2021/22 第十二屆香港中學數學創意解難比賽

2021/22 The 12th Hong Kong Mathematics Creative Problem Solving Competition for Secondary Schools

題解 Solutions

甲部 Section A

1. 等差數列 11,14,17,…,341,344,347 的總項數是 ______________

   The total number of terms in the arithmetic sequence 11, 14, 17, .., 341, 344, 347 is ______________

建議答案 Suggested solutions:

$\{11, 14, 17, ..., 341, 344, 347\}$

$= \{3+8, 6+8, 9+8, ..., 333+8, 336+8, 339+8\}$

$= \{1\times 3+8, 2\times 3+8, 3\times 3+8, ..., 111\times 3+8, 112\times 3+8, 113\times 3+8\}$

總項數是 113。

The number of terms is 113.

---

2. 解方程組：$\begin{cases} 6751x+3249y &= 26751 \\ 3249x+6751y &= 23249 \end{cases}$。

Solve the system of linear equations: $\begin{cases} 6751x+3249y &= 26751 \\ 3249x+6751y &= 23249 \end{cases}$.

建議答案 Suggested solutions:

$\begin{cases} 6751x+3249y &= 26751 \text{......(1)} \\ 3249x+6751y &= 23249 \text{......(2)} \end{cases}$

$(1)+(2)$

$10000x+10000y=50000$

$x+y=5$......(3)

$(1)-(2)$

$3502x-3502y=3502$

$x-y=1$......(4)

$(3)+(4)$

$2x=6$

$x=3$

$(3)-(4)$

$2y=4$

$y=2$

$\therefore x=3,\ y=2$

---

3. 本年度香港中學數學創意解難比賽的舉行日期為 6 月 25 日。一般來說，把月份和日期均以兩位整數表達，再把它們拼在一起，即是以 MMDD 這種型式寫出來時，如果得出數字是一個平方數的話，那天便可稱為「好日子」。例如，6 月 25 日寫成 0625，　而 625 是 25 的平方，所以 6 月 25 日是「好日子」。在 2022 年有多少天是好日子呢?

The competition day of Hong Kong Mathematics Creative Problem Solving Competition for Secondary Schools this year is 25/6. In general, if the month number and the day number are considered as two-digit integers and when the four-digit number obtained by combining the two-digit month number with the two-digit day number becomes a square number, the day is called a "Good Day". For example, consider the date 25/6, the month number is 06 and the day number is 25, the four-digit number 0625 is obtained when they are combined. As 625 is the square of 25, hence 25/6 is a "Good Day".

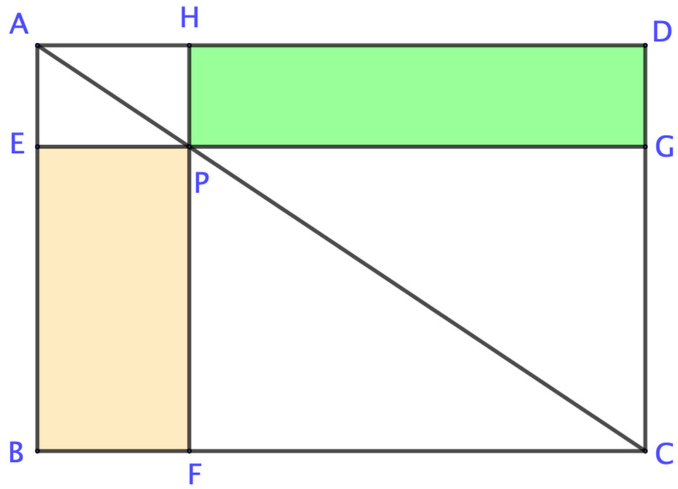
How many "Good Day" are there in the year 2022?

建議答案 Suggested solutions:

月份 Month	「好日子」 "Good Day"	「好日子」數目 Number of "Good Day"
1	21	1
2	25	1
3	24	1
4	/	0
5	29	1
6	25	1
7	29	1
8	/	0
9	/	0
10	24	1
11	/	0
12	25	1
總數 Total		8

4. 圖中，$ABCD$，$EBFP$ 及 $HPGD$ 為長方形。P 是對角線 AC 上的一點。若 $AE = 2$ cm 及 $EBFP$ 的面積 $= 18$ cm^2，求 $HPGD$ 的面積。

In the figure, $ABCD$, $EBFP$, and $HPGD$ are rectangles. P is a point on diagonal AC. If $AE = 2$ cm and area of $EBFP = 18$ cm^2, find the area of $HPGD$.



建議答案 Suggested solutions:

Area of ΔABC = Area of Δ ADC

Area of EBFP

= Area of ΔABC-Area of ΔAEP – Area of ΔPFC

= Area of ΔADC-Area of ΔAHP-Area of ΔPGC

= Area of HPGD

So Area of HPGD=18 cm^2

5. $2022 \times M \times N$ 是一個大於 20220625 的平方數，當中 M 和 N 為正整數。求 $M+N$ 的最小值。

$2022 \times M \times N$ is a square number larger than 20220625, where M and N are positive integers. Find the smallest value of $M+N$.

建議答案 Suggested solutions:

$2022 \times M \times N$

$= 2 \times 3 \times 337 \times M \times N$

若 $M \times N = 2022$ ，$2022 \times M \times N = 4088484 < 20220625$ ，不合題意。

If $M \times N = 2022$, $2022 \times M \times N = 4088484 < 20220625$, which does not satisfy the requirement of the question.

若 $M \times N = 2022 \times 2^2$ ，$2022 \times M \times N = 16353936 < 20220625$ ，不合題意。

If $M \times N = 2022 \times 2^2$, $2022 \times M \times N = 16353936 < 20220625$, which does not satisfy the requirement of the question.

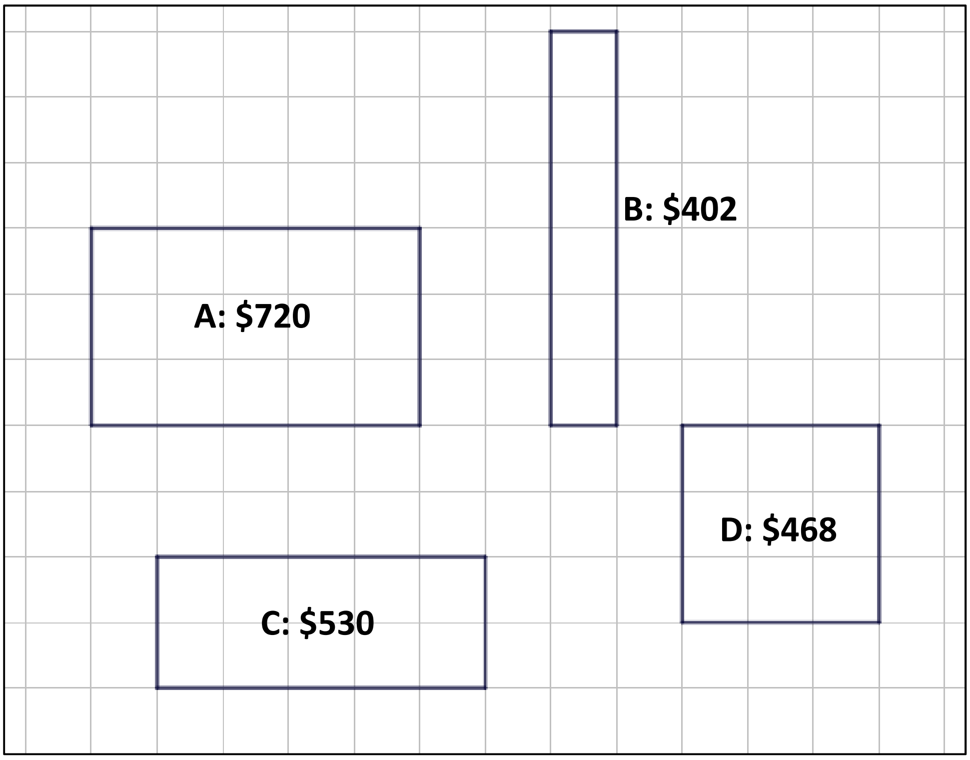
若 $M \times N = 2022 \times 3^2$ ，$2022 \times M \times N = 36796356 > 20220625$ ，符合題意；

If $M \times N = 2022 \times 3^2$, $2022 \times M \times N = 36796356 > 20220625$, which satisfies the requirement of the question;

M	N	$M+N$
1	18198	18199
2	9099	9101
3	6066	6069
6	3033	3039
9	2022	2031
18	1011	1029
27	674	701
54	337	391

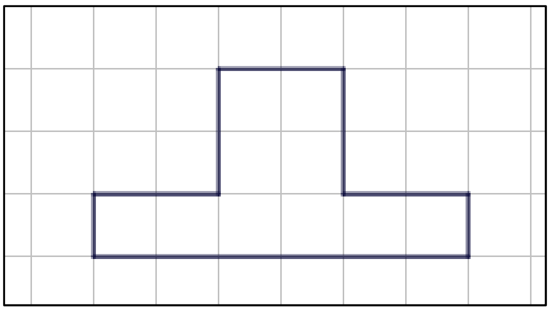
6. 有一間售賣地毯的店鋪，店主以地毯的面積和周界來定價，以下為各款地毯的售價:

In a carpet shop, the owner sets the price according to the area and perimeter of the carpet. The following are the prices of different types of carpets:



根據以上定價的規律，有客人想訂製以下款式的地毯，這款地毯的售價是多少?

A customer wants to order a carpet in the following shape. According to the information above, what is the price of this carpet?



建議答案 Suggested solutions:

每單位面積是$32，每單位長度是$15

因此這款地毯的售價是: $10 \times 32 + 15 \times 18 = \$590$

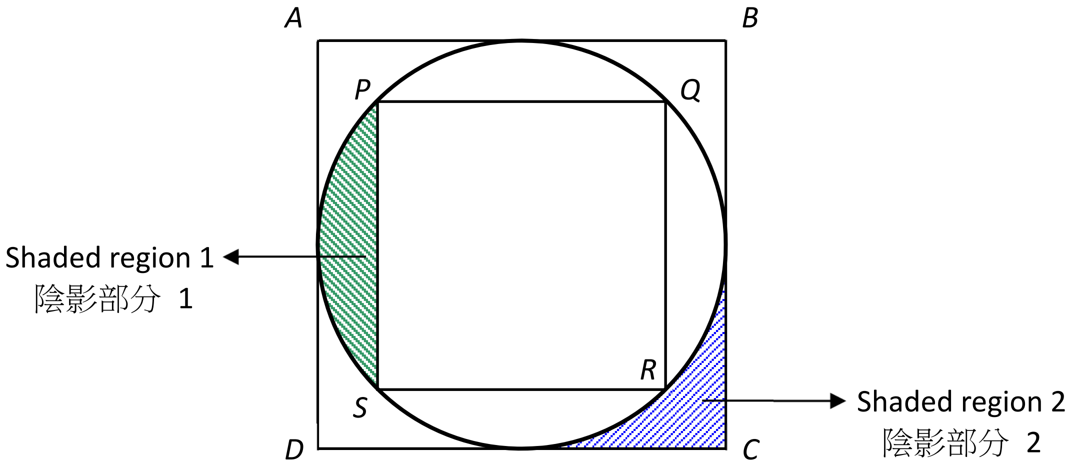
The price per unit area is $32 and the price per unit length is $15

Hence, the price of the carpet is: $10 \times 32 + 15 \times 18 = \$590$

7. In the figure, the circle circumscribes the square *ABCD*. Another square *PQRS* is inscribed in the circle. Find $\dfrac{\text{area of shaded region 1}}{\text{area of shaded region 2}}$.

圖中的圓外接正方形 *ABCD*。另一正方形 *PQRS* 內接該圓。

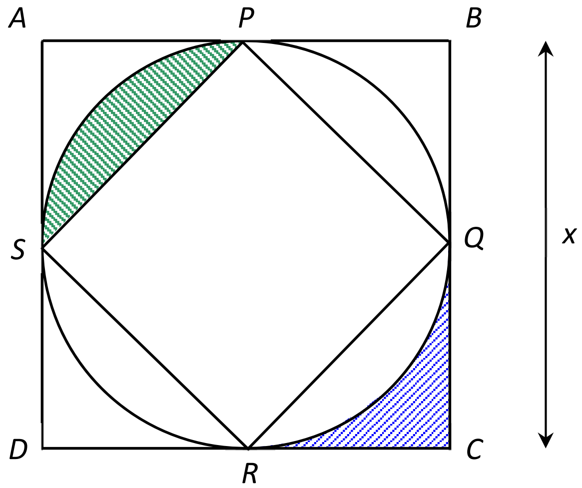
求 $\dfrac{\text{陰影部分1的面積}}{\text{陰影部分2的面積}}$。



建議答案 Suggested solutions:

Let the length of AD (length of outer square) be x . Rotate the square $PQRS$ 45° clockwise about the centre of the circle as shown.

設 AD 的長度（外接正方形的邊長）為 x。如圖，把正方形 $PQRS$ 繞圓的圓心順時針方向旋轉 45°。



PR is the diagonal of the square $PQRS$ and the length is equal to $AD = x$.

PR 為正方形 $PQRS$ 的對角線，其長度等於 $AD = x$。

$$\because \quad \text{Area of shaded region 2} = \frac{x^2 - \pi\left(\frac{x}{2}\right)^2}{4} = \frac{x^2}{16}(4-\pi)$$

$$\therefore \quad \frac{\text{area of shaded region 1}}{\text{area of shaded region 2}} = \frac{\frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{x^2}{16}(4-\pi)}{\frac{x^2}{16}(4-\pi)} = \frac{\pi-2}{4-\pi}$$

$$\because \quad \text{陰影部分 2 的面積} = \frac{x^2 - \pi\left(\frac{x}{2}\right)^2}{4} = \frac{x^2}{16}(4-\pi)$$

$$\therefore \quad \frac{\text{陰影部分1的面積}}{\text{陰影部分2的面積}} = \frac{\frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{x^2}{16}(4-\pi)}{\frac{x^2}{16}(4-\pi)} = \frac{\pi-2}{4-\pi}$$

8. 設 x 、 y 、 z 為滿足 $x+y+z+xy+yz+zx+xyz=2021$ 的非零整數。求 xyz 的最大可能值。

Let x, y, z be non-zero integers satisfying $x+y+z+xy+yz+zx+xyz=2021$. Find the greatest possible value of xyz.

建議答案 Suggested solutions:

$x+y+z+xy+yz+zx+xyz=2021$

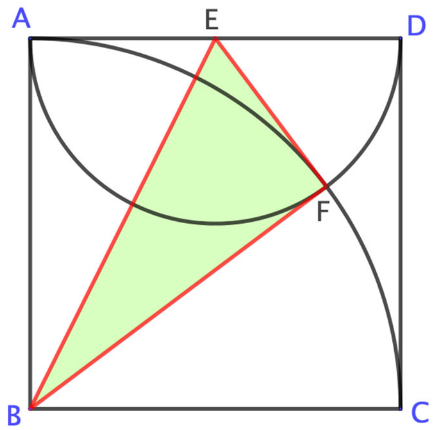
$1+x+y+z+xy+yz+zx+xyz=2022$

$(1+x)(1+y)(1+z)=2022=2\times3\times337$

$1+x$	$1+y$	$1+z$	x	y	z	xyz
2	3	337	1	2	336	672
–2	–3	337	–3	–4	336	4032
–2	3	–337	–3	2	–338	2028
2	–3	–337	1	–4	–338	1352
–1	–3	674	–2	–4	673	5384
–1	3	–674	–2	2	–675	2700
–1	–2	1011	–2	–3	1010	6060
–1	2	–1011	–2	1	–1012	2024
–1	–1	2022	–2	–2	2021	8084

9. 圖中，$ABCD$ 為一正方形。以 AD 為直徑的半圓與弧 CFA 相交於 F。已知 E 是 AD 的中點。若 ΔBEF 的面積是 25 cm^2，求正方形的面積。

In the figure, $ABCD$ is a square. Semi-circle with diameter AD and circular arc CFA meet at F. Given that E is the mid-point of AD. If the area of ΔBEF is 25 cm^2, find the area of the square.



建議答案 Suggested solutions:

$EF = EA$ $(radius)$

$AB = BF$ $(radius)$

$BE = BE$ $(common)$

$\Delta AEB \cong \Delta EFB$ (SSS)

Area of ABCD = (25)(4) = 100 cm^2

10. 以三種不同方式把 2、3、4、5、6、7、8、9 填到以下的空格內，使得不等式成立。

Fill in the following boxes by 2, 3, 4, 5, 6, 7, 8, 9 in three different ways to make the inequality valid.

(1) $\square^{\square} < 2022 < \square^{\square} < \square^{\square} < \square^{\square}$

(2) $\square^{\square} < \square^{\square} < 2022 < \square^{\square} < \square^{\square}$

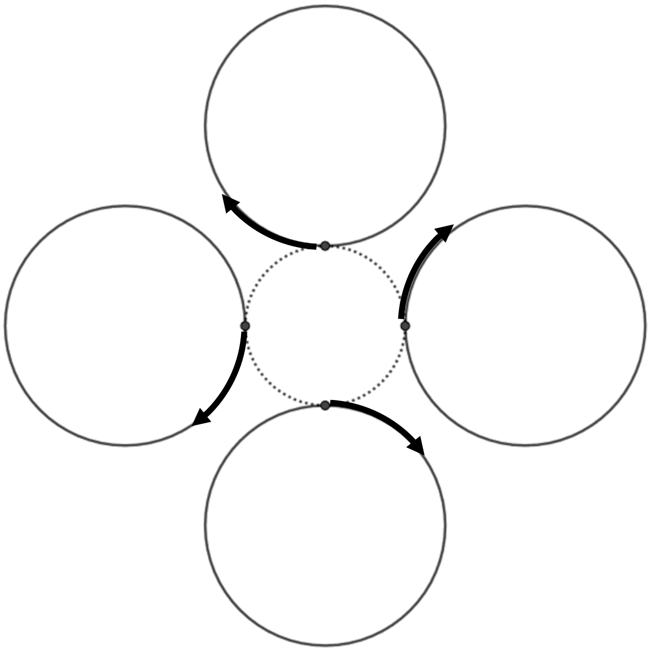
(3) $\square^{\square} < \square^{\square} < \square^{\square} < 2022 < \square^{\square}$

建議答案 Suggested solutions:

任何合理答案皆可接受/Any reasonable answer is acceptable

11. 四位單車手在各自圓周為$\frac{1}{3}$公里的圓形軌道上踏單車。他們同時在圖中的黑點上出發，車速分別是每小時 6, 9, 12 及 15 公里。在 20 分鐘的旅程上，他們會同時到達各自的黑點多少次？

Four cyclists do their act on circular paths, each $\frac{1}{3}$ km long. They start simultaneously at the black spots, with speeds of 6, 9, 12 and 15 km per hour. By the end of the act (20 minutes), how many times will they have simultaneously returned to the spots where they started?



建議答案 Suggested solutions:

每位單車手在 20 分鐘的旅程上所走的距離(公里)是

Total distance traveled (in km) by each cyclist in the act (20 minutes) are

$6\times\frac{1}{3}$ $9\times\frac{1}{3}$ $12\times\frac{1}{3}$ $15\times\frac{1}{3}$

每位單車手完成的圈數是：

No. of complete circular path finished by each cyclist:

$6\times\frac{1}{3}\div\frac{1}{3}$ $9\times\frac{1}{3}\div\frac{1}{3}$ $12\times\frac{1}{3}\div\frac{1}{3}$ $15\times\frac{1}{3}\div\frac{1}{3}$

$=6$ $=9$ $=12$ $=15$

$\because$ 6，9，12 及 15 可被 3 整除。

$\because$ 6, 9, 12 and 15 are divisible by 3.

$\therefore$ 在 20 分鐘的第一個 $\frac{1}{3}$ 、第二個 $\frac{1}{3}$ 及第三個 $\frac{1}{3}$ 的時刻，所有單車手會到達各自的黑點。

$\therefore$ At the moment of the 1^{st} $\frac{1}{3}$, the 2^{nd} $\frac{1}{3}$ and the 3^{rd} $\frac{1}{3}$ of 20 minutes, all the cyclists will return to their starting position.

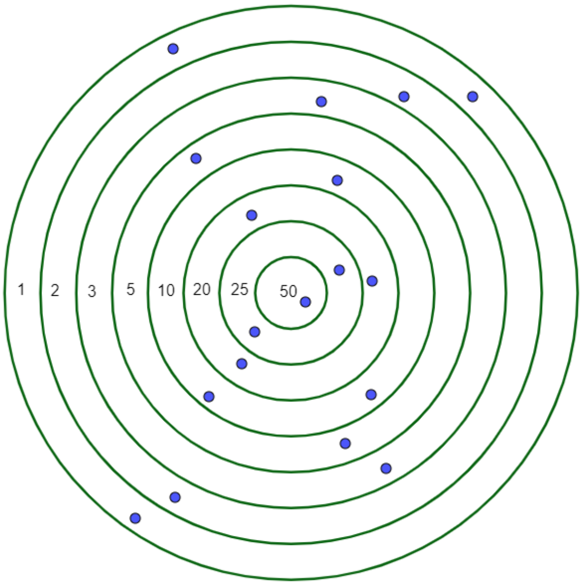
即 $\frac{1}{3}\times 20=6\frac{2}{3}$ 分鐘、$2\times\frac{1}{3}\times 20=13\frac{1}{3}$ 分鐘及 $3\times\frac{1}{3}\times 20=20$ 分鐘

i.e. $\frac{1}{3}\times 20=6\frac{2}{3}$ minutes, $2\times\frac{1}{3}\times 20=13\frac{1}{3}$ minutes and $3\times\frac{1}{3}\times 20=20$ minutes

$\therefore$ 所需答案為 3。

$\therefore$ The required answer is 3.

12. 在一個射擊比賽中，小創、小意及小蘭分別開了 6 槍並都得了 71 分。
小創頭兩槍合共得了 22 分而小蘭第一槍得了 3 分。
他們當中誰人射中了紅心呢？
In a shooting match, Ada, Ben and Christine each fired 6 shots and each got 71 points.
Ada's first two shots got 22 points and Christine's first shot got only 3 points.
Who hit the bull's-eye?



建議答案 Suggested solutions:

只有 3 種情況可得到 71 分。

Only 3 ways to get 71 points.

I) 25, 20, 20, 3, 2, 1

II) 25, 20, 10, 10, 5, 1

III) 50, 10, 5, 3, 2, 1

I 是唯一可令頭兩槍合共得 22 分的情況

∴ 情況 I 是小創。

I is the only possible to have 22 points in the first two shots.

∴ I is Ada.

摒除情枧 I 之後，III 是為一可令頭一槍得 3 分的情況

∴ 情況 III 是小蘭。

By excluding I, III is the only possible to have 3 points in the first shot.

∴ III is Christine.

由此可知，情況 II 是小意

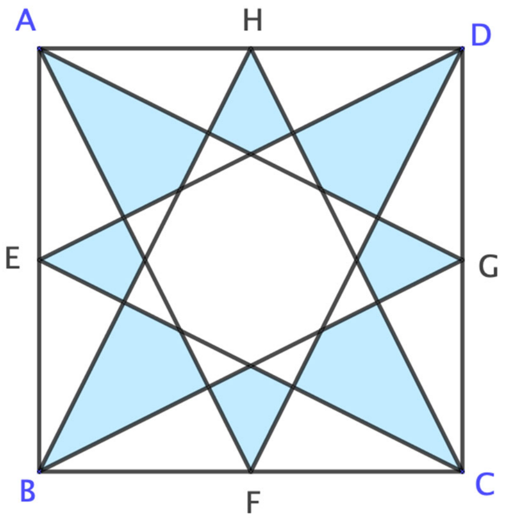
Hence, II is Ben

∴ 小蘭射中了紅心。

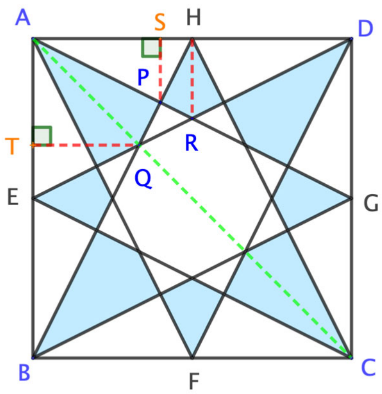
∴ Christine hit the bull's-eye.

13. 圖中，$ABCD$ 是一個正方形，E、F、G、H 分別是 AB、BC、CD、DA 的中點。若 $AB = 30$ cm，求藍色部份的面積。

In the figure, $ABCD$ is a square and E, F, G, H are mid-points of AB, BC, CD, DA respectively. If $AB = 30$ cm, find the area of the blue parts.



建議答案 Suggested solutions:



It is not difficult to show that

$AH = \frac{1}{2}AB = 15\ \ cm$

$SP = \frac{2}{5}DG = 6\ \ cm$

$HR = \frac{1}{2}DG = 7.5\ \ cm$

$TQ = \frac{1}{3}AB = 10\ \ cm$

Thus, Area of a small blue kite

$= (15)(7.5) - (15)(6) = 22.5\ \ cm^2$

Area of a big blue kite

$= (15)(10) - (15)(6) = 60\ \ cm^2$

So area of blue parts $= 4(60 + 22.5) =$

$330\ \ cm^2$

乙部（建議此部用 15 分鐘作答）

Section B (Suggested to use 15 minutes in this Section)

1. 必勝和當勞玩一個遊戲。兩人輪流放下 1 至 10 元的整數硬幣，第一個放到 100 元正的便獲勝。
必勝爭取第一個並放下 1 元後，他便滿有信心說：「我必能勝出這場比賽。」
Wynner and Donald play a game together. They put down coins of value ranges from $1 to $10 alternately. The one who puts the coins which makes the total value $100 will be the winner of the game.
Wynner is the first one to put down coins. After he puts down a $1 coin, he claims that he must be able to win this competition.

(a) 試以數學的方法解釋必勝必能獲得勝利的原因。
Explain, with mathematical arguments, why Wynner is confident with winning the game.

建議答案 Suggested solutions:

由於$100-1=99$可被 11 整除，所以必勝放下第一個一元硬幣後，只要每當當勞放下k元的硬幣，必勝便放下$(11-k)$元的硬幣，便可拼至 100 元正的目標。

As $100-1=99$ is divisible by 11, after Wynner puts down the first $1 coin, when Donald puts down coins of $\$k$ in each subsequent round, Wynner responds by putting down coins of $\$(11-k)$. As a result, Wynner can make the total of $100.

(b) 若遊戲規則不改變，但改為第一個放到 200 元正的便獲勝。試解釋必勝是否應該繼續爭取第一個放下硬幣，使得他必能勝出這場比賽。
If the other regulations are unchanged, but the one who puts the coins which makes the total value $200 will be the winner of the game. Explain whether Wynner should put down coins first to ensure that he could win the game.

建議答案 Suggested solutions:

由於$200-2=198$可被 11 整除，所以必勝應爭取第一個放下硬幣，並放下 2 元的硬幣，然後每當當勞放下k元的硬幣，必勝便放下$(11-k)$元的硬幣，便可拼至 200 元正的目標。

As $200-2=198$ is divisible by 11, Wynner should put down a $2 coin first. When Donald puts down coins of $\$k$ in each subsequent round, Wynner responds by putting down coins of $\$(11-k)$. As as result, Wynner can make the total of \$200.

(c) 若遊戲規則不改變，但改為第一個放到T 元正($T>200$)的便獲勝。試寫下 3 個不同的n並解釋原因，使得第一個放下硬幣的人必不能勝出這場比賽。

If the other regulations are unchanged, but the one who puts the coins which makes the total value $\$T$ will be the winner of the game, where $T>200$. Write down 3 different values of T, with explanation, so that the first one who put down coins must not be the winner of the game.

建議答案 Suggested solutions:

209, 220, 231, 242, 253, … 或其他可被 11 整除並大於 200 的整數。

由於這些T 的值可被 11 整除，所以每當第一人放下k元的硬幣後，第二人便放下$(11-k)$ 元的硬幣，第二人便可拼至n元正的目標。

209, 220, 231, 242, 253, … or other multiples of 11 larger than 200.

As these values of T are divisible by 11, when the first player puts down coins of $\$k$, the second player responds by putting down coins of $\$(11-k)$. As a result, the second player will make the total of $\$T$.

(d) 必勝改為與小創及小意玩遊戲，三人輪流放下硬幣，若小創及小意可放下 1 至 8 元的整數硬幣，而必勝可放下 1 至n元的整數硬幣，第一個放到T 元正($T>200$)的便獲勝。

必勝爭取第一個並放下a元後，他便滿有信心說：「我必能勝出這場比賽。」

試寫出一組n、T及a的數值，或以數學的方法表達n、T及a之間的關係，並解釋必勝如何能勝出這場比賽。

Wynner plays the game with Ada and Ben. They put down coins alternately. Ada and Ben can put down coins of values range from $\$1$ to $\$8$, while Wynner can put down coins of values range from $\$1$ to $\$n$. The one who puts the coins which makes the total value $\$T$, where $T>200$, will be the winner of the game.

Wynner is the first one to put down coins. After he puts down coins of value $\$a$, he claims that he must be able to win this competition.

Write down a set of values of n, T and a, or express the mathematical relation between n, T and a. Explain, with mathematical arguments, why Wynner is confident with winning the game.

建議答案 Suggested solutions:

$(T-a)$ 可被 $(2+n)$ 整除，即 $(2+n)\,|\,(T-a)$ ，及 $n>14$

$(T-a)$ is divisible by $(2+n)$, i.e. $(2+n)\,|\,(T-a)$, where $n>14$

n	T	a
15	205	1
16	218	2
…	…	…

必勝放下 a 元的硬幣後，小創與小意每輪合共放下的幣值為 k 元，其中 $2\le k\le 16$；必勝只需在同一輪中放下 $n+2-k$ 元的幣值，便可拼夠 $n+2$ 的幣值。由於 $T-a$ 可被 $n+2$ 整除，所以必勝最終便可拼至 T 元的目標。

Wynner puts down coins of $\$a$ first. Afterwards, when Ada and Ben put down coins of a total value of $\$k$, where $2\le k\le 16$, Wynner responds by putting down coins of $\$(n+2-k)$ to make a total of $\$(n+2)$ in each subsequent round. As $T-a$ is divisible by $n+2$, Wynner could make a total of $\$T$ at the end.
