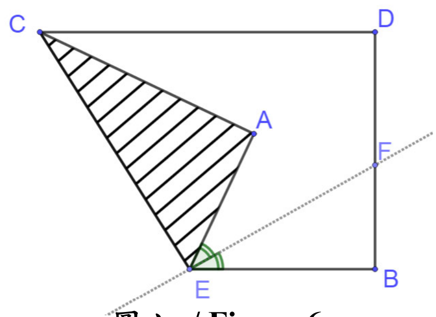
若長方形的面積是 200 cm^2，求ΔCFE 的面積。

Figure 5 shows a piece of rectangle paper ABDC with $AB: AC = \sqrt{2}: 1$. E is the mid-point of AB. Fold along the line CE and then fold along the angle bisector of ∠AEB (Figure 6). The angle bisector intersects DB at F. Unfold the paper and join CF (Figure 7).

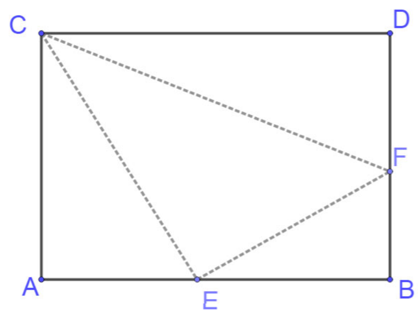
If the area of the rectangle is 200 cm^2 , find the area of ΔCFE .



圖五 / **Figure 5**

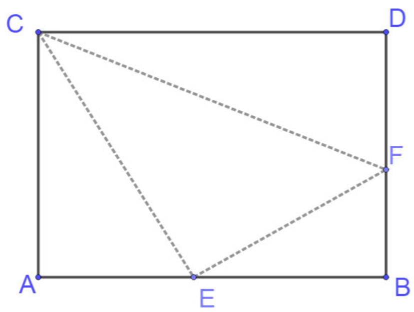


圖六 / **Figure 6**



圖七 / **Figure 7**

建議答案 Suggested solutions:



設 / Let

$\angle CEA = a$, $\angle FEB = b$

$2a + 2b = 180°$ (adj. ∠s on st. line)

$a + b = 90°$

得出/we have,

$\Delta CAE \sim \Delta EBF$ (AAA)

$$\frac{CA}{AE} = \frac{EB}{FB}$$

$$\frac{1}{\left(\frac{\sqrt{2}}{2}\right)} = \frac{\left(\frac{\sqrt{2}}{2}\right)}{FB}$$

$$FB = \frac{1}{2}$$

F is the mid-point of DB.

∴ Area of ΔCAE : Area of ΔEBF = AC: BF = 2 : 1

Area of ΔCFE = Area of ΔCAE + Area of ΔEBF

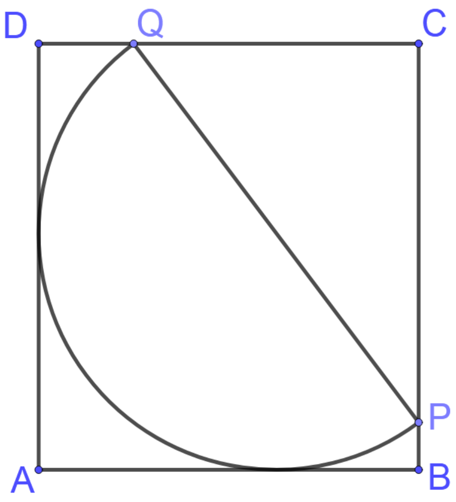
∴ Area of ΔCFE : Area of ΔCAE : Area of ΔEBF = 3 : 2 : 1

Area of ΔCDF : Area of ΔEBF = CD: EB = 2 : 1

Area of ΔCDF : Area of ΔCFE : Area of ΔCAE : Area of ΔEBF = 2 : 3 : 2 : 1

Area of ΔCFE = $\frac{3}{2+3+2+1} \times 200 = 75 \text{ cm}^2$

11.



圖八 / Figure 8

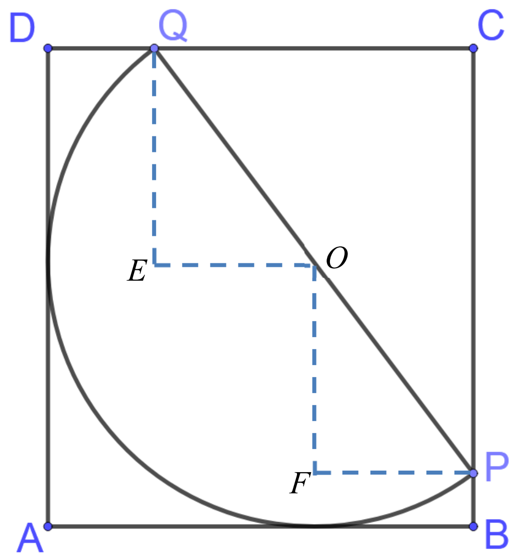
ABCD 是一個長方形。P 和 Q 分別是 BC 和 CD 上的一點。有一直徑為 PQ 的半圓與 AB 和 AD 相切(圖八)。已知 PB = 1，QD = 2 及 PQ = 10，求長方形 ABCD 的面積。

ABCD is a rectangle. P and Q are points on BC and CD respectively. A semi-circle with diameter PQ touches the side AB and AD (Figure 8). Given that PB = 1, QD = 2 and PQ = 10, find the area of rectangle ABCD.

建議答案 Suggested solutions:

Let the radius of the semi-circle be r.

設半圓的半徑為 r



$$QE = \sqrt{5^2 - (5-2)^2} = 4$$

∵ $\Delta QOE \cong \Delta OPF$ (ASA)

∴ FP = EO = 3

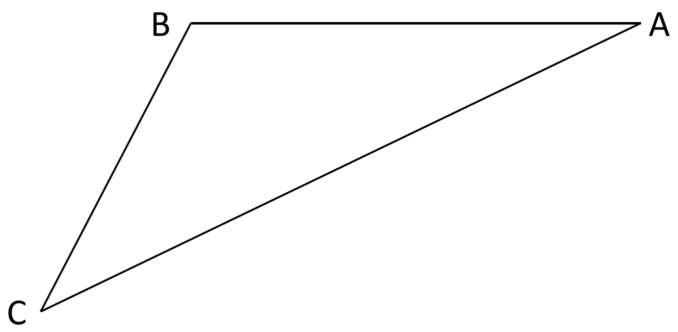
∴ 長方形 ABCD 的面積 = (4 + 5)(5 + 3) = 72

∴ Area of rectangle ABCD = (4 + 5)(5 + 3) = 72

12.

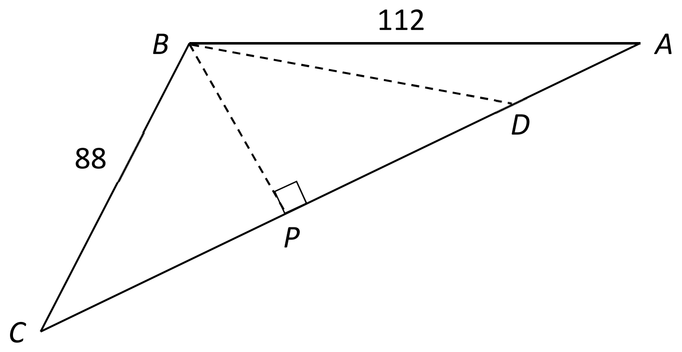
圖九顯示了三角形 ABC。AB = 112、BC = 88 及 AC = 160。D 為 AC 上的一點使得 BC = BD。求 $\frac{AD}{CD}$。

Figure 9 shows a triangle ABC. AB = 112, BC = 88 and AC = 160. D is a point on AC such that BC = BD. Find $\frac{AD}{CD}$.



圖九 / Figure 9

建議答案 Suggested solutions:



Let P be the perpendicular foot from B to AC. Then BP = h is the perpendicular distance from B to AC. Note that CP = 160 – AP. By Pythagoras' theorem,

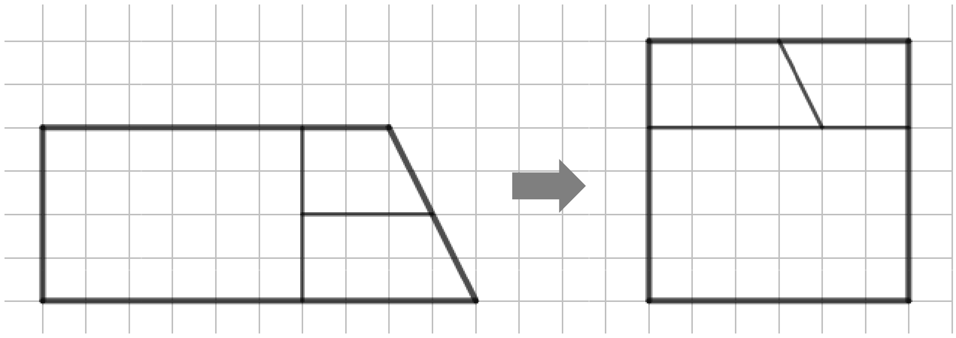
設 P 為由 B 至 AC 的垂足，則 BP = h 為 B 至 AC 的垂直距離。注意 $CP = 160 - AP$。運用畢氏定理，

$$\begin{cases} h^2 + AP^2 = 112^2 \\ h^2 + (160 - AP)^2 = 88^2 \end{cases}$$

$$\begin{cases} h^2 = 112^2 - AP^2 \\ h^2 = 88^2 - (160 - AP)^2 \end{cases}$$

$$\therefore \quad 112^2 - AP^2 = 88^2 - (160 - AP)^2$$
$$112^2 - AP^2 = 88^2 - (160^2 - 320AP + AP^2)$$
$$112^2 - AP^2 = 88^2 - 160^2 + 320AP - AP^2$$
$$320AP = 112^2 - 88^2 + 160^2$$
$$320AP = (112 + 88)(112 - 88) + 160^2$$
$$320AP = (200)(24) + (320)(80)$$
$$AP = 15 + 80 = 95$$
$$\therefore \quad CP = 160 - 95 = 65$$
$$\therefore \quad DP = CP = 65 \quad \text{(prop. of isos. } \Delta\text{) (等腰}\Delta\text{特性)}$$
$$\therefore \quad AD = AP - DP = 95 - 65 = 30$$
$$\therefore \quad \frac{AD}{CD} = \frac{30}{65 \times 2} = \frac{3}{13}$$

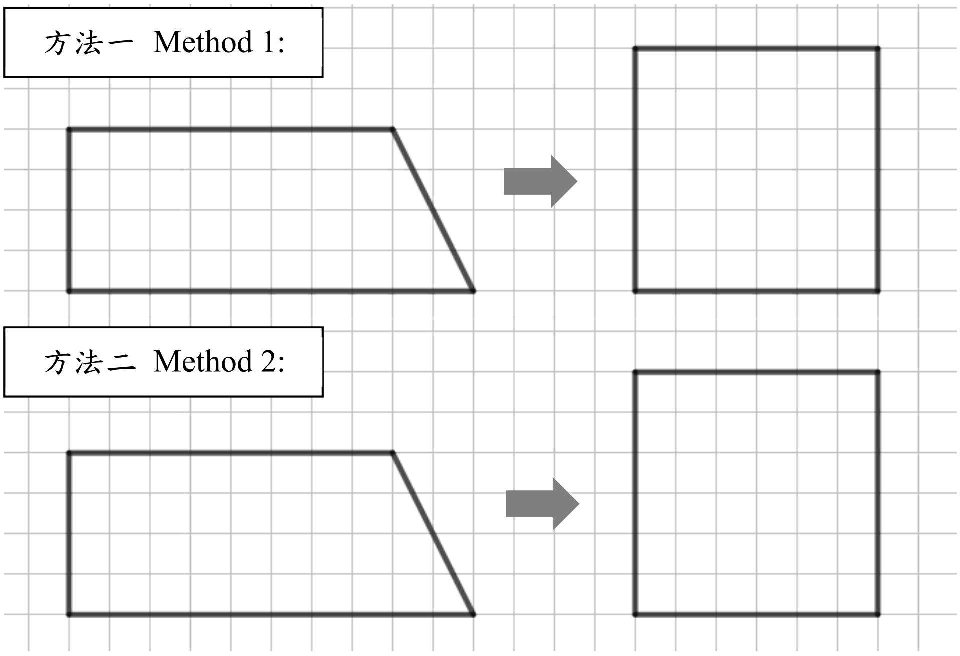
13.
我們可以把梯形加上直線，把它分割成數份，並重新組合成一個正方形。
圖十為一個分割的例子。
We can draw straight lines on the trapezium to cut it into several pieces, and rearrange the pieces to form a square. Figure 10 is an example.



圖十 / Figure 10

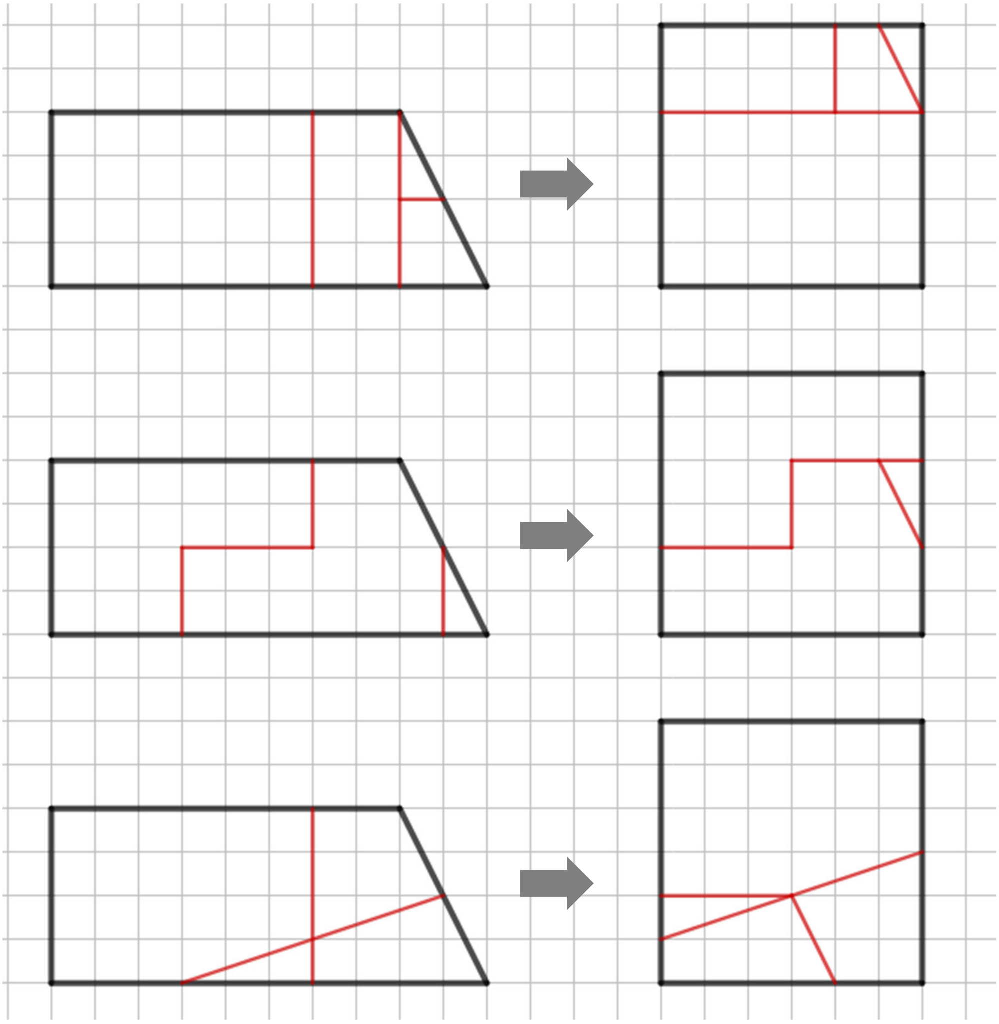
若兩種切割方法得出來的圖形組互為全等，則此兩種方法視之為相同的切割方法。
If the set of pieces obtained by two cutting methods are identical, we say these two cutting methods are the same.

試設計兩種與例子不同的分割方法，把下圖的梯形加畫直線，分割成少於 5 塊，重新組合成一個正方形，並在正方形上畫直線顯示如何組合。
Design two cutting methods different from the example. Draw straight lines on each of the trapezium below to cut it into less than 5 pieces. Rearrange the pieces to form a square and draw straight lines on the squares to show the combinations.


方法一 Method 1:
方法二 Method 2:

建議答案 Suggested solutions:

(1M – 每個方法 /each method)



接受其他合理答案。

Accept other possible answers.

乙部 Section B

(a)	(b)		
S 的可能答案 Possible answers of S	S_1	S_2	S_3
標準差： Standard deviation of $\frac{x_{n+1}}{x_n}-1$ or $\frac{x_{n+1}}{x_n}$	0.078911	0.033549	0.321305
平均離差： Mean deviation of $\frac{x_{n+1}}{x_n}-1$ or $\frac{x_{n+1}}{x_n}$	0.060308	0.058883	0.243416
標準差： Standard deviation of $\lvert x_{n+1}-x_n \rvert$	34.87406	0	24.48589633
平均離差： Mean deviation of $\lvert x_{n+1}-x_n \rvert$	27.67107	0	21.80495
在圖上畫一條最適線並求所有的「偏差」，然後測量這些「偏差」的離差。 Draw a best fit line on the graph to get all “deviations”, then measure the deviation of the “deviations”.	取決於所繪畫的線 Depend on the line drawn.		